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COMPACTNESS IN DITOPOLOGICAL TEXTURE SPACES

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Abstract

This paper considers compactness in a ditopological setting. After a brief introduction, Section 2 is devoted to compact and cocompact spaces. Results include preservation under surjective (co) continuous difunctions, and analogues of the Mrowka Charcterization and Tychonoff Product Theorem. Stability and costability are discussed in Section 3. Here generalizations of several results concerning separation are presented, characterizations of the compact and cocompact elements of the texturing given under suitable conditions and the preservation of stability and costability under surjective bicontinuous difunctions established. Finally Section 4 considers dicompactness and cumulates with a proof of the Tychonoff Product Theorem for this very important class of ditopological texture spaces.

Keywords: Ditopological texture space, Compactness, Cocompactness, Stability, Costability, Dicompactness, Mrowaka characterization theorem, Tychonoff product theorem..

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1. Introduction

Textures were introduced by the first author as a point-set setting for the study of fuzzy sets, but they have since proved useful as a framework in which to discuss complement-free mathematical concepts. There is now a considerable literature on this subject, and an adequate introduction to the theory and the motivation for its study may be obtained from [2, 3, 4, 5, 6, 7].

Briefly, if S is a set, a *texturing* $\mathcal{S}$ of S is a subset of $\mathcal{P}(S)$ which is a point-separating, complete, completely distributive lattice containing S and $\emptyset$, and for which meet coincides

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with intersection and finite joins with union. The pair $(S, \mathcal{S})$ is then called a *texture*. We regard a texture $(S, \mathcal{S})$ as a framework in which to do mathematics.

For a texture $(S, \mathcal{S})$, most properties are conveniently defined in terms of the *p-sets* $P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}$ and the *q-sets*, $Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\}$. We recall from [5] the following fundamental properties:

(1) For $A, B \in \mathcal{S}$, if $A \not\subseteq B$ then there exists $s \in S$ with $A \not\subseteq Q_s$ and $P_s \not\subseteq B$.
(2) $A = \bigcap\{Q_s \mid P_s \not\subseteq A\}$ for all $A \in \mathcal{S}$.
(3) $A = \bigvee\{P_s \mid A \not\subseteq Q_s\}$ for all $A \in \mathcal{S}$.

For $A \in \mathcal{S}$ the *core* $A^\flat$ *of* A is given by $A^\flat = \{s \in S \mid A \not\subseteq Q_s\}$. The set $A^\flat$ does not necessarily belong to $\mathcal{S}$, but we note for future reference that for $A, B \in \mathcal{S}$ we have $A \subseteq B \iff A^\flat \subseteq B^\flat$.

The following are some basic examples of textures we will need later on.

1.1. Examples. (1) If X is a set and $\mathcal{P}(X)$ the powerset of X, then $(X, \mathcal{P}(X))$ is the *discrete texture* on X. For $x \in X$, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$.

(2) Setting $\mathbb{I} = [0, 1]$, $\mathcal{I} = \{[0, r), [0, r] \mid r \in \mathbb{I}\}$ gives the *unit interval texture* $(\mathbb{I}, \mathcal{I})$. For $r \in \mathbb{I}$, $P_r = [0, r]$ and $Q_r = [0, r)$.

(3) The texture $(L, \mathcal{L})$ is defined by $L = (0, 1]$, $\mathcal{L} = \{(0, r] \mid r \in \mathbb{I}\}$. For $r \in L$, $P_r = (0, r] = Q_r$.

As noted in [1, 8] we may associate with $(S, \mathcal{S})$ the C-space (core-space) [10] $(S, \mathcal{S}^c)$, and then the frequently occurring relationship $P_{s_2} \not\subseteq Q_{s_1}$, $s_1, s_2 \in S$, is equivalent to $s_1 \,\omega_S\, s_2$, where ω_S is the *interior relation* for $(S, \mathcal{S}^c)$. For the above examples $x_1 \,\omega_X\, x_2 \iff x_1 = x_2$, $r_1 \,\omega_{\mathbb{I}}\, r_2 \iff r_1 \le r_2$ and $r_1 \,\omega_L\, r_2 \iff r_1 < r_2$, respectively.

In general a texturing $\mathcal{S}$ need not be closed under the operation of taking the set complement, so we must forego the usual relationship between the open and closed sets. In the context of a texture $(S, \mathcal{S})$ the notion of topology is therefore replaced by that of dichotomous topology. Specifically, a *dichotomous topology*, or *ditopology* for short, on a texture $(S, \mathcal{S})$ is a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of *open sets* τ satisfies

(1) $S, \emptyset \in \tau$,
(2) $G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau$ and
(3) $G_i \in \tau$, $i \in I \implies \bigvee_i G_i \in \tau$,

and the set of *closed sets* κ satisfies

(1) $S, \emptyset \in \kappa$,
(2) $K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa$ and
(3) $K_i \in \kappa$, $i \in I \implies \bigcap K_i \in \kappa$.

For $A \in \mathcal{S}$ the *closure* $[A]$ and *interior* $]A[$ of A are given by

$$[A] = \bigcap\{K \in \kappa \mid A \subseteq K\} \text{ and }]A[= \bigvee\{G \in \tau \mid G \subseteq A\}$$

respectively.

If (X, u, v) is a bitopological space [13] then (u, v^c) is a ditopology on $(X, \mathcal{P}(X))$. In particular, (u, u^c) is a ditopology on $(X, \mathcal{P}(X))$ induced by the topology u on X. It is special in the following sense. An inclusion-reversing idempotent mapping $\sigma : \mathcal{S} \to \mathcal{S}$, where one exists, is known as a *complementation* on $(S, \mathcal{S})$. Should $\kappa = \sigma(\tau)$ then $(S, \mathcal{S}, \sigma, \tau, \kappa)$ is said to be a *complemented ditopological texture space*. The ditopology (u, u^c) is clearly complemented for the complementation $\pi_X : \mathcal{P}(X) \to \mathcal{P}(X)$ given by $\pi_X(Y) = X \setminus Y$. The texture $(\mathbb{I}, \mathcal{I})$ has a natural complementation ι, $\iota([0, r)) = [0, 1-r]$, $\iota([0, r]) = [0, 1-r)$ and complemented ditopology $\tau_{\mathbb{I}} = \{[0, r) \mid r \in \mathbb{I}\} \cup \{\mathbb{I}\}$, $\kappa_{\mathbb{I}} = \{[0, r] \mid$

$r \in \mathbb{I}\} \cup \{\emptyset\}$; while $(L, \mathcal{L})$ has a natural complementation λ, $\lambda((0,r]) = (0, 1-r]$ and complemented ditopology $(\mathcal{L}, \mathcal{L})$.

In addition to the link with topological and bitopological spaces mentioned above, fuzzy sets and topologies may also be represented naturally as ditopological texture spaces [3], although we will not pursue this aspect here.

We recall the product of textures and of ditopological texture spaces. Let $(S_j, \mathcal{S}_j)$, $j \in J$, be textures and $S = \prod_{j \in J} S_j$. If $A_k \in \mathcal{S}_k$ for some $k \in J$ we write

$$E(k, A_k) = \prod_{j \in J} Y_j \text{ where } Y_j = \begin{cases} A_j, & \text{if } j = k \\ S_j, & \text{otherwise.} \end{cases}$$

Then the *product texturing* $\mathcal{S} = \bigotimes_{j \in J} \mathcal{S}_j$ of S consists of arbitrary intersections of elements of the set

$$\mathcal{E} = \left\{ \bigcup_{j \in J} E(j, A_j) \mid A_j \in \mathcal{S}_j \text{ for } j \in J \right\}.$$

Let $(S_j, \mathcal{S}_j)$, $j \in J$ be textures and $(S, \mathcal{S})$ their product. Then for $s = (s_j) \in S$,

$$P_s = \bigcap_{j \in J} E(j, P_{s_j}) = \prod_{j \in J} P_{s_j}, \text{ and } Q_s = \bigcup_{j \in J} E(j, Q_{s_j}).$$

It is easy to verify that for $A_j \in \mathcal{S}_j$, $j \in J$ we have $\prod_{j \in J} A_j \in \mathcal{S}$ and $\left(\prod_{j \in J} A_j \right)^\flat = \prod_{j \in J} A_j^\flat$.

In case (τ_j, κ_j) is a ditopology on $(S_j, \mathcal{S}_j)$, $j \in J$, the *product ditopology* on the product texture $(S, \mathcal{S})$ has subbase $\{E(j, G) \mid G \in \tau_j, j \in J\}$, cosubbase $\gamma = \{E(j, K) \mid K \in \kappa_j, j \in J\}$.

One of the most useful notions in the theory of (ditopological) texture spaces is that of difunction [5]. A difunction is a special type of direlation. Specifically, if $(S, \mathcal{S})$, $(T, \mathcal{T})$ are textures we will denote by $\overline{P}_{(s,t)}$, $\overline{Q}_{(s,t)}$ respectively the p-sets and q-sets for the texture $(S \times T, \mathcal{P}(S) \otimes \mathcal{T})$. Then:

(1) $r \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *relation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies
  R1 $r \not\subseteq \overline{Q}_{(s,t)}, P_{s'} \not\subseteq Q_s \implies r \not\subseteq \overline{Q}_{(s',t)}$.
  R2 $r \not\subseteq \overline{Q}_{(s,t)} \implies \exists s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}_{(s',t)}$.
(2) $R \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *corelation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies
  CR1 $\overline{P}_{(s,t)} \not\subseteq R, P_s \not\subseteq Q_{s'} \implies \overline{P}_{(s',t)} \not\subseteq R$.
  CR2 $\overline{P}_{(s,t)} \not\subseteq R \implies \exists s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}_{(s',t)} \not\subseteq R$.
(3) A pair (r, R), where r is a relation and R a corelation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ is called a *direlation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$.

Inverses and compositions of direlations are given in [5]. The notion of difunction is derived from that of direlation as follows.

1.2. Definition. Let (f, F) be a direlation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Then (f, F) is called a *difunction from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies the following two conditions.

DF1 For $s, s' \in S$, $P_s \not\subseteq Q_{s'} \implies \exists\, t \in T$ with $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F$.
DF2 For $t, t' \in T$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t')} \not\subseteq F \implies P_{t'} \not\subseteq Q_t$.

Difunctions are preserved under composition, and there is a natural identity difunction so one may consider the category **dfTex** of textures and difunctions [5].

Let (f, F) be a difunction from $(S, \mathcal{S})$ to $(T, \mathcal{T})$, and $B \in \mathcal{T}$. Then the *inverse image* $f^{\leftarrow}(B)$ and the *inverse co-image* $F^{\leftarrow}(B)$ of B are given by the formulae

$$\begin{aligned} f^{\leftarrow}(B) &= \bigvee\{P_s \mid \forall\, t,\ f \not\subseteq \overline{Q}_{(s,t)} \implies P_t \subseteq B\} \in \mathcal{S}, \text{ and} \\ F^{\leftarrow}(B) &= \bigcap\{Q_s \mid \forall\, t,\ \overline{P}_{(s,t)} \not\subseteq F \implies B \subseteq Q_t\} \in \mathcal{S}, \end{aligned} \tag{1.1}$$

respectively. It is known that for difunctions the inverse image and inverse co-image coincide for all $B \in \mathcal{T}$, and that they preserve arbitrary intersections and joins. In case we have ditopologies (τ_S, κ_S), (τ_T, κ_T) on $(S, \mathcal{S})$, $(T, \mathcal{T})$ respectively then (f, F) is called *continuous* if $G \in \tau_T \implies F^{\leftarrow}G \in \tau_S$, *cocontinuous* if $K \in \kappa_T \implies f^{\leftarrow}K \in \kappa_S$ and *bicontinuous* if it is both. The category of ditopological texture spaces and bicontinuous difunctions is denoted by **dfDitop**.

The *image* $f^{\rightarrow}A$ and *co-image* $F^{\rightarrow}A$ of $A \in \mathcal{S}$ under (f, F) are given by

$$\begin{aligned} f^{\rightarrow}A &= \bigcap\{Q_t \mid \forall\, s,\ f \not\subseteq \overline{Q}_{(s,t)} \implies A \subseteq Q_s\}, \text{ and} \\ F^{\rightarrow}A &= \bigvee\{P_t \mid \forall\, s,\ \overline{P}_{(s,t)} \not\subseteq F \implies P_s \subseteq A\}, \end{aligned} \tag{1.2}$$

respectively. Interrelations between these and the inverse (co-) image are given in [5]. If $(f, F) : (S, \mathcal{S}, \tau_S, \kappa_S) \to (T, \mathcal{T}, \tau_T, \kappa_T)$ is a difunction then

(1) (f, F) is *open* (*co-open*) if $G \in \tau_S \implies f^{\rightarrow}G \in \tau_T$ ($F^{\rightarrow}G \in \tau_T$).
(2) (f, F) is *closed* (*coclosed*) if $K \in \kappa_S \implies f^{\rightarrow}K \in \kappa_T$ ($F^{\rightarrow}K \in \kappa_T$).

1.3. Definition. Let (f, F) be a difunction from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Then (f, F) is called *surjective* if it satisfies the condition

SUR. For $t, t' \in T$, $P_t \not\subseteq Q_{t'} \implies \exists\, s \in S$ with $f \not\subseteq \overline{Q}_{(s,t')}$ and $\overline{P}_{(s,t)} \not\subseteq F$.

Likewise, (f, F) is called *injective* if it satisfies the condition

INJ. For $s, s' \in S$ and $t \in T$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F \implies P_s \not\subseteq Q_{s'}$.

A difunction which is both surjective and injective is called *bijective*. The bijective difunctions are precisely the isomorphisms of **dfTex** [5, Proposition 3.14 (5)]. The isomorphisms of **dfDitop** are known as *dihomeomorphisms* [1]. These are the bijective difunctions which, together with their inverse are bicontinuous.

In general difunctions are not directly related to ordinary (point) functions between the base sets, but we recall from [5, Lemma 3.4] that if $(S, \mathcal{S})$, $(T, \mathcal{T})$ are textures and $\varphi : S \to T$ an ω-compatible point function, namely one satisfying $P_s \not\subseteq Q_{s'} \implies P_{\varphi(s)} \not\subseteq Q_{\varphi(s')}$, then the formulae

$$\begin{aligned} f_\varphi &= \bigvee\{\overline{P}_{(s,t)} \mid \exists\, u \in S \text{ with } P_s \not\subseteq Q_u \text{ and } P_{\varphi(u)} \not\subseteq Q_t\}, \\ F_\varphi &= \bigcap\{\overline{Q}_{(s,t)} \mid \exists\, v \in S \text{ with } P_v \not\subseteq Q_s \text{ and } P_t \not\subseteq Q_{\varphi(v)}\}, \end{aligned} \tag{1.3}$$

define a difunction (f_φ, F_φ) from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Moreover, it is easy to verify that for each $B \in \mathcal{T}$ we have $f_\varphi^{\leftarrow}B = \varphi^{\leftarrow}B = F_\varphi^{\leftarrow}B$, where

$$\varphi^{\leftarrow}B = \bigvee\{P_u \mid \varphi(u) \in B\} = \bigcap\{Q_v \mid \varphi(v) \notin B\}. \tag{1.4}$$

Breaking the link between the open and closed sets means that certain results in the theory of topological spaces cannot hold in the theory of general ditopological texture spaces. For example, while every open cover of a topological space has a finite subcover if and only if every family of closed sets with the finite intersection has a non-empty intersection, this does not hold for ditopologies in general. The first statement is taken as the definition of *compactness*, and the second as a *dual concept* called *cocompactness*. Such pairs of dual properties occur often in the theory of ditopological texture spaces, although some properties, such as normality, turn out to be *self-dual*.

Compactness and cocompactness in ditopological texture spaces are investigated in Section 2. Preservation under surjective (co) continuous difunctions is established, generalizations of the Mrowka Characterization Theorem [9, 15] are proved and versions of the Tychonoff product theorem are given for both compact and cocompact ditopological texture spaces.

Compact ditopological texture spaces lack many of the nice properties of compact spaces, and the same is true of cocompact spaces. For instance, closed elements of the texturing need not be compact, and non of the results concerning separation axioms in topological spaces hold in the ditopological case. After giving some counterexamples, Section 3 presents the dual notions of *stability* and *costability* for ditopological texture spaces. It is shown that such spaces have some of the pleasant properties of compact topological spaces. In particular many results concerning separation axioms are shown to have appropriate generalizations. Under suitable conditions the compact sets of a stable ditopological space are characterized in terms of *pseudo-closed* sets, and dual results are given for costable spaces. Finally, it is shown that stability and costability are preserved under bicontinuous surjective difunctions.

Ditopological texture spaces which have all of the four properties compact, stable, cocompact and costable are discussed in Section 4 under the name *dicompact*. Characterizations in terms of *dicovers* and in terms of difamilies satisfying the *finite exclusion property* are recalled [2], and a version of the Tychonoff Product Theorem is presented.

The reader is referred to [11] for concepts from lattice theory not defined here.

This paper is largely based on previously unpublished work from the PhD thesis of the second author [12], but several results have been reformulated and extended, and new material added.

2. Compactness and cocompactness

We begin by considering a direct generalization of the topological notion of compactness. Let (τ, κ) be a ditopology on the texture space $(S, \mathcal{S})$ and take $A \in \mathcal{S}$. The family $\{G_i \mid i \in I\}$ is said to be an *open cover of* A if $G_i \in \tau$ for all $i \in I$ and $A \subseteq \bigvee_{i \in I} G_i$. Dually we may speak of a *closed cocover of* A, namely a family $\{F_i \mid i \in I\}$ with $F_i \in \kappa$ for all $i \in I$ satisfying $\bigcap_{i \in I} F_i \subseteq A$. Let us now recall [2],

2.1. Definition. Let (τ, κ) be a ditopology on the texture $(S, \mathcal{S})$ and $A \in \mathcal{S}$.

(i) A is called *compact* if whenever $\{G_i \mid i \in I\}$ is an open cover of A then there is a finite subset J of I with $A \subseteq \bigcup_{j \in J} G_j$. In particular the ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is called *compact* if S is compact.

(ii) A is *cocompact* if whenever $\{F_i \mid i \in I\}$ is a closed cocover of A there is a finite subset J of I with $\bigcap_{j \in J} F_j \subseteq A$. In particular the ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is called *cocompact* if $\emptyset$ is cocompact.

In general compactness and cocompactness are independent, as the following examples show.

2.2. Example. Consider the texture $(L, \mathcal{L})$ of Example 1.1 (3).

(1) Define the ditopology (τ, κ) by $\tau = \{\emptyset, L\}$ and $\kappa = \mathcal{L}$. Since τ is finite, (τ, κ) is compact. However, it is not cocompact since, for example, the family $\mathcal{F} = \{(0, 1/n] \mid n = 1, 2, \ldots\}$ of closed sets satisfies $\bigcap \mathcal{F} = \emptyset$, but no finite subset of $\mathcal{F}$ has an empty intersection.

(2) Dually let $\tau = \mathcal{L}$ and $\kappa = \{\emptyset, L\}$. Then the ditopology (τ, κ) is cocompact but not compact.

On the other hand, for complemented ditopological texture spaces these two properties are equivalent.

2.3. Proposition. *Let (τ,κ) be a complemented ditopology on $(S,\mathcal{S},\sigma)$. Then $(S,\mathcal{S},\sigma,\tau,\kappa)$ is compact if and only if it is cocompact.*

Proof. Suppose that (τ,κ) is compact and let $\mathcal{F}=\{F_i \mid i\in I\}$ be a family of closed sets with $\bigcap\mathcal{F}=\emptyset$. Consider the family $\mathcal{G}=\{\sigma(F_i)\mid i\in I\}$ of open sets. Then

$$\bigvee\mathcal{G}=\bigvee\{\sigma(F_i)\mid i\in I\}=\sigma(\bigcap\{F_i\mid i\in I\})=\sigma(\emptyset)=S$$

and so we have $J\subseteq I$ finite with $\bigvee\{\sigma(F_i)\mid i\in J\}=S$, whence $\bigcap\{F_i\mid i\in J\}=\emptyset$ and we see that (τ,κ) is cocompact.

In just the same way, if (τ,κ) is cocompact then it is compact. □

2.4. Example. Let $(\mathbb{I},\mathcal{I})$ be the unit interval texture of Examples 1.1 (2) with complementation ι and complemented ditopology $(\tau_{\mathbb{I}},\kappa_{\mathbb{I}})$ as defined above. Then $(\tau_{\mathbb{I}},\kappa_{\mathbb{I}})$ is compact because $\mathbb{I}$ is the only open set containing 1, and hence must belong to any open cover of $\mathbb{I}$ since for this texture join coincides with union. It follows by Proposition 2.3 that $(\tau_{\mathbb{I}},\kappa_{\mathbb{I}})$ is also cocompact.

As is well known, if X and Y are topological spaces, $f:X\to Y$ a continuous function and $A\subseteq X$ is compact then $f(A)\subseteq Y$ is compact. Below we generalize this result to ditopological texture spaces.

2.5. Theorem. *Let $(f,F):(S_1,\mathcal{S}_1,\tau_1,\kappa_1)\to(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ be a continuous difunction. If $A\in\mathcal{S}_1$ is (τ_1,κ_1)-compact then $f^{\to}A\in\mathcal{S}_2$ is (τ_2,κ_2)-compact.*

Proof. Take $f^{\to}A\subseteq\bigvee_{j\in J}G_j$, where $G_j\in\tau_2$, $j\in J$. Now by [5, Theorem 2.24 (2 a) and Corollary 2.12 (2)] we have

$$A\subseteq F^{\leftarrow}(f^{\to}A)\subseteq F^{\leftarrow}\Big(\bigvee_{j\in J}G_j\Big)=\bigvee_{j\in J}F^{\leftarrow}G_j.$$

Also, $F^{\leftarrow}G_j\in\tau_1$ since (f,F) is continuous, so by the compactness of A there exists $J'\subseteq J$ finite such that $A\subseteq\bigcup_{j\in J'}F^{\leftarrow}G_j$. Hence

$$f^{\to}A\subseteq f^{\to}\Big(\bigcup_{j\in J'}F^{\leftarrow}G_j\Big)=\bigcup_{j\in J'}f^{\to}(F^{\leftarrow}G_j)\subseteq\bigcup_{j\in J'}G_j$$

by [5, Corollary 2.12 (2) and Theorem 2.24 (2 b)]. This establishes that $f^{\to}A$ is compact. □

2.6. Proposition. *Let $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$ and $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ be ditopological texture spaces and $(f,F):S_1\to S_2$ a continuous surjective difunction. Then if $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$ is compact so is $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$.*

Proof. This follows by taking $A=S_1$ in Theorem 2.5 and noting that $f^{\to}S_1=f^{\to}(F^{\leftarrow}S_2)=S_2$ by [5, Proposition 2.28 (1 c) and Corollary 2.33 (1)]. □

2.7. Corollary. *Let $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$ and $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ be ditopological texture spaces and $\varphi:S_1\to S_2$ a continuous surjective ω-preserving point function. If $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$ is compact so is $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$.*

Proof. If we show that the associated difunction (f_φ, F_φ) is surjective the result will follow at once from Proposition 2.6. Hence, take $t, t' \in \mathcal{S}_2$ with $P_t \not\subseteq Q_{t'}$. We may choose $w \in S_2$ satisfying $P_t \not\subseteq Q_w$ and $P_w \not\subseteq Q_{t'}$. Since φ is surjective there exists $s \in S_1$ with $w = \varphi(s)$. Hence $P_{\varphi(s)} \not\subseteq Q_{t'}$, so $\overline{P}_{(s,\varphi(s))} \not\subseteq \overline{Q}_{(s,t')}$, and since $f = \bigvee\{\overline{P}_{(s,\varphi(s))} \mid s \in S_1\}$ we have $f \not\subseteq \overline{Q}_{(s,t')}$. In just the same way $\overline{P}_{(s,t)} \not\subseteq F$, and we have established that (f_φ, F_φ) is surjective. □

As expected, we have dual results for cocompactness. We omit the proofs.

2.8. Theorem. *Let $(f,F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be a cocontinuous difunction. If $A \in \mathcal{S}_1$ is (τ_1, κ_1)-cocompact then $F^{\to}A$ is (τ_2, κ_2)-cocompact.* □

2.9. Proposition. *Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ and $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be ditopological texture spaces and $(f,F) : S_1 \to S_2$ a cocontinuous surjective difunction. If $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is cocompact so is $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$.* □

2.10. Corollary. *Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ and $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be ditopological texture spaces and $\varphi : S_1 \to S_2$ a cocontinuous surjective ω-preserving point function. Then if $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is cocompact so is $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$.* □

It should be noted that Proposition 2.6 and Proposition 2.9 are strictly more powerful than their respective corollaries. Indeed, even if the surjective difunction postulated in these propositions corresponds to a point function, this point function need not be surjective. An example is provided by [6, Example 2.14]. Here we recall that $(M_{\mathcal{I}}, \mathcal{M}_{\mathcal{I}}, \mu_{\mathcal{I}}, \tau_{\mathcal{I}}, \kappa_{\mathcal{I}})$, the Hutton texture of $(\mathbb{I}, \mathcal{I}, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$, is given by

$$\begin{aligned}
M_{\mathcal{I}} &= (\mathbb{I} \times \{0\}) \cup ((\mathbb{I} \setminus \{0\}) \times \{1\}),\\
\mathcal{M}_{\mathcal{I}} &= \{A_r \mid r \in \mathbb{I}\} \cup \{B_r \mid r \in \mathbb{I}\},\\
\mu_{\mathcal{I}}(A_r) &= B_{1-r},\ \mu_{\mathcal{I}}(B_r) = A_{1-r},\ r \in \mathbb{I},
\end{aligned}$$

where

$$\begin{aligned}
A_r &= \{(s,0) \mid 0 \le s \le r\} \cup \{(s,1) \mid 0 < s \le r\}\\
B_r &= \{(s,0) \mid 0 \le s < r\} \cup \{(s,1) \mid 0 < s \le r\}.
\end{aligned}$$

It is easy to see that

$$P_{(r,0)} = A_r,\ Q_{(r,0)} = B_r\ (0 \le r \le 1) \text{ and } P_{(r,1)} = B_r,\ Q_{(r,1)} = B_r\ (0 < r \le 1).$$

The complemented ditopology $(\tau_{\mathcal{I}}, \kappa_{\mathcal{I}})$ is given by $\tau_{\mathcal{I}} = \{B_r \mid r \in \mathbb{I}\} \cup \{M_{\mathcal{I}}\}$, $\kappa_{\mathcal{I}} = \{A_r \mid r \in \mathbb{I}\} \cup \{\emptyset\}$. The ω-preserving bicontinuous point function $\varphi : \mathbb{I} \to M_{\mathcal{I}}$ defined by $\varphi(r) = (r, 0)$, $0 \le r \le 1$ is injective but clearly not surjective. On the other hand, it is shown in [6, Example 2.14] that the corresponding difunction (f_φ, F_φ) is a **dfDitop** isomorphism between $(\mathbb{I}, \mathcal{I}, \iota, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ and $(M_{\mathcal{I}}, \mathcal{M}_{\mathcal{I}}, \mu_{\mathcal{I}}, \tau_{\mathcal{I}}, \kappa_{\mathcal{I}})$, hence in particular a continuous surjection. Since $(\mathbb{I}, \mathcal{I}, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ is compact by Example 2.4 we deduce by Proposition 2.6 that $(M_{\mathcal{I}}, \mathcal{M}_{\mathcal{I}}, \mu_{\mathcal{I}}, \tau_{\mathcal{I}}, \kappa_{\mathcal{I}})$ is compact. In just the same way (f_φ, F_φ) is cocontinuous, so by Proposition 2.9 the space $(M_{\mathcal{I}}, \mathcal{M}_{\mathcal{I}}, \mu_{\mathcal{I}}, \tau_{\mathcal{I}}, \kappa_{\mathcal{I}})$ is also cocompact. Of course, in this particular case it is also easy to verify these results directly.

Now let us consider a family $(S_i, \mathcal{S}_i, \tau_i, \kappa_i)$, $i \in I$, of non-empty ditopological texture spaces and denote by $(S, \mathcal{S}, \tau, \kappa)$ the product of these spaces. For each $k \in I$,

$$\pi_k = \bigvee\{\overline{P}_{(s,s_k)} \mid s = (s_i) \in S\},\ \Pi_k = \bigcap\{\overline{Q}_{(s,s_k)} \mid s = (s_i) \in S^\flat\} \tag{2.1}$$

define the k-th projection difunction (π_k, Π_k) on $(S, \mathcal{S})$ to $(S_k, \mathcal{S}_k)$ [6, Lemma 3.9]. Moreover:

2.11. Lemma. *Let $(S, \mathcal{S})$ be the product of non-empty textures $(S_i, \mathcal{S}_i)$, $i \in I$. Then the k-th projection difunction (π_k, Π_k) is surjective.*

Proof. Here $(\pi_k, \Pi_k) : (S, \mathcal{S}) \to (S_k, \mathcal{S}_k)$. Take $t, t' \in S_k$ with $P_t \not\subseteq Q_{t'}$. Take any $s \in S$ with $P_t \not\subseteq Q_{s_k}$ and $P_{s_k} \not\subseteq Q_{t'}$. This is possible since $S_k \neq \emptyset$ for all $k \in I$. It is clear that $\pi_k \not\subseteq \overline{Q}_{(s,t')}$ and $\overline{P}_{(s,t)} \not\subseteq \Pi_k$, so (π_k, Π_k) is surjective. □

Finally (π_k, Π_k) is bicontinuous by the definition of the product ditopology, so Proposition 2.6 and Proposition 2.9 immediately give the following:

2.12. Corollary. *Let $(S_i, \mathcal{S}_i, \tau_i, \kappa_i)$, $i \in I$, be non-empty ditopological texture spaces and $(S, \mathcal{S}, \tau, \kappa)$ their product.*

(1) *If $(S, \mathcal{S}, \tau, \kappa)$ is compact then $(S_k, \mathcal{S}_k, \tau_k, \kappa_k)$ is compact for each $k \in I$.*
(2) *If $(S, \mathcal{S}, \tau, \kappa)$ is cocompact then $(S_k, \mathcal{S}_k, \tau_k, \kappa_k)$ is cocompact for each $k \in I$.* □

The above corollary gives us one direction of a Tychonoff Theorem for compact and cocompact ditopological texture spaces. To prove the opposite direction we require appropriate generalizations of the Alexander subbase theorem. We begin with a definition (c.f. [16]).

2.13. Definition. Let $(S, \mathcal{S})$ be a texture.

(1) $\mathcal{A} \subseteq \mathcal{S}$ is *inadequate* provided $\bigvee \mathcal{A} \neq S$. It is *finitely inadequate* provided no finite subcollection covers S.
(2) $\mathcal{A} \subseteq \mathcal{S}$ is *co-inadequate* provided $\bigcap \mathcal{A} \neq \emptyset$. It is *finitely co-inadequate* provided no finite subcollection cocovers $\emptyset$.

Clearly, finite co-inadequacy is equivalent to the finite intersection property. Now we give:

2.14. Theorem. *Let (τ, κ) be a ditopology on the texture space $(S, \mathcal{S})$ and γ a subbase for τ. Then (τ, κ) is compact iff every open cover of S by elements of γ has a finite subcover.*

Proof. $\Longrightarrow$. Clear since $\gamma \subseteq \tau$.

$\Longleftarrow$. Suppose that every cover of S by members of γ has a finite subcover. We prove that (τ, κ) is compact by proving that every finitely inadequate collection of open sets is inadequate. Let $\mathcal{B}$ be a finitely inadequate collection of open sets and P the set of all finitely inadequate collections $\mathcal{G}$ of open sets such that $\mathcal{B} \subseteq \mathcal{G}$. Then $(P, \subseteq)$ is a poset. Let C be a chain in P, and let $\mathcal{G}^*$ be the union of all the member of C. Then $\mathcal{G}^*$ is a collection of open sets and clearly $\mathcal{B} \subseteq \mathcal{G}^*$. Furthermore for each $\mathcal{G} \in C$, $\mathcal{G} \subseteq \mathcal{G}^*$. Suppose there is a finite subcollection $U_1, U_2, \ldots U_n$ of $\mathcal{G}^*$ that covers S. For each $i = 1, 2 \ldots n$ there exists $\mathcal{G}_i \in C$ such that $U_i \in \mathcal{G}_i$. Since C is a chain there exists k with $1 \leq k \leq n$ such that $\mathcal{G}_k$ contains all the sets $\mathcal{G}_i$, $i = 1, 2, \ldots, n$, and so for each $i = 1, 2, \ldots n$ we have $U_i \in \mathcal{G}_k$. However, $\mathcal{G}_k$ is finitely inadequate and we have a contradiction. Therefore $\mathcal{G}^*$ is finitely inadequate. Hence $\mathcal{G}^* \in P$ and $\mathcal{G}^*$ is an upper bound of C. By Zorn's Lemma, P has a maximal element $\mathcal{A}$. We want to show that $\mathcal{B}$ is inadequate. Since $\mathcal{B} \subseteq \mathcal{A}$, it is sufficient to show that $\mathcal{A}$ is inadequate. We establish that $\mathcal{A}$ has the following properties.

(a) If $U \in \tau$ and $U \notin \mathcal{A}$, then there exists a finite subcollection $U_1, U_2, \ldots, U_n$ of $\mathcal{A}$ such that $S = U \cup (\bigcup_{i=1}^n U_i)$
(b) If $U_1, U_2, \ldots, U_n$ is a finite collection of open sets none of which belong to $\mathcal{A}$ then $\bigcap_{i=1}^n U_i \notin \mathcal{A}$
(c) If $U_1, U_2, \ldots, U_n$ is a finite collection of open sets and $V \in \mathcal{A}$ satisfies $\bigcap_{i=1}^n U_i \subseteq V$ then there exists j with $1 \leq j \leq n$ such that $U_j \in \mathcal{A}$.

Proof of (a). Clear since $\mathcal{A} \cup \{U\}$ cannot be finitely inadequate.

Proof of (b). It is sufficient to show (b) for the case of two sets $U_1, U_2 \notin \mathcal{A}$. By (a) there are finite subcollections $V_1, V_2, \ldots, V_n$ and $W_1, W_2, \ldots, W_m$ of $\mathcal{A}$ such that $S = U_1 \cup \bigcup_{i=1}^n V_i = U_2 \cup \bigcup_{j=1}^m W_j$ Hence

$$S = (U_1 \cap U_2) \cup \bigcup_{i=1}^{n} V_i \cup \bigcup_{j=1}^{m} W_j.$$

Since $\mathcal{A}$ is finitely inadequate, $U_1 \cap U_2 \notin \mathcal{A}$. Hence (b) is proved.

Proof of (c). Suppose $U_1, U_2, \ldots, U_n$ are open sets and $V \in \mathcal{A}$ satisfies $\bigcap_{i=1}^n U_i \subseteq V$. If non of the sets U_i, $i = 1, 2, \ldots, n$ belongs to $\mathcal{A}$ then by (b), $\bigcap_{i=1}^n U_i \notin \mathcal{A}$. By (a) there is a finite subcollection $V_1, V_2, \ldots, V_m$ of $\mathcal{A}$ such that $S = (\bigcap_{i=1}^n U_i) \cup \bigcup_{j=1}^m V_j$, whence,

$$S = V \cup \Big(\bigcup_{j=1}^{m} V_j \Big).$$

This is a contradiction since $\mathcal{A}$ is finitely inadequate, therefore (c) is true.

The collection $\gamma \cap \mathcal{A}$ is finitely inadequate because $\gamma \cap \mathcal{A} \subseteq \mathcal{A}$ and $\mathcal{A}$ is finitely inadequate. Let us show that $\bigvee \mathcal{A} \subseteq \bigvee(\mathcal{A} \cap \gamma)$. Take $s \in S$ with $\bigvee \mathcal{A} \not\subseteq Q_s$. Then we have $A \in \mathcal{A}$ with $A \not\subseteq Q_s$. Since $A \in \tau$ and γ is a subbase for τ, by [6, Theorem 3.2 (1 ii)] there exists $G_1, G_2, \ldots, G_n \in \gamma$ so that $\bigcap_{i=1}^n G_i \subseteq A$ and $\bigcap_{i=1}^n G_i \not\subseteq Q_s$. By (c) there exists k, $1 \leq k \leq n$, so that $G_k \in \mathcal{A} \cap \gamma$, and $G_k \not\subseteq Q_s$, so $\bigvee(\mathcal{A} \cap \gamma) \not\subseteq Q_s$. This verifies that $\bigvee \mathcal{A} \subseteq \bigvee(\mathcal{A} \cap \gamma)$. If $\bigvee \mathcal{A} = S$ then $\bigvee(\mathcal{A} \cap \gamma) = S$, and since by hypothesis every cover of S by sets in γ has a finite subcover, we obtain a contradiction to the fact that the family $\mathcal{A} \cap \gamma$ is finitely inadequate. Hence $\mathcal{A}$, and therefore $\mathcal{B}$ is inadequate, and we deduce that (τ, κ) is compact. □

The corresponding result for cocompactness is given below. Since the proof is essentially dual to the above it is omitted.

2.15. Theorem. *Let (τ, κ) be a ditopology on the texture space $(S, \mathcal{S})$ and γ a subbasis for the closed sets κ. Then (τ, κ) is cocompact iff every closed cocover of $\emptyset$ by member of γ has a finite subcocover* □.

We may now give the Tychonoff Theorem for compactness and cocompactness.

2.16. Theorem. *Let $(S_i, \mathcal{S}_i, \tau_i, \kappa_i)$, $i \in I$, be non-empty ditopological texture spaces and $(S, \mathcal{S}, \tau, \kappa)$ their product.*

(i) *$(S, \mathcal{S}, \tau, \kappa)$ is compact if and only if $(S_i, \mathcal{S}_i, \tau_i, \kappa_i)$ is compact for all $i \in I$.*
(ii) *$(S, \mathcal{S}, \tau, \kappa)$ is cocompact if and only if $(S_i, \mathcal{S}_i, \tau_i, \kappa_i)$ is cocompact for all $i \in I$.* □

Proof. (i) $\Longrightarrow$. This is just Corollary 2.12.

$\Longleftarrow$. Let γ be the subbase $\{E(i, G) \mid i \in I, G \in \tau_i\}$ and $\mathcal{C}$ an open cover of S by sets in γ. For $i \in I$ let

$$C_i = \bigvee \{G \mid E(i, G) \in \mathcal{C}\}.$$

By [3, Lemma 2.3] we have $S = \bigvee \mathcal{C} = \bigcup_{i \in I} E(i, C_i)$. If we had $C_i \neq S_i$ for all $i \in I$ then we could choose $s_i \in S_i$ with $s_i \notin C_i$ and then $s = (s_i) \in S$ would satisfy $s \notin \bigcup_{i \in I} E(i, C_i)$, which contradicts the above. Hence there exists $k \in I$ so that $C_k = S_k$, whence $\{G \mid E(i, G) \in \mathcal{C}\}$ is an open cover of S_k in $(S_k, \mathcal{S}_k, \tau_k, \kappa_k)$. Since this space is compact we have $G_1, G_2, \ldots, G_n$ with $E(k, G_j) \in \mathcal{C}$ and $S_k = G_1 \cup G_2 \cup \ldots \cup G_n$. Hence

$$\bigcup_{j=1}^{n} E(k, G_j) = E(k, S_k) = S,$$

which shows that $\{E(k,G_1),E(k,G_2),\dots,E(k,G_n)\}$ is a finite subcover of $\mathcal{C}$. Hence, by Theorem 2.14, the product space $(S,\mathcal{S},\tau,\kappa)$ is compact.

(ii) $\Longrightarrow$. This is dual to (i) and is omitted. □

We now recall [6, Proposition 3.21], which states that the projection difunctions on a product ditopological space are always open and coclosed. This should be compared with the following:

2.17. Theorem. *Let $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$ be compact, $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ any ditopological texture space and $(S,\mathcal{S},\tau,\kappa)$ the product of these spaces. Then the projection*

$$(\pi_2,\Pi_2):(S,\mathcal{S},\tau,\kappa)\to(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$$

is co-open.

Proof. We must show that $G\in\tau \implies \Pi_2^{\to}G\in\tau_2$. Assume that for some $G\in\tau$ we have $\Pi_2^{\to}G\notin\tau_2$. Then $\Pi_2^{\to}G\not\subseteq\,]\Pi_2^{\to}G[$, so there exists $t\in S_2$ with $\Pi_2^{\to}G\not\subseteq Q_t$ and

(2.2) $\quad P_t\not\subseteq\,]\Pi_2^{\to}G[.$

Since $\Pi_2^{\to}G=\bigvee\{P_t\mid \overline{P}_{((u,v),t)}\not\subseteq\Pi_2 \implies P_{(u,v)}\subseteq G\}$, there exists t' so that $P_{t'}\not\subseteq Q_t$ and

(2.3) $\quad \overline{P}_{((u,v),t')}\not\subseteq\Pi_2 \implies P_{(u,v)}\subseteq G.$

Let us choose $v\in S_2$ with $P_{t'}\not\subseteq Q_v$ and $P_v\not\subseteq Q_t$, and take any $s\in S_1^\flat$. Note that $P_{t'}\not\subseteq Q_v$ implies $Q_v\neq S_2$, so $v\in S_2^\flat$ and we have $(s,v)\in S_1^\flat\times S_2^\flat=S^\flat$. Hence $\Pi_2\subseteq\overline{Q}_{((s,v),v)}$ by (2.1), and since $\overline{P}_{((s,v),t')}\not\subseteq\overline{Q}_{((s,v),v)}$ we obtain $\overline{P}_{((s,v),t')}\not\subseteq\Pi_2$. The implication (2.3) now gives us $P_{(s,v)}\subseteq G$, whence $G\not\subseteq\overline{Q}_{(s,t)}$ since $P_v\not\subseteq Q_t$.

By definition the family of sets $\{(G^1\times S_2)\cap(S_1\times G^2)\}$ for $G^1\in\tau_1$ and $G^2\in\tau_2$ is a base for τ. Hence, since $G\in\tau$ and for each $s\in S_1^\flat$ we have $G\not\subseteq Q_{(s,t)}$, there exists $G_s^1\in\tau_1$ and $G_s^2\in\tau_2$ so that

(2.4) $\quad (G_s^1\times S_2)\cap(S_1\times G_s^2)\subseteq G,$

(2.5) $\quad (G_s^1\times S_2)\cap(S_1\times G_s^2)\not\subseteq Q_{(s,t)}.$

We wish to show that

(2.6) $\quad S_1=\bigvee\{G_s^1\mid s\in S^b\}.$

Clearly it is sufficient to show $S_1\subseteq\bigvee\{G_s^1\mid s\in S^b\}$, so we assume this is false, whence we have $u\in S_1$ satisfying $S_1\not\subseteq Q_u$ and $P_u\not\subseteq\bigvee\{G_s^1\mid s\in S^b\}$. Clearly $u\in S_1^\flat$, so $P_u\not\subseteq G_u^1$. Hence $G_u^1\times S_2\subseteq Q_u\times S_2\subseteq(Q_u\times S_2)\cup(S_1\times Q_t)=Q_{(u,t)}$, which contradicts (2.5) for $s=u$, and so establishes (2.6).

As the space $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$ is compact there exists $s_1,s_2\dots s_n\in S_1^\flat$ such that $S_1=\bigcup_{i=1}^n G_{s_i}^1$. We prove that

(2.7) $\quad P_t\subseteq G_{s_1}^2\cap G_{s_2}^2\dots\cap G_{s_n}^2\subseteq\Pi_2^{\to}G.$

The first inclusion is clear since $G_{s_i}^2\not\subseteq Q_t,\ i=1,2,\dots,n$ by (2.5). Assume that $\bigcap_{i=1}G_{s_i}^2\not\subseteq\Pi_2^{\to}G$. Then $\exists z\in S_2$ such that $\bigcap_{i=1}G_{s_i}^2\not\subseteq Q_z$ and $P_z\not\subseteq\Pi_2^{\to}G$. Now since $P_z\not\subseteq\Pi_2(G)=\bigvee\{P_t\mid\overline{P}_{((u,v),t)}\not\subseteq\Pi_2 \implies P_{(u,v)}\subseteq G\}$ there exists $u\in S_1$, $v\in S_2$, such that $\overline{P}_{((u,v),z)}\not\subseteq\Pi_2$ and $P_{(u,v)}\not\subseteq G$. Again we may choose u',v' so that $P_{(u,v)}\not\subseteq Q_{(u',v')}$ and $P_{(u',v')}\not\subseteq G$. Clearly $\overline{P}_{((u',v'),z)}\not\subseteq\Pi_2$ by condition *CR1* for the corelation Π_2. Also, $(u',v')\in S^\flat$, and since $\Pi_2=\bigcap\{\overline{Q}_{((u',v'),v')}\mid(u',v')\in S^\flat\}$ we see that $P_z\not\subseteq Q_{v'}$. This implies that $P_{v'}\subseteq P_z$, and since $P_{(u',v')}\not\subseteq G$ we obtain

(2.8) $\quad P_{(u',z)}\not\subseteq G.$

Now $P_{u'} \subseteq S_1 = \bigcup_{i=1}^n G^1_{s_i}$ by (2.6), so there exists k, $1 \le k \le n$, so that $P_{u'} \subseteq G^1_{s_k}$. Also $\bigcap_{i=1}^n G^2_{s_i} \not\subseteq Q_z$ implies $G^2_{s_k} \not\subseteq Q_z$ and so $P_z \subseteq G^2_{s_k}$. Hence

$$P_{(u',z)} \subseteq G^1_{s_k} \times G^2_{s_k} \subseteq G$$

by (2.4), and this contradicts (2.8) and hence proves (2.7). Since $G^2_{s_1} \cap G^2_{s_2} \ldots \cap G^2_{s_n} \in \tau_2$ the inclusions (2.7) imply that $P_t \subseteq \,]\Pi^{\to} G[$ which contradicts (2.2). This completes the proof of the theorem. $\square$

The above property of compact ditopological texture spaces is in fact characteristic, as the following theorem shows.

2.18. Theorem. *Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ be a ditopological texture space. The following are equivalent.*

(i) *$(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is compact.*

(ii) *For all ditopological texture spaces $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$, the projection difunction (π_2, Π_2) on the product of $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ and $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is co-open.*

(iii) *For all ditopological texture spaces $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ with $\mathcal{S}_2 = \mathcal{P}(S_2)$ and $\kappa_2 = \mathcal{P}(S_2)$, the projection difunction (π_2, Π_2) on the product of $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ and $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is co-open.*

Proof. (i) $\Longrightarrow$ (ii). This is just Theorem 2.17.

(ii) $\Longrightarrow$ (iii). Clear since (iii) is a special case of (ii).

(iii) $\Longrightarrow$ (i). Suppose (iii) is true and (i) is false. Then $\exists\, G_i \in \tau$, $i \in I$, such that $S_1 = \bigvee_{i \in I} G_i$ but $S_1 \ne \bigcup_{i \in I'} G_i$ for all finite $I' \subseteq I$. We must show the existence of a ditopological texture space $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ which contradicts (iii).

Choose $t_0 \notin S_1$ and make the following definitions:

$$S_2 = S_1^\flat \cup \{t_0\},$$
$$\mathcal{S}_2 = \mathcal{P}(S_2),$$
$$\tau_2 = \Big\{ H \subseteq S_2 \mid \exists\, i_1, i_2, \ldots, i_n \in I \text{ so that } H \cup \Big(\bigcup_{k=1}^n G^\flat_{i_k} \Big) = S_2 \Big\} \cup \{\emptyset\},$$
$$\kappa_2 = \mathcal{P}(S_2).$$

Clearly $(S_2, \mathcal{S}_2)$ is a texture and κ_2 a cotopology on this texture. We must verify that τ_2 is a topology. Clearly $S_2 \in \tau_2$, and by definition $\emptyset \in \tau_2$. Suppose that $H_1, H_2 \in \tau_2$. If either of H_1, H_2 is empty so is $H_1 \cap H_2$ and so $H_1 \cap H_2 \in \tau_2$. Hence suppose that $H_1 \ne \emptyset \ne H_2$. Now we have $i_1, i_2, \ldots, i_n \in I$ and $i_{n+1}, i_{n+2}, \ldots, i_m \in I$ so that

$$H_1 \cup \Big(\bigcup_{k=1}^n G^\flat_{i_k} \Big) = S_2 = H_2 \cup \Big(\bigcup_{k=n+1}^m G^\flat_{i_k} \Big).$$

It follows that $(H_1 \cap H_2) \cup \Big(\bigcup_{k=1}^m G^\flat_{i_k} \Big) = S_2$, whence again $H_1 \cap H_2 \in \tau_2$. Since in $(S_2, \mathcal{S}_2)$ join is the same as union, it remains to show that if $H_\alpha \in \tau_2$ then $H = \bigcup_\alpha H_\alpha \in \tau_2$. If $H_\alpha = \emptyset$ for all α there is nothing to prove, so assume we have α_0 with $H_{\alpha_0} \ne \emptyset$. Then we have $i_1, i_2, \ldots, i_n \in I$ so that $H_{\alpha_0} \cup \Big(\bigcup_{k=1}^n G^\flat_{i_k} \Big) = S_2$. But $H_{\alpha_0} \subseteq H$, so $H \cup \Big(\bigcup_{k=1}^n G^\flat_{i_k} \Big) = S_2$, and we have proved that $H \in \tau_2$. Hence τ_2 is indeed a topology, so $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is a ditopological texture space which satisfies the hypotheses of (iii).

Let $(S, \mathcal{S}, \tau, \kappa)$ be the product of the spaces $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ and $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$. We show that there exists $G \in \tau$ for which $\Pi_2^{\to} G \notin \tau_2$.

For $i \in I$ note that $(S_2 \setminus G^\flat_i) \cup G^\flat_i = S_2$, whence $S_2 \setminus G^\flat_i \in \tau_2$. Define

$$G = \bigvee \{ (G_i \times S_2) \cap (S_1 \times (S_2 \setminus G^\flat_i)) \mid i \in I \} \in \tau. \tag{2.9}$$

Let us prove that $\Pi^{\to}G = \{t_0\}$.

First assume that $\Pi_2^{\to}G \not\subseteq \{t_0\}$. Hence

$$\{t \in S_2 \mid \overline{P}_{((s_1,s_2),t)} \not\subseteq \Pi_2 \implies P_{(s_1,s_2)} \subseteq G\} \not\subseteq \{t_0\}$$

since in the texture $(S_2, \mathcal{S}_2)$ we have $P_t = \{t\}$. Hence

(2.10) $\quad \exists\, t_1 \in S_2,\ t_1 \neq t_0$ and $\overline{P}_{((s_1,s_2),t_1)} \not\subseteq \Pi_2 \implies P_{(s_1,s_2)} \subseteq G$.

By the definition of S_2 we have $t_1 \in S_1^\flat$, so $Q_{t_1} \neq S_1$ and we may choose $t_2 \in S_1$ so that $P_{t_2} \not\subseteq Q_{t_1}$. Without loss of generality we may assume $t_2 \in S_1^\flat$, and $S_2^\flat = S_2$ so $t_1 \in S_2^\flat$. Hence, by the definition of Π_2, we have $\Pi_2 \subseteq \overline{Q}_{((t_2,t_1),t_1)}$, whence $\overline{P}_{((t_2,t_1),t_1)} \not\subseteq \Pi_2$ since $P_{t_1} \not\subseteq Q_{t_1}$ in the texture $(S_2, \mathcal{S}_2)$. By the implication (2.10) we obtain $P_{(t_2,t_1)} \subseteq G$, and since $P_{t_2} \not\subseteq Q_{t_1}$ in $(S_1, \mathcal{S}_1)$ and $P_{t_1} \not\subseteq Q_{t_1}$ in $(S_2, \mathcal{S}_2)$ we obtain

$$G = \bigvee_{i\in I} \{(G_i \times S_2) \cap (S_1 \times (S_2 \setminus G_i^\flat))\} \not\subseteq Q_{(t_1,t_1)}.$$

Then there exists $i \in I$ with

$$(G_i \times S_2) \cap (S_1 \times (S_2 \setminus G_i^\flat)) \not\subseteq Q_{(t_1,t_1)} = (S_1 \times (S_2 \setminus \{t_1\})) \cup (Q_{t_1} \times S_2).$$

In particular $G_i \times S_2 \not\subseteq Q_{t_1} \times S_2$ and $S_1 \times (S_2 \setminus G_i^\flat) \not\subseteq S_1 \times (S_2 \setminus \{t_1\})$. From the first of these we obtain $G_i \not\subseteq Q_{t_1}$, whence $t_1 \in G_i^\flat$, while from the second we have $S_2 \setminus G_i^\flat \not\subseteq S_2 \setminus \{t_1\}$, which gives the contradiction $t_1 \in G_i^\flat$. So our supposition is wrong and $\Pi_2^{\to}G \subseteq \{t_0\}$.

Now assume that $\{t_0\} \not\subseteq \Pi_2^{\to}G$. Then there exists $s = (s_1, s_2) \in S$ such that $\overline{P}_{((s_1,s_2),t_0))} \not\subseteq \Pi_2$ and $P_{(s_1,s_2)} \not\subseteq G$. But $s_2 \in S_2 = S_2^\flat$, and without loss of generality we may assume $s_1 \in S_1^\flat$. Hence, from the formula for Π_2 we obtain $P_{t_0} \not\subseteq Q_{s_2}$. Since $Q_{s_2} = S_2 \setminus \{s_2\}$ we obtain $s_2 = t_0$ and so

(2.11) $\quad P_{(s_1,t_0)} \not\subseteq G$.

On the other hand $s_1 \in S_1^\flat$ implies that $\bigvee_{i\in I} G_i = S_1 \neq Q_{s_1}$ so there exists $i_0 \in I$ with $G_{i_0} \neq Q_{s_1}$. Hence $P_{(s_1,t_0)} \subseteq G_{i_0} \times S_2$. Since $P_{(s_1,t_0)} \not\subseteq (G_{i_0} \times S_2) \cap (S_1 \times (S_2 \setminus G_{i_0}^\flat))$ by (2.9) and (2.11), we have $P_{(s_1,t_0)} \not\subseteq S_1 \times (S_2 \setminus G_{i_0}^\flat)$ and so $t_0 \in G_{i_0}^\flat \subseteq S_1^\flat \subseteq S_1$, which contradicts the choice of t_0. Hence $\{t_0\} \subseteq \Pi_2^{\to}G$.

This verifies that $\Pi_2^{\to}G = \{t_0\}$, as claimed. If we had $\{t_0\} \in \tau_2$ then we should have $i_1, i_2, \ldots, i_n \in I$ so that $\{t_0\} \cup (\bigcup_{k=1}^n G_{i_k})^\flat = S_2 = S_1^\flat \cup \{t_0\}$. This would imply that $S_1^\flat = (\bigcup_{k=1}^n G_{i_k})^\flat$, and so $S_1 = \bigvee_{k=1}^n G_{i_k}$. This would contradict our supposition about the family G_i, $i \in I$. Therefore $\{t_0\} \notin \tau_2$ and so $\Pi_2^{\to}G \notin \tau_2$. This contradicts the conclusion of (iii), and proves that $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is compact. □

2.19. Corollary. *Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ be a ditopological texture space. Then the following are equivalent.*

(i) *$(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is compact.*
(ii) *For all ditopological texture spaces $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$, the projection difunction (π_2, Π_2) on the product of $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ and $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is co-open.*
(iii) *For all normal ditopological texture spaces $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ the projection difunction (π_2, Π_2) on the product of $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ and $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is co-open.*

Proof. A ditopological texture space $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ with $\mathcal{S}_2 = \mathcal{P}(S_2)$ and $\kappa_2 = \mathcal{P}(S_2)$ is normal. □

It follows that Theorem 2.17 generalizes the characterization of compact topological spaces due to Mrówka [15] (see also [9]).

As expected the dual results for cocompactness also hold. We summarize these in the following theorem and corollary. The proofs are essentially dual to those for compactness, and we just sketch an outline.

2.20. Theorem. *Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ be a ditopological texture space. The following are equivalent.*

(i) $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ *is cocompact.*
(ii) *Given any ditopological texture space $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$, the projection difunction (π_2, Π_2) on the product of $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ and $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is closed.*
(iii) *Given any ditopological texture space $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ with $\mathcal{S}_2 = \mathcal{P}(S_2)$ and $\tau_2 = \mathcal{P}(S_2)$, the projection difunction (π_2, Π_2) on the product of $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ and $(S_2, \mathcal{P}(S_2), \mathcal{P}(S_2), \kappa_2)$ is closed.*

Proof. (i) $\Longrightarrow$ (ii). Suppose that K is closed for the product topology, but that $\pi_2^{\to} K$ is not closed. Then we may take $t \in S_2$ with $[\pi_2(K)] \not\subseteq Q_t$, $P_t \not\subseteq \pi_2^{\to} K$ and obtain sets $K_z^1 \in \kappa_1$, $K_z^2 \in \kappa_2$ for $z \in S_1$ for which $K \subseteq (K_z^1 \times S_2) \cup (S_1 \times K_z^2)$ and $P_{(z,t)} \not\subseteq (K_z^1 \times S_2) \cup (S_1 \times K_z^2)$. From the cocompactness of (τ_1, κ_1) we see that there exist points $z_k \in S_1$, $k = 1, 2, \ldots, n$ for which $\bigcap_{k=1}^n K_{z_k}^1 = \emptyset$. If now we let $F = \bigcup_{k=1}^n K_{z_k}^2 \in \kappa_2$ then it may be shown that $\pi_2^{\to} K \subseteq F \subseteq Q_t$, so giving the contradiction $[\pi_2^{\to} K] \subseteq Q_t$. Hence (π_2, Π_2) is closed.

(ii) $\Longrightarrow$ (iii). Immediate.

(iii) $\Longrightarrow$ (i). Suppose (iii) holds but that (i) is false. Then there are sets $F_i \in \kappa_1$, $i \in I$, with the finite intersection property but having an empty intersection. We define the ditopological texture space $(S_2, \mathcal{P}(S_2), \mathcal{P}(S_2), \kappa_2)$ by taking $t_0 \notin S_1$ and setting

$$S_2 = S_1 \cup \{t_0\},$$
$$\kappa_2 = \{B \subseteq S_2 \mid B \cap \bigcap_{i \in I'} K_i = \emptyset \text{ for some finite } I' \subseteq I\} \cup \{S_2\}.$$

It is clear that κ_2 is a cotopology on $(S_2, \mathcal{P}(S_2))$. Moreover, $S_2 \setminus K_i \in \kappa_2$ for each $i \in I$ and so $K = \bigcap_{i \in I} [(K_i \times S_2) \cup (S_1 \times (S_2 \setminus K_i))]$ is closed for the product ditopology. However it may be verified that $\pi_2^{\to} K = S_1$, and it is clear that $S_1 \notin \kappa_2$ since $\bigcap_{i \in I'} F_i \neq \emptyset$ for each finite $I' \subseteq I$. This contradiction to (iii) establishes that (τ_1, κ_1) is cocompact. □

2.21. Corollary. *Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ be a ditopological texture space. The following are equivalent.*

(i) $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ *is cocompact.*
(ii) *Given any ditopological texture space $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$, the projection difunction (π_2, Π_2) on the product of $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ and $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is closed.*
(iii) *Given any normal ditopological texture space $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$, the projection difunction (π_2, Π_2) on the product of $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ and $(S_2, \mathcal{P}(S_2), \mathcal{P}(S_2), \kappa_2)$ is closed.* □

3. Stability and Costability

In a compact topological space every closed subset is compact. The following example shows that this is not true for compact ditopological texture spaces. Dually we give an example of a cocompact ditopological space in which not every open subset is cocompact.

3.1. Example. Consider the texture $(L, \mathcal{L})$ of Examples 1.1 (3).

(1) Let $\tau = \{(0,r] \mid 0 \le r \le 1/2\} \cup \{L\}$ and $\kappa = \{\emptyset, (0,1/2], L\}$. It is trivial to verify that (τ,κ) is a ditopology on $(L,\mathcal{L})$. This ditopology is compact because any open cover of L must contain L, and so $\{L\}$ is a finite subcover. In fact it is cocompact also, because κ is finite. However, the closed subset $K = (0,1/2]$ is not compact because $\mathcal{C} = \{(0,1/2-1/n] \mid n = 3,4,5,\ldots\}$ is an open cover of K with no finite subcover.

(2) Dually, let $\tau = \{\emptyset, (0,1/2], L\}$ and $\kappa = \{(0,r] \mid 1/2 \le r \le 1\} \cup \{\emptyset\}$. The ditopology (τ,κ) on $(L,\mathcal{L})$ is cocompact since every cocover of $\emptyset$ must contain $\emptyset$, whence $\{\emptyset\}$ is a finite subcocover. It is also compact because τ is finite. However the open set $G = (0,1/2]$ is not cocompact because $\mathcal{D} = \{(0,1/2+1/n] \mid n = 3,4,5,\ldots\}$ is a closed cocover of G with no finite subcocover.

This leads to the following concepts.

3.2. Definition. [2]. Let (τ,κ) be a ditopology on the texture $(S,\mathcal{S})$. Then (τ,κ) is called,

(i) *Stable* if every $K \in \kappa$ with $K \ne S$ is compact.
(ii) *Costable* if every $G \in \tau$ with $G \ne \emptyset$ is cocompact.

These concepts generalize analogous concepts for bitopological spaces introduced by R. D. Kopperman in [14]. Examples 3.1 (1) shows that a compact ditopological texture space need not be stable, while (2) shows that a cocompact ditopological texture space need not be costable. The following examples show that the converses also hold.

3.3. Example. Consider the texture $(L,\mathcal{L})$ as in the previous examples.

(1) Let $\tau = \mathcal{L}$ and $\kappa = \{\emptyset, L\}$. The ditopology (τ,κ) is not compact because $\mathcal{C} = \{(0,1-1/n] \mid n = 2,3,4,\ldots\}$ is an open cover of L which has no finite subcover. On the other hand (τ,κ) is stable because the only closed set different from L is $\emptyset$, and this set is trivially compact.

(2) Dually let $\tau = \{\emptyset, L\}$ and $\kappa = \mathcal{L}$. This ditopology is costable but not cocompact.

(3) Let $\tau = \mathcal{L}$ and $\kappa = \{\emptyset, (0,1/2], L\}$. Since κ is finite every open set is cocompact. Hence (τ,κ) is costable and cocompact. However the closed set $(0,1/2]$ is not compact since $\mathcal{C} = \{(0,1/2-1/n] \mid n = 3,4,5,\ldots\}$ is an open cover with no finite subcover. Hence (τ,κ) is not stable. It is also not compact.

(4) Dually, let $\tau = \{\emptyset, (0,1/2], L\}$ and $\kappa = \mathcal{L}$. The ditopology (τ,κ) is stable and compact but neither costable nor cocompact.

The last two examples show that in general stability and costability are independent of one another. However for complemented ditopological texture spaces these concepts are equivalent, as we now show.

3.4. Proposition. *Let $(S,\mathcal{S},\sigma)$ be a texture with complement σ and let (τ,κ) be a complemented ditopology on $(S,\mathcal{S},\sigma)$. Then (τ,κ) is stable if and only if (τ,κ) is costable.*

Proof. Let (τ,κ) be stable, take $G \in \tau$ with $G \ne \emptyset$ and let $\mathcal{D}$ be a closed cocover of G. Set $K = \sigma(G)$. Then $K \in \kappa$ satisfies $K \in \kappa$, $K \ne S$. Hence K is compact. Let $\mathcal{C} = \{\sigma(F) \mid F \in \mathcal{D}\}$. Since $\bigcap \mathcal{D} \subseteq G$ we have $K \subseteq \bigvee \mathcal{C}$, i.e. $\mathcal{C}$ is an open cover of K so there exists $F_1, F_2, \ldots, F_n \in \mathcal{D}$ so that $K \subseteq \sigma(F_1) \cup \sigma(F_2) \cup \ldots \cup \sigma(F_n) = \sigma(F_1 \cap F_2 \cap \ldots \cap F_n)$. This gives $F_1 \cap F_2 \cap \ldots \cap F_n \subseteq \sigma(K) = G$, so G is cocompact. Hence (τ,κ) is costable.

The proof that costable implies stable is the dual of the above, and is omitted. □

3.5. Theorem. *A regular stable ditopological texture space is normal.*

Proof. Let $F \in \kappa$, $G \in \tau$ with $F \subseteq G$. By [7, Definition 5.18] we must prove that there exists $H \in \tau$ satisfying

$$F \subseteq H \subseteq [H] \subseteq G.$$

There are two cases to consider.

(i) $G = S$. In this case we may take $H = S$, and then $F \subseteq H \subseteq [H] \subseteq G$.

(ii) $G \neq S$. Take any $s \in S$ with $G \not\subseteq Q_s$. Then by regularity [7, Definition 3.1 (e)] there exists $H_s \in \tau$ with $H_s \not\subseteq Q_s$ and $[H_s] \subseteq G$. Now we prove that $F \subseteq \bigvee\{H_s \mid G \not\subseteq Q_s\}$.

Assume that $F \not\subseteq \bigvee\{H_s \mid G \not\subseteq Q_s\}$. Then $\exists\, u \in S$ such that $F \not\subseteq Q_u$ and $P_u \not\subseteq \bigvee\{H_s \mid G \not\subseteq Q_s\}$. Now $F \not\subseteq Q_u$ and $F \subseteq G$ implies $G \not\subseteq Q_u$, so we have $H_u \in \tau$ with $H_u \not\subseteq Q_u$ and $[H_u] \subseteq G$. But now $P_u \subseteq H_u \subseteq \bigvee\{H_s \mid G \not\subseteq Q_s\}$, which is a contradiction. Hence $F \subseteq \bigvee\{H_s \mid G \not\subseteq Q_s\}$, that is $\{H_s \mid G \not\subseteq Q_s\}$ is an open cover of F. But $F \neq S$ since $F \subseteq G$ and $G \neq S$, whence F is compact since (τ, κ) is stable. Hence there exists $s_1, s_2, \ldots, s_n \in S$ such that

$$F \subseteq H_{s_1} \cup H_{s_2} \cup \ldots \cup H_{s_n} \subseteq [H_{s_1} \cup H_{s_2} \cup \ldots \cup H_{s_n}] = [H_{s_1}] \cup [H_{s_2}] \cup \ldots \cup [H_{s_n}] \subseteq G.$$

If we define $H = H_{s_1} \cup H_{s_2} \cup \ldots \cup H_{s_n}$ then we see that $H \in \tau$ and $F \subseteq H \subseteq [H] \subseteq G$.

From (i) and (ii) we have that (τ, κ) is normal. □

3.6. Theorem. *A coregular costable ditopological texture space is normal.*

Proof. Dual to the proof of Theorem 3.5. □

3.7. Theorem. *A R_1 costable ditopological texture space is regular.*

Proof. Let $G \in \tau$, $G \not\subseteq Q_s$. We must show that there exists $H \in \tau$ satisfying

$$H \not\subseteq Q_s \text{ and } [H] \subseteq G.$$

First take $v \in S$ satisfying $G \not\subseteq Q_v$ and $P_v \not\subseteq Q_s$. Since (τ, κ) is R_1, by [7, Definition 3.1 (c)] for any $t \in S$ with $P_t \not\subseteq G$ there exists $H_t \in \tau$ with $H_t \not\subseteq Q_v$ and $P_t \not\subseteq [H_t]$. Let us show that

$$\bigcap\{[H_t] \mid P_t \not\subseteq G\} \subseteq G.$$

Assume this is false. Then we have $u \in S$ for which $\bigcap\{[H_t] \mid P_t \not\subseteq G\} \not\subseteq Q_u$ and $P_u \not\subseteq G$. As above we have $H_u \in \tau$ with $H_u \not\subseteq Q_v$ and $P_u \not\subseteq [H_u]$, so $\bigcap\{[H_t] \mid P_t \not\subseteq G\} \subseteq [H_u] \subseteq Q_u$, which is a contradiction. Hence the above inclusion is valid, which means that $\{[H_t] \mid P_t \not\subseteq G\}$ is a closed cocover of G.

By hypothesis $G \not\subseteq Q_s$, and so $G \not\subseteq \emptyset$. Since (τ, κ) is costable, G is cocompact and so there exists $t_1, t_2 \ldots t_n$ with $P_{t_i} \not\subseteq G$ for which $[H_{t_1}] \cap [H_{t_2}] \ldots \cap [H_{t_n}] \subseteq G$. Let $H = H_{t_1} \cap H_{t_2} \ldots \cap H_{t_n}$. Then $H \in \tau$, $H_{t_i} \not\subseteq Q_v \implies P_v \subseteq H_{t_i}$, $i = 1, 2, \ldots, n$, whence $P_v \subseteq H$ and so $H \not\subseteq Q_s$ since $P_v \not\subseteq Q_s$. Finally,

$$[H] = [H_{t_1} \cap H_{t_2} \ldots \cap H_{t_n}] \subseteq [H_{t_1}] \cap [H_{t_2}] \cap \ldots \cap [H_{t_n}] \subseteq G,$$

which shows that H has the required properties. Hence (τ, κ) is regular. □

3.8. Theorem. *A co-R_1 stable ditopological texture space is coregular.*

Proof. Dual to the proof of Theorem 3.7. □

On the other hand the R_0 axiom [7, Definition 3.1 (a)] in the presence of costability does not imply R_1, as the following example shows.

3.9. Example. Let X be an infinite set and consider the complemented texture $(X, \mathcal{P}(X), \pi_X)$. Define the complemented ditopology (τ, κ) by taking κ to consist of X and the set of finite subsets of X and τ to be the cofinite topology on X. It is well known [16] that τ is a compact topology, so all the closed subsets are compact also. This means that the ditopology (τ, κ) is compact and stable. Since (τ, κ) is a complemented ditopology, it is also cocompact and costable by Proposition 2.3 and Proposition 3.4, respectively.

Take $G \in \tau$ and $G \not\subseteq Q_x = X \setminus \{x\}$ for $x \in X$. Then $P_x = \{x\} \subseteq G$, and since $\{x\}$ is a finite set we have $P_x \in \kappa$, so $[P_x] = P_x \subseteq G$. This proves that (τ, κ) is R_0, whence (τ, κ) is also co-R_0 by [7, Corollary 3.5]

On the other hand, (τ, κ) is not R_1. To see this take $G \in \tau$ and $x, y \in X$ with $G \not\subseteq Q_x$ and $P_y \not\subseteq G$. Suppose that we have $H \in \tau$ with $H \not\subseteq Q_x$ and $P_y \not\subseteq [H]$, i.e. $x \in H$ and $y \notin [H]$. Now $H \neq \emptyset$ so H is the complement of a finite set, and so is an infinite set since X is infinite. It follows that $[H] = X$, and this contradicts $y \notin [H]$.

By definition every closed set in a compact stable ditopological texture space is compact, and likewise every open set in a cocompact costable space is cocompact. In order to investigate in greater detail precisely which sets are compact or cocompact in such spaces the following definitions will prove useful.

3.10. Definition. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $A \in \mathcal{S}$.

(1) (a) $Q(A) = \bigcap\{\,]Q_s[\; \mid P_s \not\subseteq A\}$.
 (b) A is called *pseudo open* if $Q(A) = \,]A[$.

(2) (a) $P(A) = \bigvee\{[P_s] \mid A \not\subseteq Q_s\}$.
 (b) A is called *pseudo closed* if $P(A) = [A]$.

3.11. Lemma. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $A \in \mathcal{S}$. Then*

(1) $]A[\, \subseteq Q(A) \subseteq A$.
(2) $A \subseteq P(A) \subseteq [A]$.

Proof. (1) Suppose that $]A[\, \not\subseteq Q(A)$. Then there exists $s \in S$ with $]A[\, \not\subseteq]Q_s[$ and $P_s \not\subseteq A$. From $P_s \not\subseteq A$ we have $A \subseteq Q_s$, and so $]A[\, \subseteq]Q_s[$, which is a contradiction. Hence $]A[\, \subseteq Q(A)$. On the other hand, $Q(A) = \bigcap\{\,]Q_s[\; \mid P_s \not\subseteq A\} \subseteq \bigcap\{Q_s \mid P_s \not\subseteq A\} = A$ by [5, Theorem 1.2 (6)].

(2) $A = \bigvee\{P_s \mid A \not\subseteq Q_s\} \subseteq \bigvee\{[P_s] \mid A \not\subseteq Q_s\} = P(A)$ by [5, Theorem 1.2 (7)]. On the other hand, suppose that $P(A) \not\subseteq [A]$. Then there exists $s \in S$ with $[P_s] \not\subseteq [A]$ and $A \not\subseteq Q_s$. From $A \not\subseteq Q_s$ we have $P_s \subseteq A$, and so $[P_s] \subseteq [A]$, which is a contradiction. Hence $P(A) \subseteq [A]$. □

3.12. Corollary. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space.*

(1) *Every set $A \in \tau$ is pseudo open.*
(2) *Every set $A \in \kappa$ is pseudo closed.*

Proof. (1) If $A \in \tau$ then $A = \,]A[$ so by Lemma 3.11 (1) we have $]A[\, \subseteq Q(A) \subseteq \,]A[$, whence $Q(A) = \,]A[$ which means that A is pseudo open.

(2) If $A \in \kappa$ then $A = [A]$ so by Lemma 3.11 (2) we have $[A] \subseteq P(A) \subseteq [A]$, whence $P(A) = [A]$ which means that A is pseudo closed. □

On the other hand not every pseudo open set need be open, and not every pseudo closed set need be closed, as the following examples show.

3.13. Examples. Consider again the texture $(L, \mathcal{L})$.

(1) Consider a ditopology (τ, κ) on $(L, \mathcal{L})$ for which $\kappa = \{\emptyset, (0, 1/2], L\}$. Put, for example, $A = (0, 1/4] \in \mathcal{L}$. Clearly, $A \notin \kappa$ so A is not closed. However,

$$P(A) = \bigvee\{[(0, s]] \mid (0, 1/4] \not\subseteq (0, s]\} = \bigvee\{[(0, s]] \mid s < 1/4\} = (0, 1/2] = [A],$$

so A is pseudo closed.

(2) Consider a ditopology (τ, κ) on $(L, \mathcal{L})$ for which $\tau = \{\emptyset, (0, 1/2], L\}$. Put, for example, $A = (0, 3/4] \in \mathcal{L}$. Clearly, $A \notin \tau$ so A is not open. However

$$Q(A) = \bigcap\{](0, s][\mid (0, s] \not\subseteq (0, 3/4]\} = \bigcap\{](0, s][\mid s > 3/4\} = (0, 1/2] =]A[,$$

so A is pseudo open.

3.14. Theorem. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space.*

(1) *Suppose that (τ, κ) is compact, stable and R_0. Then every pseudo closed set $A \in \mathcal{S}$ is compact.*

(2) *Suppose that (τ, κ) is cocompact, costable and co-R_0. Then every pseudo open set $A \in \mathcal{S}$ is cocompact.*

Proof. (1) Let $A \in \mathcal{S}$ be pseudo closed. We must prove that A is compact. To this end, let $G_i \in \tau$, $i \in I$ satisfy the condition $A \subseteq \bigvee_{i \in I} G_i$. We prove first that

$$P(A) \subseteq \bigvee_{i \in I} G_i.$$

Assume $P(A) \not\subseteq \bigvee G_i$. By the definition of $P(A)$ there exists $s \in S$ with $A \not\subseteq Q_s$ and $[P_s] \not\subseteq \bigvee G_i$. Now $A \not\subseteq Q_s \implies \bigvee G_i \not\subseteq Q_s \implies [P_s] \subseteq \bigvee G_i$ since $\bigvee G_i \in \tau$ and (τ, κ) is R_0. So our supposition is wrong and $P(A) \subseteq \bigvee G_i$. As A is pseudo closed we have $P(A) = [A]$ and so $[A] \subseteq \bigvee G_i$. There are two cases to consider.

Case 1. $[A] = S$. As S is compact, we have $[A]$ is compact. Then there exists $i_1, i_2, \dots i_n$ such that $[A] \subseteq G_{i_1} \cup G_{i_2} \cup \dots \cup G_{i_n}$. Since $A \subseteq [A]$ we have $A \subseteq \cup_{i=1}^n G_i$ which shows that A is compact.

Case 2. $[A] \neq S$. Again $[A]$ is compact, this time because (τ, κ) is stable. Exactly as in Case 1 we see that A is compact.

(2) Dual to (1), and hence omitted.

□

3.15. Theorem. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space.*

(1) *Suppose (τ, κ) is co-R_1. Then if A is compact it is pseudo closed.*

(2) *Suppose (τ, κ) is R_1. Then if A is cocompact it is pseudo open.*

Proof. (1) Let (τ, κ) be co-R_1 and $A \in \mathcal{S}$ compact. We must prove that $P(A) = [A]$. By Lemma 3.11 (2) we know that $P(A) \subseteq [A]$, so it is sufficient to prove that

(3.1) $\quad [A] \subseteq P(A).$

Assume $[A] \not\subseteq P(A)$. Then we have $s \in S$ satisfying $[A] \not\subseteq Q_s$ and $P_s \not\subseteq P(A)$. Take any $u \in S$ with $A \not\subseteq Q_u$. Then, since $P(A) = \bigvee\{[P_u] \mid A \not\subseteq Q_u\}$, we have $P_s \not\subseteq [P_u] \in \kappa$. If we take any $t \in S$ with $[P_u] \not\subseteq Q_t$ we may apply the co-R_1 axiom [7, Definition 3.1 (d)] to give a set $K_t^u \in \kappa$ satisfying $P_s \not\subseteq K_t^u$ and $]K_t^u[\not\subseteq Q_t$. Let us prove that

(3.2) $\quad A \subseteq \bigvee\{]K_t^u[\mid A \not\subseteq Q_u \text{ and } [P_u] \not\subseteq Q_t\}.$

Suppose that (3.2) is false. Then there exists $w \in S$ with $A \not\subseteq Q_w$ and $P_w \not\subseteq \bigvee\{]K_t^u[\mid A \not\subseteq Q_u \text{ and } [P_u] \not\subseteq Q_t\}$. Hence we have $v \in S$ satisfying $P_w \not\subseteq Q_v$ and $P_v \not\subseteq \bigvee\{]K_t^u[\mid A \not\subseteq Q_u \text{ and } [P_u] \not\subseteq Q_t\}$. Applying the above argument with $u = w$ we have $P_s \not\subseteq [P_w]$, and if we note that $[P_w] \not\subseteq Q_v$ we have $K_v^w \in \kappa$ with $P_s \not\subseteq K_v^w$ and $]K_v^w[\not\subseteq Q_v$. We now obtain $P_v \subseteq]K_v^w[\subseteq \bigvee\{]K_t^u[\mid A \not\subseteq Q_u \text{ and } [P_u] \not\subseteq Q_t\}$, which is a contradiction. Hence (3.2) is proved.

Since $]K_t^u[\in \tau$ for all $A \not\subseteq Q_u$, $[P_u] \not\subseteq Q_t$, and A is compact we have $u_1, u_2, \dots, u_n \in S$ and $t_1, t_2, \dots, t_n \in S$ so that $[P_{u_i}] \not\subseteq Q_{t_i}$, $i = 1, 2, \dots, n$ and

$$A \subseteq]K_{t_1}^{u_1}[\cup]K_{t_2}^{u_2}[\cup \ldots \cup]K_{t_n}^{u_n}[\subseteq K_{t_1}^{u_1} \cup K_{t_2}^{u_2} \cup \ldots \cup K_{t_n}^{u_n} \in \kappa,$$

from which we deduce $[A] \subseteq K_{t_1}^{u_1} \cup K_{t_2}^{u_2} \cup \ldots \cup K_{t_n}^{u_n}$. But $[A] \not\subseteq Q_s$ implies $P_s \subseteq [A]$, so $P_s \subseteq K_{t_k}^{u_k}$ for some k, $1 \le k \le n$, since P_s is a molecule. However this contradicts $P_s \not\subseteq K_{t_i}^{u_i}$ for all $i = 1, 2, \dots, n$, and (3.1) is proved. Hence A is pseudo closed.

(2) Dual to (1), and we omit the details. □

If we assume the stronger bi-T_2 axiom we obtain the following improved result.

3.16. Theorem. *Let $(S, \mathcal{S}, \tau, \kappa)$ be bi-T_2. Then every compact set in $\mathcal{S}$ is closed and every cocompact set in $\mathcal{S}$ is open.*

Proof. We prove the first result, leaving the dual proof of the second result to the interested reader. Hence, let $A \in \mathcal{S}$ be compact. Take $s \in S$ with $P_s \not\subseteq A$. Then for any $t \in S$ satisfying $A \not\subseteq Q_t$ we have $Q_s \not\subseteq Q_t$, so by the bi-T_2 axiom we have $H^t \in \tau$, $K^t \in \kappa$ so that $H^t \subseteq K^t$, $H^t \not\subseteq Q_t$ and $P_s \not\subseteq K^t$ [7, Theorem 4.17 (2)]. Now $A \subseteq \bigvee_{A \not\subseteq Q_t} H^t$, so by compactness we have $t_1, t_2, \dots t_n \in S$ with $A \not\subseteq Q_{t_k}$, $k = 1, 2, \dots, n$ and $A \subseteq \bigcup_{k=1}^n H^{t_k} \subseteq \bigcup_{k=1}^n K^{t_k} \in \kappa$. Since $P_s \not\subseteq \bigcup_{k=1}^n K^{t_k}$ this establishes that $A \in \kappa$ by [6, Theorem 3.2 (4)] applied to $\beta = \kappa$. □

Let us now investigate the preservation of stability and costability under surjective difunctions.

3.17. Theorem. *Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$, $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be ditopological texture spaces with $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ stable, and $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ a bicontinuous surjective difunction. Then $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is stable.*

Proof. Take $K \in \kappa_2$ with $K \ne S_2$. Since (f, F) is cocontinuous, $f^{\leftarrow} K \in \kappa_1$. Let us prove that $f^{\leftarrow} K \ne S_1$. Assume the contrary. Since $f^{\leftarrow} S_2 = S_1$ by [5, Lemma 2.28 (1 c)] we have $f^{\leftarrow} S_2 \subseteq f^{\leftarrow} K$, whence $S_2 \subseteq K$ by [5, Corollary 2.33 (1 ii)] as (f, F) is surjective. This is a contradiction, so $f^{\leftarrow}(K) \ne S_1$. Hence $f^{\leftarrow}(K)$ is compact in $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ by stability. As (f, F) is continuous, $f^{\rightarrow}(f^{\leftarrow} K)$ is compact for the ditopology (τ_2, κ_2) by Theorem 2.5, and by [5, Corollary 2.33 (1)] this set is equal to K. This establishes that $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is stable. □

3.18. Theorem. *Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$, $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be ditopological texture spaces with $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ costable, and $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ a bicontinuous surjective difunction. Then $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is costable.*

Proof. Dual to the proof of Theorem 3.16, and we omit the details. □

In view of Lemma 2.11 the following is an immediate consequence of Theorem 3.17 and Theorem 3.18.

3.19. Corollary. *Let $(S_i, \mathcal{S}_i, \tau_i, \kappa_i)$, $i \in I$, be non-empty ditopological texture spaces and $(S, \mathcal{S}, \tau, \kappa)$ their product.*

(1) *If $(S, \mathcal{S}, \tau, \kappa)$ is stable then $(S_k, \mathcal{S}_k, \tau_k, \kappa_k)$ is stable for each $k \in I$.*
(2) *If $(S, \mathcal{S}, \tau, \kappa)$ is costable then $(S_k, \mathcal{S}_k, \tau_k, \kappa_k)$ is costable for each $k \in I$.* □

It is currently an open question as to whether the converse implications hold or not.

Finally we have the following.

3.20. Theorem. *Let* $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ *be a difunction and* $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ *bi-*T_2*. Then:*

(1) *If* (f, F) *is continuous,* $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ *compact and stable then* (f, F) *is closed.*
(2) *If* (f, F) *is cocontinuos,* $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ *cocompact and costable then* (f, F) *is co-open*

Proof. (1) Any $K \in \kappa_1$ is (τ_1, κ_1)-compact, whence $f^{\to}K$ is (τ_2, κ_2)-compact by Theorem 2.5. By Theorem 3.16 we deduce that $f^{\to}K \in \kappa_2$, so (f, F) is closed.

(2) Dual to (1). □

4. Dicompactness

In the previous sections we have considered compact and cocompact, stable and costable ditopological texture spaces. The notion of dicompacness combines all four of these properties.

4.1. Definition. A ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is called *dicompact* if it is compact, cocompact, stable and costable.

We may state at once:

4.2. Proposition. *Let* (τ_1, κ_1) *be a dicompact ditopology on the texture* $(S_1, \mathcal{S}_1)$*. Then if* $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ *is a second ditopological texture space for which there is a surjective bicontinuous difunction* $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ *the ditopology* (τ_2, κ_2) *is dicompact.*

Proof. The compactness, cocompactness, stability and costability of (τ_2, κ_2) follow at once from Proposition 2.6, Proposition 2.9, Theorem 3.17 and Theorem 3.18, respectively. Hence (τ_2, κ_2) is dicompact by Definition 4.1. □

4.3. Proposition. *Let* $(S, \mathcal{S}, \tau, \kappa)$ *be a dicompact ditopological texture space.*

(1) *If* (τ, κ) *is* R_1 *then* (τ, κ) *is regular.*
(2) *If* (τ, κ) *is co-*R_1 *then* (τ, κ) *is coregular.*
(3) *If* (τ, κ) *is regular then* (τ, κ) *is normal.*
(4) *If* (τ, κ) *is coregular then* (τ, κ) *is normal.*

Proof. (1) This follows by Theorem 3.7 since a dicompact space is costable.

(2) This follows from Theorem 3.8 since a dicompact space is stable.

(3) This follows from Theorem 3.5 since a dicompact space is stable.

(4) This follows from Theorem 3.6 since a dicompact space is costable. □

4.4. Proposition. *Let* $(S, \mathcal{S}, \tau, \kappa)$ *be a dicompact ditopological texture space.*

(1) *If* (τ, κ) *is* R_0 *then every pseudo closed set* $A \in \mathcal{S}$ *is compact.*
(2) *If* (τ, κ) *is co-*R_0 *then every pseudo open set* $A \in \mathcal{S}$ *is cocompact.*

Proof. (1) Since a dicompact space is compact and stable this is a consequence of Theorem 3.14 (1).

(2) Since a dicompact space is cocompact and costable this is a consequence of Theorem 3.14 (2). □

The following result is now immediate from Theorem 3.20.

4.5. Theorem. *Let* $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ *be dicompact,* $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ *bi-*T_2 *and* $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ *bicontinuous. Then* (f, F) *is closed an co-open.* □

4.6. Corollary. *Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ be dicompact, $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ bi-T_2 and $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ a bicontinuous bijection. Then (f, F) is a dihomeomorphism.*

Proof. This follows at once from Theorem 4.5 and [1, Proposition 4.4]. $\square$

Applying this to the identity difunction $(i_S, I_S) : (S, \mathcal{S}, \tau_1, \kappa_1) \to (S, \mathcal{S}, \tau_2, \kappa_2)$, where $(S, \mathcal{S}, \tau_1, \kappa_1)$ is dicompact, $(S, \mathcal{S}, \tau_2, \kappa_2)$ bi-T_2 and $\tau_2 \subseteq \tau_1$, $\kappa_2 \subseteq \kappa_1$ we obtain $(\tau_1, \kappa_1) = (\tau_2, \kappa_2)$. This shows that a bi-$T_2$ dicompact ditopological texture space is minimally bi-T_2 and maximally dicompact, so generalizing the well known result that a Hausdorff compact topological space is minimally Hausdorff and maximally compact.

The unit interval ditopological texture space $(\mathbb{I}, \mathcal{I}, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ of Examples 1.1 (2) is an important example of a bi-T_2 dicompact space. The reader is referred to [17] for some interesting results on bi-T_2 dicompact spaces obtained from quite a different perspective.

Work on this notion of ditopological compactness in the literature has used various different characterizations. To describe these we need the following concepts.

4.7. Definition. Let (τ, κ) be a ditopology on $(S, \mathcal{S})$.

(1) A set $\mathcal{D} \subseteq \mathcal{S} \times \mathcal{S}$ is called a *difamily* on $(S, \mathcal{S})$. A difamily $\mathcal{D}$ satisfying $\mathcal{D} \subseteq \tau \times \kappa$ is *open and co-closed*, one satisfying $\mathcal{D} \subseteq \kappa \times \tau$ is *closed and co-open*.

(2) A difamily $\mathcal{D}$ has the *finite exclusion property* (fep) if whenever $(F_i, G_i) \in \mathcal{D}$, $i = 1, 2, \dots, n$ we have $\bigcap_{i=1}^n F_i \not\subseteq \bigcup_{i=1}^n G_i$.

(3) A closed, co-open difamily $\mathcal{D}$ with $\bigcap\{F \mid F \in \operatorname{dom} \mathcal{D}\} \not\subseteq \bigvee\{G \mid G \in \operatorname{ran} \mathcal{D}\}$ is said to be *bound* in $(S, \mathcal{S}, \tau, \kappa)$.

(4) A difamily $\mathcal{D} = \{(G_i, F_i) \mid i \in I\}$ is called a *dicover* of $(S, \mathcal{S})$ if for all partitions I_1, I_2 of I (including the trivial partitions) we have

$$\bigcap_{i \in I_1} F_i \subseteq \bigvee_{i \in I_2} G_i.$$

(5) A difamily $\mathcal{D}$ is called *finite* (*co-finite*) if $\operatorname{dom} \mathcal{D}$ (resp. *ran*$\mathcal{D}$) is finite.

4.8. Theorem. *The following are equivalent for $(S, \mathcal{S}, \tau, \kappa)$.*

(1) *$(S, \mathcal{S}, \tau, \kappa)$ is dicompact.*

(2) *Every closed, co-open difamily with the finite exclusion property is bound.*

(3) *Every open, co-closed dicover has a sub-dicover which is finite and co-finite.* $\square$

The non-trivial proof of this theorem may be found in [2]. In view of the characterization in terms of dicovers, dicompactness has also been known as *dicover bicompactness* in the literature. On the other hand the characterization in terms of difamilies with the finite exclusion property enables us to state and prove a dual version of the Alexander subbase theorem (cf. [9]) for dicompactness.

4.9. Theorem. *Let s_τ be a subbase for τ and s_κ a subbase for κ. Then (τ, κ) is dicompact if and only if every difamily $\mathcal{B} \subseteq s_\kappa \times s_\tau$ with the finite exclusion property is bound.*

Proof. $\Longrightarrow$. Clear.

$\Longleftarrow$. Suppose that the condition is satisfied but that (τ, κ) is not dicompact. Then, by Theorem 4.8, we have $\mathcal{B} \subseteq \kappa \times \tau$ with the fep which satisfies $\bigcap \operatorname{dom} \mathcal{B} \subseteq \bigvee \operatorname{ran} \mathcal{B}$. The set of all such families $\mathcal{B}$, ordered by inclusion, is easily seen to be inductive so by Zorn's Lemma there exists a maximal element $\mathcal{B}_0$ in this set. We establish the following properties of $\mathcal{B}_0$.

(1) For $A \in \kappa$, $B \in \tau$, if

$$A \cap \bigcap_{i=1}^n A_i \not\subseteq B \cup \bigcup_{i=1}^n B_i \quad \forall\, (A_1, B_1), \dots, (A_n, B_n) \in \mathcal{B}_0,\ n \in \mathbb{N}^+,$$

then $(A,B)\in\mathcal{B}_0$.

Indeed, under the given conditions it is trivial to verify that $\mathcal{B}'=\mathcal{B}_0\cup\{(A,B)\}\subseteq\kappa\times\tau$ has the fep and so $(A,B)\in\mathcal{B}'=\mathcal{B}_0$ by the maximality of $\mathcal{B}_0$.

(2) $\mathcal{B}_0=\operatorname{dom}\mathcal{B}_0\times\operatorname{ran}\mathcal{B}_0$.

If we take $C_1\in\operatorname{dom}\mathcal{B}_0$ and $D_2\in\operatorname{ran}\mathcal{B}_0$ then we have (C_1,D_1), $(C_2,D_2)\in\mathcal{B}_0$ for some $D_1\in\tau, C_2\in\kappa$. If $(C_1,D_2)\notin\mathcal{B}_0$ then by (1) we have $(A_i,B_i)\in\mathcal{B}_0$, $i=1,2,\dots,n$ for which

$$C_1\cap\bigcap_{i=1}^n A_i\subseteq D_2\cup\bigcup_{i=1}^n B_i.$$

From this we deduce $C_1\cap C_2\cap\bigcap_{i=1}^n A_i\subseteq D_1\cup D_2\cup\bigcup_{i=1}^n B_i$, which contradicts the fep for $\mathcal{B}_0$.

(3) $(A_1,B_1),(A_2,B_2)\in\mathcal{B}_0\implies(A_1\cap A_2,B_1\cup B_2)\in\mathcal{B}_0$.

This follows trivially from (1).

(4) $(A,B)\in\mathcal{B}_0$, $A\subseteq A'\in\kappa$, $B\supseteq B'\in\tau\implies(A',B')\in\mathcal{B}_0$.

Clear from (1).

(5) $\mathcal{F}=\operatorname{dom}\mathcal{B}_0$ is a filter in κ and $\mathcal{G}=\operatorname{ran}\mathcal{B}_0$ a dual filter in τ.

Immediate from (3), (4) and the evident fact that $(\emptyset,S)\notin\mathcal{B}_0$.

(6) For $A_1,\dots,A_n\in\kappa$ and $B_1,\dots,B_m\in\tau$,

$$\left(\bigcup_{i=1}^n A_i,\ \bigcap_{j=1}^m B_j\right)\in\mathcal{B}_0\implies\exists\, i,j\text{ with }(A_i,B_j)\in\mathcal{B}_0.$$

First we prove that $(\bigcup_{i=1}^n A_i,B)\in\mathcal{B}_0\implies\exists\, i$ with $(A_i,B)\in\mathcal{B}_0$. If $(A_i,B)\notin\mathcal{B}_0$, $i=1,2,\dots,n$ then by (1) for each i we have $(C_k^i,D_k^i)\in\mathcal{B}_0$ for $k=1,2,\dots,n_i$ with

$$A_i\cap\bigcap_{k=1}^{n_i}C_k^i\subseteq B\cup\bigcup_{k=1}^{n_i}D_k^i.$$

From this we obtain

$$(A_1\cup\ldots\cup A_n)\cap\bigcap_{i=1}^n\bigcap_{k=1}^{n_i}C_k^i\subseteq B\cup\bigcup_{i=1}^n\bigcup_{k=1}^{n_i}D_k^i,$$

which again contradicts the fep for $\mathcal{B}_0$. In the same way

$$\left(A,\bigcap_{j=1}^m B_j\right)\in\mathcal{B}_0\implies\exists\, j\text{ with }(A,B_j)\in\mathcal{B}_0.$$

Applying these implications now gives the required result.

(7) $\mathcal{F}$ is a prime filter and $\mathcal{G}$ a prime dual filter.

To complete the proof let us now set $\mathcal{B}=\mathcal{B}_0\cap(s_\kappa\times s_\tau)=(\mathcal{F}\cap s_\kappa)\times(\mathcal{G}\cap s_\tau)$. Since $\mathcal{B}\subseteq\mathcal{B}_0$ it has the fep. For each $A\in\mathcal{F}$, $B\in\mathcal{G}$ we may write

$$A=\bigcap_\alpha(F_1^\alpha\cup\ldots\cup F_{n_\alpha}^\alpha)\quad B=\bigvee_\beta(G_1^\beta\cap\ldots\cap G_{m_\beta}^\beta)$$

where $F_i^\alpha\in s_\kappa$ and $G_j^\beta\in s_\tau$. By (4) we have $(F_1^\alpha\cup\ldots\cup F_{n_\alpha}^\alpha,G_1^\beta\cap\ldots\cap G_{m_\beta}^\beta)\in\mathcal{B}_0$ for all α and β. By (6) we have i_α, j_β for which $(F_{i_\alpha}^\alpha,G_{j_\beta}^\beta)\in\mathcal{B}_0$. But now $(F_{i_\alpha}^\alpha,G_{j_\beta}^\beta)\in\mathcal{B}$, $\bigcap_\alpha F_{i_\alpha}^\alpha\subseteq A$ and $B\subseteq\bigvee_\beta G_{j_\beta}^\beta$ which gives $\bigcap\operatorname{dom}\mathcal{B}\subseteq\bigcap\operatorname{dom}\mathcal{B}_0\subseteq\bigvee\operatorname{ran}\mathcal{B}_0\subseteq\bigvee\operatorname{ran}\mathcal{B}$. This contradicts the hypothesis of the theorem. □

The following theorem is the Tychonoff Product Theorem for dicompactness.

4.10. Theorem. *A product of non-empty ditopological texture spaces is dicompact if and only each of the spaces is dicompact.*

Proof. Sufficiency. Consider dicompact ditopological texture spaces $(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha)$, $\alpha \in A$, and let $(S, \mathcal{S}, \tau, \kappa)$ be their product. Denote by s_τ the subbase of τ consisting of the sets $E(\alpha, G_\alpha)$, $\alpha \in A$, $G_\alpha \in \tau_\alpha$ and by s_κ the subbase of κ consisting of the sets $E(\alpha, K_\alpha)$, $\alpha \in A$, $K_\alpha \in \kappa_\alpha$. By Theorem 4.8, it will be sufficient to show that every difamily $\mathcal{B} \subseteq s_\kappa \times s_\tau$ with the fep is bound. Suppose, on the contrary, that there is such a $\mathcal{B}$ with the fep for which $\bigcap \operatorname{dom} \mathcal{B} \subseteq \bigvee \operatorname{ran} \mathcal{B}$. By hypothesis we may write

$$\mathcal{B} = \{(E(\alpha_i, F^i_{\alpha_i}),\ E(\beta_i,\ G^i_{\beta_i})) \mid i \in I\},$$

where $\alpha_i, \beta_i \in A$, $F^i_{\alpha_i} \in \kappa_{\alpha_i}$ and $G^i_{\beta_i} \in \tau_{\beta_i}$. Now

$$\bigcap_{i \in I} E(\alpha_i, F^i_{\alpha_i}) = \prod_{\alpha \in A} Y_\alpha,$$

where

$$Y_\alpha = \begin{cases} \bigcap\{F^i_{\alpha_i} \mid \alpha_i = \alpha\}, & \text{if } \exists\, i \text{ with } \alpha_i = \alpha, \\ S_\alpha & \text{otherwise,} \end{cases}$$

and

$$\bigvee_{i \in I} E(\beta_i, G^i_{\beta_i}) = \bigcup_{\beta \in \{\beta_i \mid i \in I\}} E(\beta, \bigvee\{G^i_{\beta_i} \mid \beta_i = \beta\}).$$

by [3, Lemma 2.3]. Now $\bigcap \operatorname{dom} \mathcal{B} \subseteq \bigvee \operatorname{ran} \mathcal{B}$ and the form of the sets involved gives $\beta \in \{\beta_i \mid i \in I\}$ satisfying $Y_\beta \subseteq \bigvee\{G^i_{\beta_i} \mid \beta_i = \beta\}$.

There are two cases to consider:

(1) $\exists\, i \in I$ with $\alpha_i = \beta$. In this case

$$\bigcap\{F^i_{\alpha_i} \mid \alpha_i = \beta\} \subseteq \bigvee\{G^i_{\beta_i} \mid \beta_i = \beta\}.$$

If we define

$$\begin{aligned} \mathcal{B}_\beta = \{(F^i_{\alpha_i}, \emptyset) \mid \alpha_i = \beta,\ \beta_i \neq \beta\} &\cup \{(F^i_{\alpha_i}, G^i_{\beta_i}) \mid \alpha_i = \beta_i = \beta\} \\ &\cup \{(S_\beta, G^i_{\beta_i}) \mid \beta_i = \beta,\ \alpha_i \neq \beta\} \subseteq \kappa_\beta \times \tau_\beta, \end{aligned}$$

then $\bigcap \operatorname{dom} \mathcal{B}_\beta \subseteq \bigvee \operatorname{ran} \mathcal{B}_\beta$, and since $(\tau_\beta, \kappa_\beta)$ is dicompact $\exists\, i_1, \ldots, i_p$ with $\beta_{i_k} = \beta,\ \alpha_{i_k} \neq \beta,\ k = 1, \ldots, p$; $i_{p+1}, \ldots i_q$ with $\alpha_{i_k} = \beta_{i_k} = \beta,\ k = p+1, \ldots, q$ and $i_{q+1}, \ldots, i_r$ with $\alpha_{i_k} = \beta,\ \beta_{i_k} \neq \beta,\ k = q+1, \ldots, r$ so that

$$\bigcap_{k=1}^{q} F^{i_k}_{\alpha_{i_k}} \subseteq \bigcup_{i=p+1}^{r} G^{i_k}_{\alpha_{i_k}}.$$

But then

$$\bigcap_{k=1}^{r} E(\beta, F^{i_k}_{\alpha_{i_k}}) \subseteq \bigcup_{k=1}^{r} E(\beta, G^{i_k}_{\alpha_{i_k}}),$$

which contradicts the fact that $\mathcal{B}$ has the fep.

(2) $\alpha_i \neq \beta\ \forall\, i \in I$. In this case

$$S_\beta \subseteq \bigvee\{G^i_{\alpha_i} \mid \alpha_i = \beta\},$$

and letting

$$\mathcal{B}_\beta = \{(S_\beta, G^i_{\alpha_i}) \mid \alpha_i = \beta\}$$

we obtain a contradiction to the fep for $\mathcal{B}$ as before.

This completes the proof that $(S, \mathcal{S}, \tau, \kappa)$ is dicompact.

Necessity. Suppose that $(S, \mathcal{S}, \tau, \kappa)$ is dicompact and take $k \in I$. Then $(S, \mathcal{S}, \tau, \kappa)$ is compact, cocompact, stable and costable, whence by Corollary 2.12, $(S_k, \mathcal{S}_k, \tau_k, \kappa_k)$ is compact and cocompact, while by Corollary 3.19 it is stable and costable. This proves that $(S_k, \mathcal{S}_k, \tau_k, \kappa_k)$ is dicompact. □

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