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BAYESIAN ESTIMATION OF THE SHAPE PARAMETER OF A GENERALIZED PARETO DISTRIBUTION UNDER ASYMMETRIC LOSS FUNCTIONS

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Abstract

In this paper Bayes estimators of the shape parameter of the generalized Pareto distribution have been obtained by taking quasi, inverted gamma and uniform prior distributions using the linex, precautionary and entropy loss functions. These are compared with the corresponding Bayes estimators under the squared error loss function.

Keywords: Pareto Distribution, Shape parameter, Asymmetric loss function.

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1. Introduction

Let us consider the generalized Pareto distribution (GPD) whose cumulative distribution function is defined by

$$(1.1)\qquad F(x;\sigma,\theta) = 1-(1-\frac{x}{\sigma})^{\frac{1}{\theta}};\ \theta>0,\ 0<x<\sigma,$$

see E. Castillo and A. S. Hadi [5]. Thus the probability density function (pdf) of the GPD is given by

$$(1.2)\qquad f(x;\sigma,\theta) = \frac{1}{\sigma\theta}(1-\frac{x}{\sigma})^{\frac{1}{\theta}-1};\ \theta>0,\ 0<x<\sigma,$$

where σ and θ are the scale and shape parameters, respectively.

The object of the present paper is to obtain a Bayes estimator of θ under various loss functions using a number of prior distributions.

A commonly used loss function is the squared error loss function (SELF)

$$(1.3)\qquad L(\widehat{\theta},\theta) = (\widehat{\theta},\theta)^2,$$

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which is a symmetrical loss function that assigns equal losses to over estimation and underestimation. The Bayes estimator under the above loss function is the posterior mean given by

$$\widehat{\theta}_s = \mathrm{E}_\pi(\theta), \tag{1.4}$$

where E_π denotes the posterior expectation.

The SELF is often used also because it does not lead to extensive numerical computation, but several authors (Canfield [4], Varian [10], Berger [2], Zellner [11], Basu and Ebrahimi [1], Dey and Liu [7], Calabria and Pulcini [3], and Norstrom [8]) have recognized the inappropriateness of using a symmetric loss function in several estimation problems. They use various asymmetric loss functions, given as follows:

(a) The Linex loss function:

Basu and Ebrahimi [1] considered the linex (linear-exponential) loss function $L(\Delta)$ given by

$$L(\Delta) = b[\mathrm{e}^{a\Delta} - a\Delta - 1],\ a \neq 0,\ b > 0, \tag{1.5}$$

where

$$\Delta = \frac{\widehat{\theta}}{\theta} - 1,$$

and studied Bayesian estimation under this asymmetric loss function for an exponential lifetime distribution. This loss function is suitable for situations where overestimation of θ is more costly than its underestimation.

This Bayes estimator under asymmetric loss (Δ), denoted by $\widehat{\theta}_A$, is the solution of the following equation

$$\mathrm{E}_\pi\Big[\frac{1}{\theta}\exp(\frac{a\widehat{\theta}_A}{\theta})\Big] = \mathrm{e}^a\mathrm{E}_\pi(\frac{1}{\theta}). \tag{1.6}$$

(b) The precautionary loss function:

Norstrom [8] introduced an alternative asymmetric precautionary loss function, and also presented a general class of precautionary loss functions as a special case. These loss functions approach infinitely near the origin to prevent underestimation, thus giving conservative estimators, especially when low failure rates are being estimated. These estimators are very useful when underestimation may lead to serious consequences. A very useful and simple asymmetric precautionary loss function is

$$L(\widehat{\theta}, \theta) = \frac{(\widehat{\theta} - \theta)^2}{\widehat{\theta}}. \tag{1.7}$$

The Bayes estimator under a precautionary loss function is denoted by $\hat{\theta}_p$, and is given by the following equation.

$$\widehat{\theta}_p = [\mathrm{E}_\pi(\theta^2)]^{\frac{1}{2}}. \tag{1.8}$$

(c) The entropy loss function:

In many practical situations, it appears to be more realistic to express the loss in terms of the ratio $\frac{\widehat{\theta}}{\theta}$. In this case, Calabria and Pulcini [3] point out that a useful asymmetric loss function is the entropy loss function:

$$L(\delta) \propto [\delta^P - p\log_{\mathrm{e}}(\delta) - 1],$$

where

$$\delta = \frac{\widehat{\theta}}{\theta},$$

whose minimum occurs at $\widehat{\theta} = \theta$. Also, the loss function $L(\delta)$ has been used in Dey *et al* [6] and Dey and Liu [7], in the original form having $p = 1$. Thus, $L(\delta)$ can be written as:

$$L(\delta) = b[\delta - \log_e(\delta) - 1];\ b > 0. \tag{1.9}$$

The Bayes estimator under the entropy loss function is denoted by $\widehat{\theta}_e$, and is given by the following equation:

$$\widehat{\theta}_e = \left[E_\pi\left(\frac{1}{\theta}\right)\right]^{-1}. \tag{1.10}$$

Let us obtain the Bayes estimators of the shape parameter of GPD under three prior distributions of θ.

(i) **The quasi-prior**: For the situation where the experimenter has no prior information about the parameter θ, one may use the quasi density as given by

$$g_1(\theta) = \frac{1}{\theta^d};\ \theta > 0,\ d > 0. \tag{1.11}$$

Here $d = 0$ leads to a diffuse prior and $d = 1$ to a non-informative prior.

(ii) **The inverted gamma prior**: The most widely used prior distribution of θ is the inverted gamma distribution with parameters α and $\beta(> 0)$ with p.d.f. given by

$$g_2(\theta) = \begin{cases} \dfrac{\beta^\alpha}{\Gamma(\alpha)}\theta^{-(\alpha+1)}e^{-\frac{\beta}{\theta}}, & \text{if } \theta > 0,\ (\alpha, \beta) > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1.12}$$

The main reason for its general acceptability is the mathematical tractability resulting from the fact that the inverted gamma distribution is the conjugate prior for θ.

(iii) **The uniform prior**: It frequently happens that the life tester knows in advance that the probable values of θ lies over a finite range $[\alpha, \beta]$, but he does not have any strong opinion about any subset of values over this range. In such a case a uniform distribution over$[\alpha, \beta]$ may be a good approximation:

$$g_2(\theta) = \begin{cases} \dfrac{1}{\beta - \alpha}, & \text{if } \theta < \alpha \le \theta \le \beta, \\ 0, & \text{otherwise.} \end{cases} \tag{1.13}$$

2. The Bayes estimator under $g_1(\theta)$

Let us suppose that n items are put to a life test and that the experiment is terminated when $r(< n)$ items have failed. If $x_1, \ldots, x_r$ denote the first r observations having a common density function as given in (1.2), where σ is known, then the joint probability density function is given by

$$f(\underline{x}/\theta) = \frac{n!}{(n-r)!}\left(\frac{1}{\sigma\theta}\right)^r e^{-(\frac{1}{\theta}-1)T_r}, \tag{2.1}$$

where

$$T_r = -\sum_{i=1}^{r} \log\left(1 - \frac{x_i}{\sigma}\right) + (n-r)\log\left(1 - \frac{x_{(r)}}{\sigma}\right)$$

The maximum likelihood estimator (MLE) of θ is given by

$$\widehat{\theta} = \frac{T_r}{r}.$$

The posterior pdf of θ is obtained as

$$f(\theta/\underline{x}) = \frac{T_r^{r+d-1}}{\Gamma(r+d-1)}\,\theta^{-(r+d)}\mathrm{e}^{-(T_r)/\theta};\ \theta > 0,\ r+d > 1. \tag{2.2}$$

The Bayes estimator under the squared error loss function is given by

$$\widehat{\theta}_s = \frac{T_r}{r+d-2}; r+d > 2, \tag{2.3}$$

and the Bayes estimator under the linex loss function by

$$\widehat{\theta}_A = \left(\frac{1-\mathrm{e}^{-a/(r+d)}}{a}\right)T_r \tag{2.4}$$

(Srivastava [9]). Using (1.8), the Bayes estimator under the precautionary loss function comes out to be

$$\widehat{\theta}_P = \frac{T}{[(r+d-2)(r+d-3)]^{1/2}}. \tag{2.5}$$

Also, using (1.10), the Bayes estimator under the entropy loss function is obtained as

$$\widehat{\theta}_{\mathrm{e}} = \frac{T_r}{(r+d-1)}. \tag{2.6}$$

2.1. The Risk Functions: The risk functions of the estimators $\widehat{\theta}_S$, $\widehat{\theta}_A$, $\widehat{\theta}_P$ and $\widehat{\theta}_{\mathrm{e}}$ relative to SELF are denoted by $R_S(\widehat{\theta}_S)$, $R_S(\widehat{\theta}_A)$, $R_S(\widehat{\theta}_P)$ and $R_S(\widehat{\theta}_{\mathrm{e}})$, respectively, and are given by Basu and Ebrahimi [1] as:

$$R_S(\widehat{\theta}_S) = \theta^2\left[\frac{r(r+1)}{(r+d-2)^2} - \frac{2r}{(r+d-2)} + 1\right], \tag{2.7}$$

$$R_S(\widehat{\theta}_A) = \theta^2\left[\frac{r(r+1)}{a^2}\big(1-\mathrm{e}^{-a/(r+d)}\big)^2 - \frac{2r}{a}(1-\mathrm{e}^{-a/(r+d)}) + 1\right], \tag{2.8}$$

$$R_S(\widehat{\theta}_P) = \theta^2\left[\frac{r(r+1)}{[(r+d-2)(r+d-3)]} - \frac{2r}{\big[(r+d-2)(r+d-3)\big]^{1/2}} + 1\right], \tag{2.9}$$

$$R_S(\widehat{\theta}_{\mathrm{e}}) = \theta^2\left[\frac{r(r+1)}{[(r+d-1)]^2} - \frac{2r}{(r+d-1)} + 1\right]. \tag{2.10}$$

The risk functions of the estimators $\widehat{\theta}_S$, $\widehat{\theta}_A$, $\widehat{\theta}_P$ and $\widehat{\theta}_{\mathrm{e}}$, relative to the linex loss function are denoted by $R_A(\widehat{\theta}_S)$, $R_A(\widehat{\theta}_A)$, $R_A(\widehat{\theta}_P)$ and $R_A(\widehat{\theta}_{\mathrm{e}})$, respectively, and are given by:

$$R_A(\widehat{\theta}_S) = b\left[\mathrm{e}^{-a}\Big(1-\frac{a}{r+d-2}\Big)^{-r} - \Big(\frac{ar}{r+d-2}\Big) + a - 1\right], \tag{2.11}$$

$$R_A(\widehat{\theta}_A) = b\left[\mathrm{e}^{-ad/(r+d)} - r\Big(1-\mathrm{e}^{-a/(r+d)}\Big) + a - 1\right], \tag{2.12}$$

$$\begin{aligned} R_A(\widehat{\theta}_P) = b\Bigg[&\mathrm{e}^{-a}\Big(1-\frac{a}{[(r+d-2)(r+d-3)]^{1/2}}\Big)^{-r} \\ &- \frac{ar}{[(r+d-2)(r+d-3)]^{1/2}} + a - 1\Bigg], \end{aligned} \tag{2.13}$$

$$R_A(\widehat{\theta}_{\mathrm{e}}) = b\left[e^{-a}\Big(1-\frac{a}{(r+d-1)}\Big)^{-r} - \frac{ar}{(r+d-1)} + a - 1\right] \tag{2.14}$$

The risk functions of the estimators $\widehat{\theta}_S,\ \widehat{\theta}_A,\ \widehat{\theta}_P$ and $\widehat{\theta}_e$, relative to the precautionary loss function are denoted by $R_P(\widehat{\theta}_S),\ R_P(\widehat{\theta}_A),\ R_P(\widehat{\theta}_P)$ and $R_P(\widehat{\theta}_e)$, respectively, and are given by:

$$R_P(\widehat{\theta}_S) = \theta^2\left[\frac{r(r+1)}{(r+d-2)^2} - \frac{2r}{(r+d-2)} + 1\right], \tag{2.15}$$

$$R_P(\widehat{\theta}_A) = \theta^2\left[\frac{r(r+1)}{a^2}\left(1-\mathrm{e}^{-a/(r+d)}\right)^2 - \frac{2r}{a}\left(1-\mathrm{e}^{-a/(r+d)}\right) + 1\right], \tag{2.16}$$

$$R_P(\widehat{\theta}_P) = \theta^2\left[\frac{r(r+1)}{[(r+d-2)(r+d-3)]} - \frac{2r}{[(r+d-2)(r+d-3)]^{1/2}} + 1\right], \tag{2.17}$$

$$R_P(\widehat{\theta}_e) = \theta^2\left[\frac{r(r+1)}{[(r+d-1)]^2} - \frac{2r}{(r+d-1)} + 1\right] \tag{2.18}$$

The risk functions of the estimators $\widehat{\theta}_S,\ \widehat{\theta}_A,\ \widehat{\theta}_P$ and $\widehat{\theta}_e$, relative to the entropy loss function are denoted by $R_e(\widehat{\theta}_S),\ R_e(\widehat{\theta}_A),\ R_e(\widehat{\theta}_P)$ and $R_e(\widehat{\theta}_e)$, respectively, and are given by:

$$R_e(\widehat{\theta}_S) = b\left[\frac{2-d}{(r+d-2)} - \mathrm{E_e}\log_e\left(\frac{\widehat{\theta}_S}{\theta}\right)\right], \tag{2.19}$$

$$R_e(\widehat{\theta}_A) = b\left[\frac{r(1-\mathrm{e}^{-a/(r+d)})}{a} - \mathrm{E_e}\log_e\left(\frac{\widehat{\theta}_A}{\theta}\right) - 1\right], \tag{2.20}$$

$$R_e(\widehat{\theta}_P) = b\left[\frac{r}{[(r+d-2)(r+d-3)]^{1/2}} - \mathrm{E_e}\log_e\left(\frac{\widehat{\theta}_P}{\theta}\right) - 1\right], \tag{2.21}$$

$$R_e(\widehat{\theta}_e) = b\left[\frac{1-d}{(r+d-1)} - \mathrm{E_e}\log_e\left(\frac{\widehat{\theta}_e}{\theta}\right)\right]. \tag{2.22}$$

3. The Bayes estimator under $g_2(\theta)$

It can be easily verified that $g_2(\theta)$, i.e. the inverted gamma family, is the natural conjugate prior for the parameter θ with respect to GPD. Using (2.1), we obtain the posterior distribution

$$f(\theta/\underline{x}) = \frac{(\beta+T_r)^{r+\alpha}}{\Gamma(r+\alpha)}\theta^{-(r+\alpha+1)}\mathrm{e}^{-(\beta+T_r)/\theta};\ \theta>0, \tag{3.1}$$

which is again an inverted gamma family of parameters $(\beta+T_r,\ r+\alpha)$. Thus, the Bayes estimator of θ under the squared error loss function is given by:

$$\widehat{\theta}_S = \frac{\beta+T_r}{(r+\alpha-1)}. \tag{3.2}$$

The Bayes estimator under the linex loss function is given by:

$$\widehat{\theta}_A = \frac{1-\mathrm{e}^{-a/(r+\alpha+1)}}{a}(\beta+T_r). \tag{3.3}$$

Using (1.8), the Bayes estimator under the precautionary loss function comes out to be:

$$\widehat{\theta}_P = \frac{\beta+T_r}{[(r+\alpha-1)(r+\alpha-2)]^{1/2}}. \tag{3.4}$$

Using (1.10), the Bayes estimator under the entropy loss function is obtained as

$$\widehat{\theta}_e = \frac{\beta+T_r}{(r+\alpha)}. \tag{3.5}$$

3.1. The Risk Functions: The risk functions of the estimators $\widehat{\theta}_S,\ \widehat{\theta}_A,\ \widehat{\theta}_P$ and $\widehat{\theta}_{\mathrm{e}}$, relative to SELF are denoted by $R_S(\widehat{\theta}_S),\ R_S(\widehat{\theta}_A),\ R_S(\widehat{\theta}_P)$ and $R_S(\widehat{\theta}_{\mathrm{e}})$, respectively and are given by:

$$R_S(\widehat{\theta}_S) = \theta^2\left[\left(\frac{r(r+1)+\frac{2r\beta}{\theta}+\frac{\beta^2}{\theta^2}}{(r+\alpha-1)^2}\right) - \frac{2\left(r+\frac{\beta}{\theta}\right)}{(r+\alpha-1)}+1\right], \tag{3.6}$$

$$R_S(\widehat{\theta}_A) = \theta^2\left[C^2\left\{r(r+1)+\frac{2r\beta}{\theta}+\frac{\beta^2}{\theta^2}\right\} - 2C\left(r+\frac{\beta}{\theta}\right)+1\right], \tag{3.7}$$

$$R_S(\widehat{\theta}_P) = \theta^2\left[K^2\left\{r(r+1)+2r\left(\frac{\beta}{\theta}\right)+\left(\frac{\beta}{\theta}\right)^2\right\} - 2K\left\{r+\left(\frac{\beta}{\theta}\right)\right\}+1\right], \tag{3.8}$$

$$R_S(\widehat{\theta}_{\mathrm{e}}) = \theta^2\left[\frac{r(r+1)+2r\left(\frac{\beta}{\theta}\right)+\left(\frac{\beta}{\theta}\right)^2}{[(r+\alpha)]^2} - \frac{2(r+\frac{\beta}{\theta})}{(r+\alpha)}+1\right], \tag{3.9}$$

where $C = (\dfrac{1-\mathrm{e}^{-a/(r+\alpha+1)}}{a})$, and $K = \dfrac{1}{[(r+\alpha-1)(r+\alpha-2)]^{1/2}}$.

The risk functions of the estimators $\widehat{\theta}_S,\ \widehat{\theta}_A,\ \widehat{\theta}_P$ and $\widehat{\theta}_{\mathrm{e}}$, relative to the linex function are denoted by $R_A(\widehat{\theta}_S),\ R_A(\widehat{\theta}_A),\ R_A(\widehat{\theta}_P)$ and $R_A(\widehat{\theta}_{\mathrm{e}})$, respectively, and are given by

$$R_A(\widehat{\theta}_S) = b\left[\left(\mathrm{e}^{-a(1-\frac{\beta}{\theta(r+\alpha-1)})}\right)\left(1-\frac{a}{r+\alpha-1}\right)^{-r} - \left(\frac{a(r+\frac{\beta}{\theta})}{r+\alpha-1}\right)+a-1\right], \tag{3.10}$$

$$\begin{aligned} R_A(\widehat{\theta}_A) = b\Big[&\left(\mathrm{e}^{-a(\alpha+1)/(r+\alpha+1)}\right)\left(\mathrm{e}^{\frac{\beta}{\theta}\left(1-\mathrm{e}^{-a/(r+\alpha+1)}\right)}\right) \\ &- \left(1-\mathrm{e}^{-a/(r+\alpha+1)}\right)\left(r+\frac{\beta}{\theta}\right)+a-1\Big], \end{aligned} \tag{3.11}$$

$$R_A(\widehat{\theta}_P) = b\left[(1-aK)^{-r}\mathrm{e}^{-a\left(1-\frac{a\beta K}{\theta}\right)} - aK\left(r+\frac{\beta}{\theta}+a-1\right)\right], \tag{3.12}$$

$$R_A(\widehat{\theta}_{\mathrm{e}}) = b\Big[\left(1-\frac{a}{r+\alpha}\right)^{-r}\exp\left\{-a\left(1-\frac{\beta}{\theta(r+\alpha)}\right)\right\} - \frac{a(r+\frac{\beta}{\theta})}{r+\alpha}+a-1] \tag{3.13}$$

The risk functions of the estimators $\widehat{\theta}_S,\ \widehat{\theta}_A,\ \widehat{\theta}_P$ and $\widehat{\theta}_{\mathrm{e}}$, relative to the entropy loss function are denoted by $R_{\mathrm{e}}(\widehat{\theta}_S),\ R_{\mathrm{e}}(\widehat{\theta}_A),\ R_{\mathrm{e}}(\widehat{\theta}_P)$ and $R_{\mathrm{e}}(\widehat{\theta}_{\mathrm{e}})$, respectively, and are given by:

$$R_{\mathrm{e}}(\widehat{\theta}_S) = b\left[\frac{r+\left(\frac{\beta}{\theta}\right)}{(r+\alpha-1)} - \mathrm{E}_\theta \log_{\mathrm{e}}\left(\frac{\widehat{\theta}_S}{\theta}\right)-1\right], \tag{3.14}$$

$$R_{\mathrm{e}}(\widehat{\theta}_A) = b\left[\frac{\left(1-\mathrm{e}^{-a/(r+\alpha+1)}\right)\left(r+\frac{\beta}{\theta}\right)}{a} - \mathrm{E}_\theta \log_{\mathrm{e}}\left(\frac{\widehat{\theta}_A}{\theta}\right)-1\right], \tag{3.15}$$

$$R_{\mathrm{e}}(\widehat{\theta}_P) = b\left[\left(r+\frac{\beta}{\theta}\right)K - \mathrm{E}_\theta \log_{\mathrm{e}}\left(\frac{\widehat{\theta}_P}{\theta}-1\right)\right], \tag{3.16}$$

$$R_{\mathrm{e}}(\widehat{\theta}_{\mathrm{e}}) = b\left[\frac{r+\left(\frac{\beta}{\theta}\right)}{(r+\alpha)} - \mathrm{E}_\theta \log_{\mathrm{e}}\left(\frac{\widehat{\theta}_{\mathrm{e}}}{\theta}\right)-1\right] \tag{3.17}$$

3.2. The Bayes Risks: The Bayes risks for the estimators $\widehat{\theta}_S,\ \widehat{\theta}_A,\ \widehat{\theta}_P$ and $\widehat{\theta}_{\mathrm{e}}$ are the prior expectations of the risks obtained above. Thus, the Bayes risks relative to SELF are denoted by $r_S(\widehat{\theta}_S),\ r_S(\widehat{\theta}_A),\ r_S(\widehat{\theta}_P)$ and $r_S(\widehat{\theta}_{\mathrm{e}})$, respectively, and are given by:

$$(3.18)\qquad r_S(\widehat{\theta}_S) = \frac{\beta^2}{(\alpha-1)(\alpha-2)(r+\alpha-1)},$$

$$(3.19)\qquad r_S(\widehat{\theta}_A) = \beta^2\left[\frac{r(r+1)C^2-2rC+1}{(\alpha-1)(\alpha-2)} + \frac{2C(rC-1)}{(\alpha-1)} + C^2\right],$$

$$(3.20)\qquad r_S(\widehat{\theta}_P) = \beta^2\left[\frac{r(r+1)K^2-2rK+1}{(\alpha-1)(\alpha-2)} + \frac{2K(rK-1)}{(\alpha-1)} + K^2\right],$$

$$(3.21)\qquad r_S(\widehat{\theta}_{\mathrm{e}}) = \beta^2\left[\frac{r(r+1)B^2-2rB+1}{(\alpha-1)(\alpha-2)} - \frac{2B(rB-1)}{(\alpha-1)} + B^2\right],$$

where $B = \frac{1}{(r+\alpha)}$.

Similarly, the Bayes risks for the estimators $\widehat{\theta}_S$, $\widehat{\theta}_A$, $\widehat{\theta}_P$ and $\widehat{\theta}_{\mathrm{e}}$ relative to the linex loss function are denoted by $r_A(\widehat{\theta}_S)$, $r_A(\widehat{\theta}_A)$, $r_A(\widehat{\theta}_P)$ and $r_A(\widehat{\theta}_{\mathrm{e}})$, respectively, and are given by:

$$(3.22)\qquad r_A(\widehat{\theta}_S) = b\left[\mathrm{e}^{-a}\left(1-\frac{a}{r+\alpha-1}\right)^{-(r+\alpha)} - \left(1-\frac{a}{r+\alpha-1}\right)\right],$$

$$(3.23)\qquad r_A(\widehat{\theta}_A) = b\left[a-(r+\alpha+1)\left(1-\mathrm{e}^{-a/(r+\alpha+1)}\right)\right],$$

$$(3.24)\qquad r_A(\widehat{\theta}_P) = b\left[\mathrm{e}^{-a}(1-aK)^{-(r+\alpha)} - aK(r+\alpha)+a-1\right],$$

$$(3.25)\qquad r_A(\widehat{\theta}_{\mathrm{e}}) = b\left[\mathrm{e}^{-a}\left(1-\frac{a}{r+\alpha}\right)^{-(r+\alpha)} + \frac{a(r+1)}{(r+\alpha)} + a - 1\right].$$

Also, the Bayes risks for the estimators $\widehat{\theta}_S$, $\widehat{\theta}_A$, $\widehat{\theta}_P$ and $\widehat{\theta}_{\mathrm{e}}$ relative to the entropy loss function are denoted by $r_A(\widehat{\theta}_S)$, $r_A(\widehat{\theta}_A)$, $r_A(\widehat{\theta}_P)$ and $r_A(\widehat{\theta}_{\mathrm{e}})$, respectively, and are given by:

$$(3.26)\qquad r_{\mathrm{e}}(\widehat{\theta}_S) = b\left[\frac{1}{(r+\alpha-1)} - \mathrm{E}\left\{\mathrm{E}_{\mathrm{e}}\left(\log_{\mathrm{e}}\frac{\widehat{\theta}_S}{\theta}\right)\right\}\right],$$

$$(3.27)\qquad r_{\mathrm{e}}(\widehat{\theta}_A) = b\left[\left(r+\frac{\beta^2}{\alpha-1}\right)\left(\frac{1-\mathrm{e}^{-a/(r+\alpha+1)}}{a}\right) - \mathrm{E}\left\{\mathrm{E}_\theta(\log_{\mathrm{e}}\frac{\widehat{\theta}_A}{\theta})\right\} - 1\right],$$

$$(3.28)\qquad r_{\mathrm{e}}(\widehat{\theta}_P) = b\left[K(r+\alpha) - \mathrm{E}\left\{\mathrm{E}_{\mathrm{e}}\left(\log_{\mathrm{e}}\frac{\widehat{\theta}_P}{\theta}\right)\right\} - 1\right],$$

$$(3.29)\qquad r_{\mathrm{e}}(\widehat{\theta}_{\mathrm{e}}) = -b\left[\mathrm{E}\left\{\mathrm{E}_\theta\left(\log_{\mathrm{e}}\frac{\widehat{\theta}_{\mathrm{e}}}{\theta}\right)\right\}\right].$$

Under $g_2(\theta)$, the risk functions and corresponding Bayes risks relative to the precautionary loss function cannot be obtained in closed form.

4. The Bayes estimator under $g_3(\theta)$

Under $g_3(\theta)$, using (2.1), the posterior distribution is given by

$$(4.1)\qquad f(\theta/\underline{x}) = \frac{T_r^{r-1}\theta^{-r}\mathrm{e}^{T_r/\theta}}{I_g(\frac{T_r}{\alpha}, r-1) - I_g(\frac{T_r}{\beta}, r-1)},$$

where

$$I_g(x,n) = \int_0^x \mathrm{e}^{-t}t^{n-1}\,dt.$$

The Bayes estimator of θ under SELF is given by

$$\widehat{\theta}_S = \left(\frac{I_g\left(\frac{T_r}{\alpha}, r-2\right) - I_g\left(\frac{T_r}{\beta}, r-2\right)}{I_g\left(\frac{T_r}{\alpha}, r-1\right) - I_g\left(\frac{T_r}{\beta}, r-1\right)} \right) T_1. \tag{4.2}$$

Using (1.6), the Bayes estimator of θ under the linex loss function is $\widehat{\theta}_A$, where $\widehat{\theta}_A$ is the solution of the following equation:

$$\mathrm{e}^{-a} \left(\frac{I_g\left(\frac{T_r}{\alpha}, r\right) - I_g\left(\frac{T_r}{\beta}, r\right)}{I_g\left(\frac{T_r^{-a}\widehat{\theta}_A}{\alpha}, r\right) - I_g\left(\frac{T_r^{-a}\widehat{\theta}_A}{\beta}, r\right)} \right) = \left(\frac{T_r}{T_r - a\widehat{\theta}_A} \right)^r. \tag{4.3}$$

Using (1.8), the Bayes estimator of θ under the precautionary loss function turns out to be

$$\widehat{\theta}_P = \left[\frac{I_g\left(\frac{T_r}{\alpha}, r-3\right) - I_g\left(\frac{T_r}{\beta}, r-3\right)}{I_g\left(\frac{T_r}{\alpha}, r-1\right) - I_g\left(\frac{T_r}{\beta}, r-1\right)} \right]^{1/2} T_r. \tag{4.4}$$

Using (1.10), the Bayes estimator of θ under the entropy loss function is obtained as

$$\widehat{\theta}_{\mathrm{e}} = \left[\frac{I_g\left(\frac{T_r}{\alpha}, r-1\right) - I_g\left(\frac{T_r}{\beta}, r-1\right)}{I_g\left(\frac{T_r}{\alpha}, r\right) - I_g\left(\frac{T_r}{\beta}, r\right)} \right] T_r. \tag{4.5}$$

In this case risk functions and the Bayes risks cannot be obtained in a closed form.

5. Conclusion

It is evident from Equations (2.3), (2.4), (2.5), (2.6), (3.2), (3.3), (3.4), (3.5), (4.2), (4.3), (4.4) and (4.5) that the Bayes estimators of the shape parameter of the generalized Pareto distribution, under the squared error, linex, precautionary and entropy loss function using quasi, inverted gamma and uniform priors, are given by different expressions. The Bayes estimators are seen to depend upon the parameters of prior distributions.

In Figure 1 (a–d) we have plotted the ratio of the risk functions to θ^2, i.e.

$$\frac{R_S(\widehat{\theta}_S)}{\theta^2} = B_1, \ \frac{R_S(\widehat{\theta}_A)}{\theta^2} = B_2, \ \frac{R_S(\widehat{\theta}_P)}{\theta^2} = B_3, \ \text{and} \ \frac{R_S(\widehat{\theta}_{\mathrm{e}})}{\theta^2} = B_4$$

for the Bayes estimators $\widehat{\theta}_S, \widehat{\theta}_A$, $\widehat{\theta}_P$ and $\widehat{\theta}_{\mathrm{e}}$, respectively, under the squared error loss function, as given in Equations (2.7) to (2.10), for $a = 1$, $r = 5\,(5)\,20$ and $d = 0.5\,(0.5)\,5.0$.

Figure 1 (a). Ratio of the risk functions to θ^2 for r = 5

Legend: B_1 ◊ B_2 □ B_3 △ B_4 ×

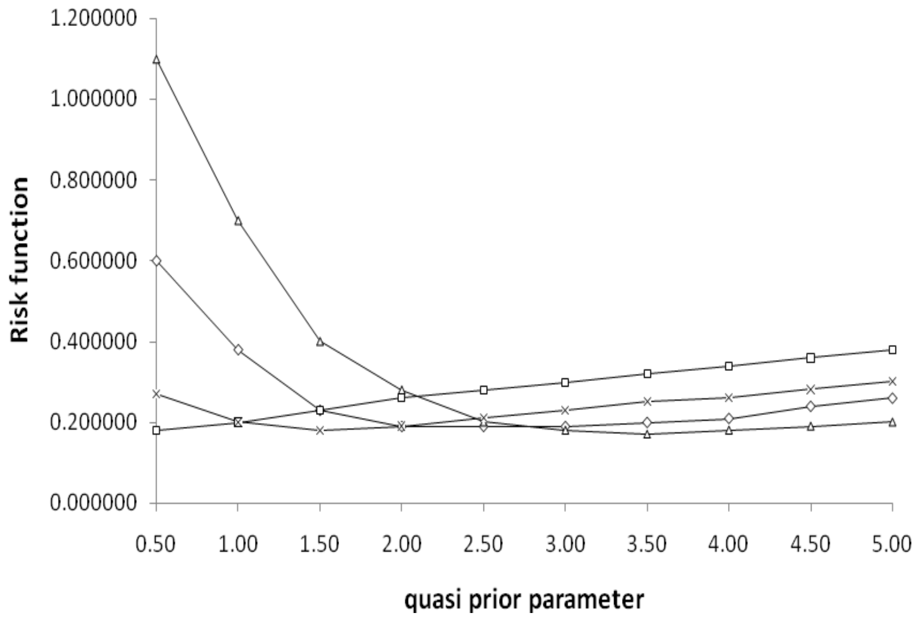


Figure 1 (b). Ratio of the risk functions to θ^2 for r = 10

Legend: B_1 ◊ B_2 □ B_3 △ B_4 ×

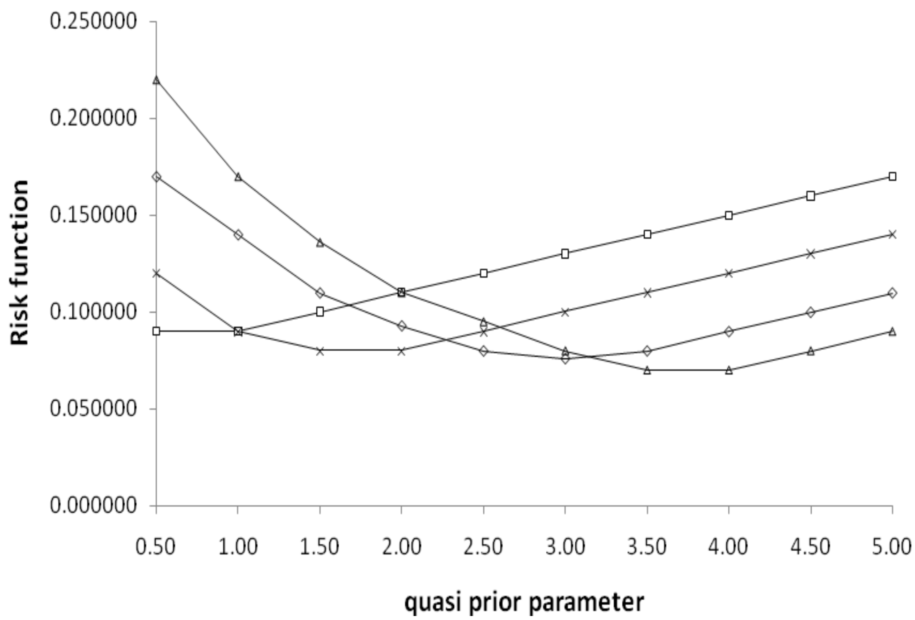


Figure 1 (c). Ratio of the risk functions to θ^2 for r = 15

Legend: B_1 ◊ B_2 □ B_3 △ B_4 ×

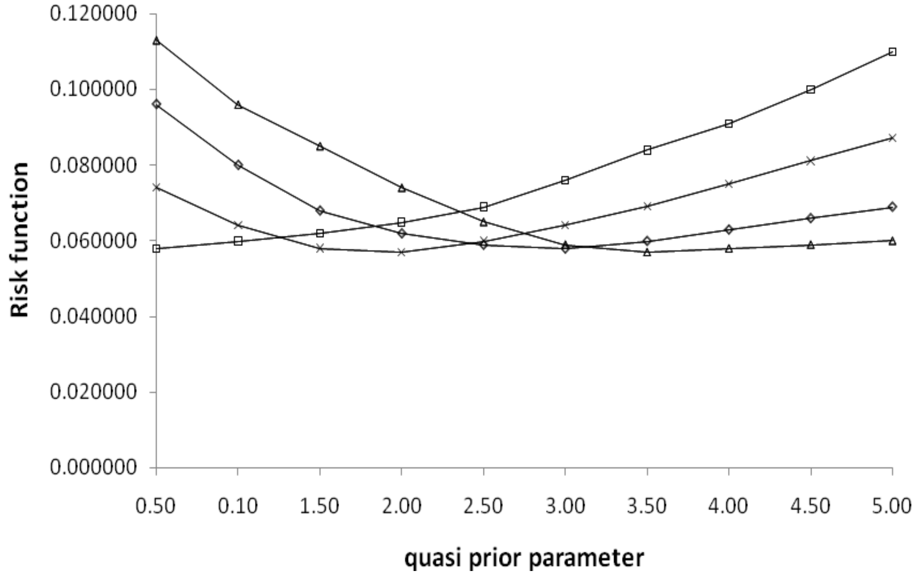
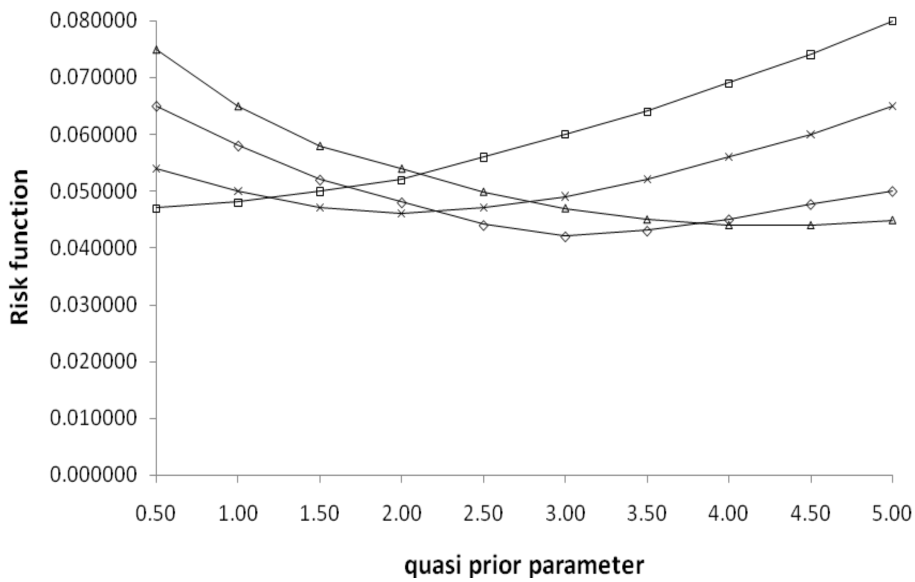


Figure 1 (d). Ratio of the risk functions to θ^2 for r = 20

Legend: B_1 ◊ B_2 □ B_3 △ B_4 ×



In Figure 2 (a–d) we have plotted the ratio of the risk functions to b i.e.

$$\frac{R_A(\widehat{\theta}_S)}{b} = C_1,\ \frac{R_A(\widehat{\theta}_A)}{b} = C_2,\ \frac{R_A(\widehat{\theta}_P)}{b} = C_3 \text{ and } \frac{R_A(\widehat{\theta}_{\mathrm{e}})}{b} = C_4$$

for the Bayes estimators $\widehat{\theta}_S$, $\widehat{\theta}_A$, $\widehat{\theta}_P$ and $\widehat{\theta}_e$, respectively, under the linex loss function $L(\Delta)$, as given in Equations (2.11) to (2.14), for $a = 1$, $r = 5\,(5)\,20$ and $d = 0.5\,(0.5)\,5.0$.

Figure 2 (a). Ratio of the risk functions to b for r = 5

Legend: C_1 ◊ C_2 □ C_3 △ C_4 ×

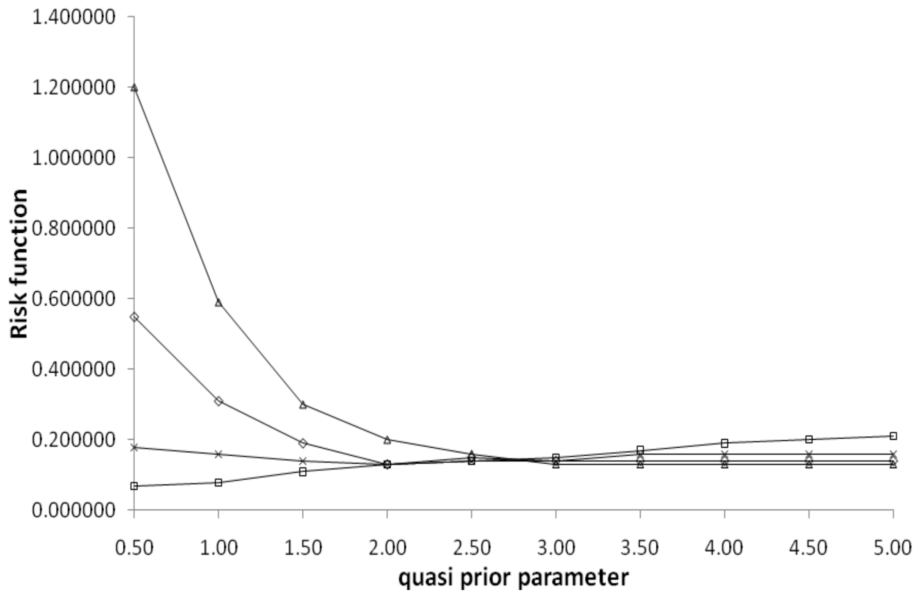


Figure 2 (b). Ratio of the risk functions to b for r = 10

Legend: C_1 ◊ C_2 □ C_3 △ C_4 ×

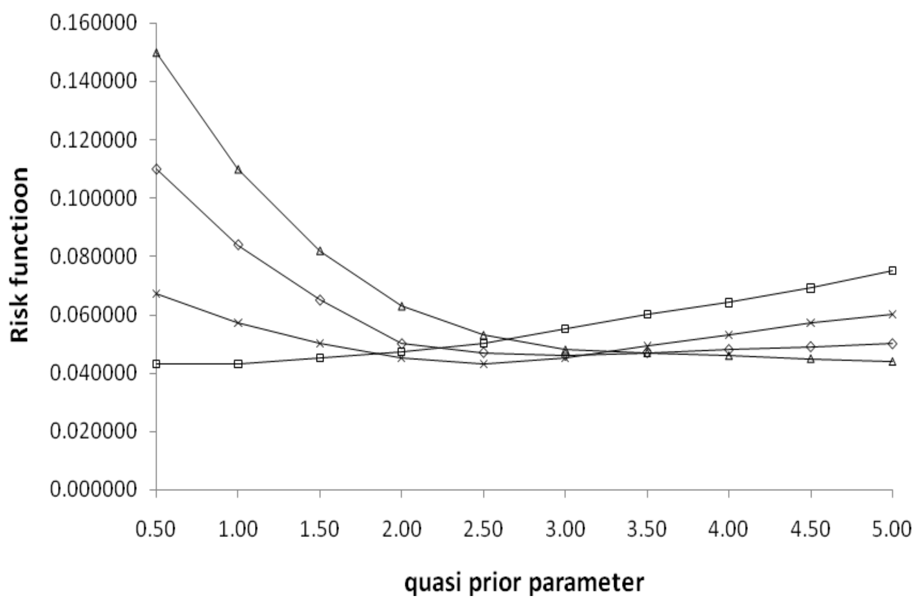


Figure 2 (c). Ratio of the risk functions to b for r = 15
Legend: C_1 ◊ C_2 □ C_3 △ C_4 ×

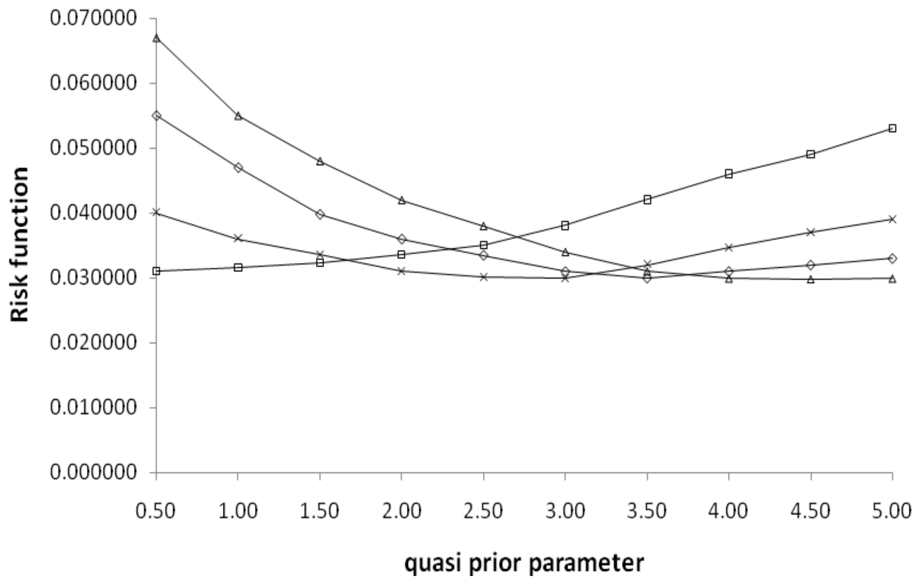
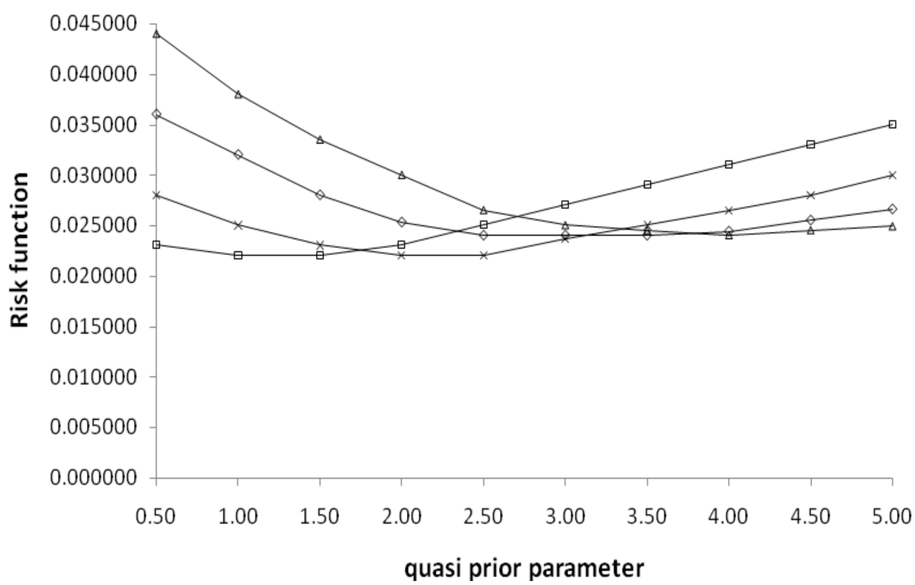


Figure 2 (d). Ratio of the risk functions to b for r = 20
Legend: C_1 ◊ C_2 □ C_3 △ C_4 ×



In Figure 3 (a–d) we have plotted the ratio of the risk functions to θ i.e.

$$\frac{R_P(\widehat{\theta}_S)}{\theta} = D_1,\ \frac{R_P(\widehat{\theta}_A)}{\theta} = D_2, \frac{R_P(\widehat{\theta}_P)}{\theta} = D_3,\ \text{and}\ \frac{R_P(\widehat{\theta}_e)}{\theta} = D_4$$

for the Bayes estimators $\widehat{\theta}_S$, $\widehat{\theta}_A$, $\widehat{\theta}_P$ and $\widehat{\theta}_e$, respectively, under the precautionary loss function, as given in Equations (2.15) to (2.18), for $a = 1$, $r = 5\,(5)\,20$ and $d = 0.5\,(0.5)\,5.0$.

Figure 3 (a). Ratio of the risk functions to θ for r = 5

Legend: D_1 ◊ D_2 □ D_3 △ D_4 ×

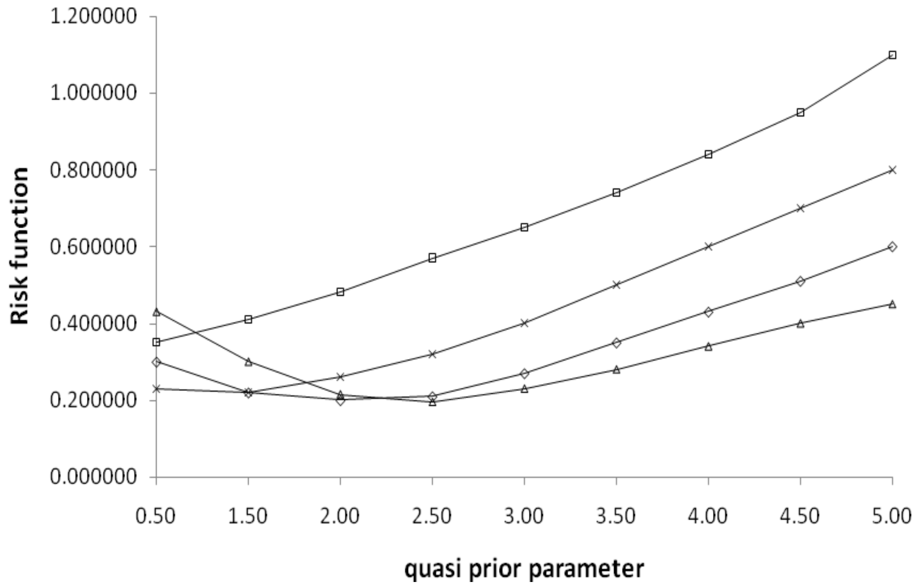


Figure 3 (b). Ratio of the risk functions to θ for r = 10

Legend: D_1 ◊ D_2 □ D_3 △ D_4 ×

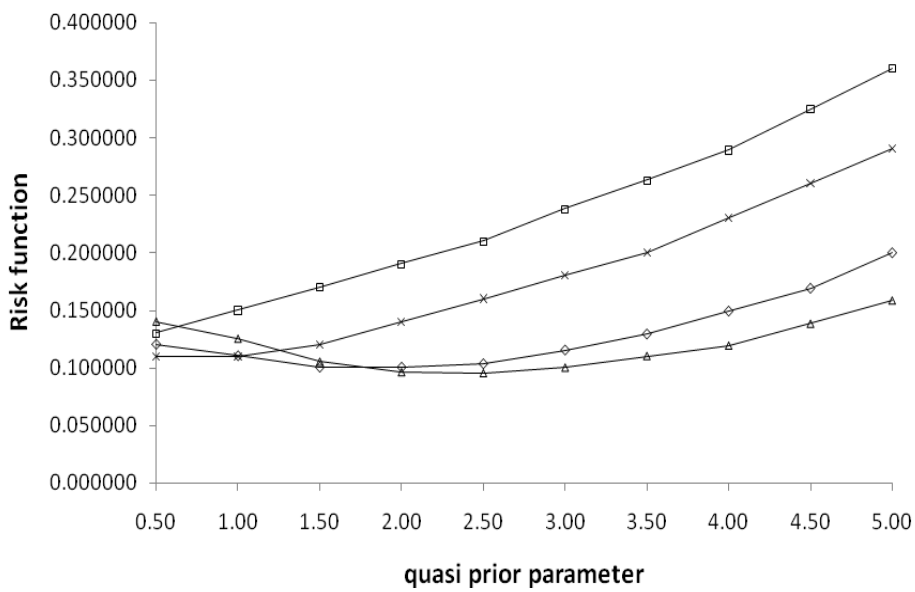


Figure 3 (c). Ratio of the risk functions to θ for r = 15

Legend: D_1 ◊ D_2 □ D_3 △ D_4 ×

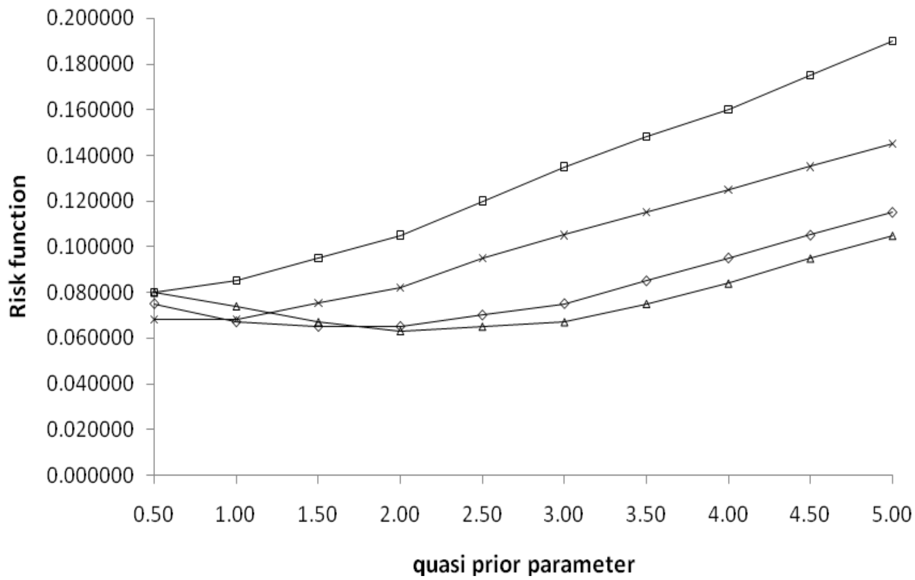
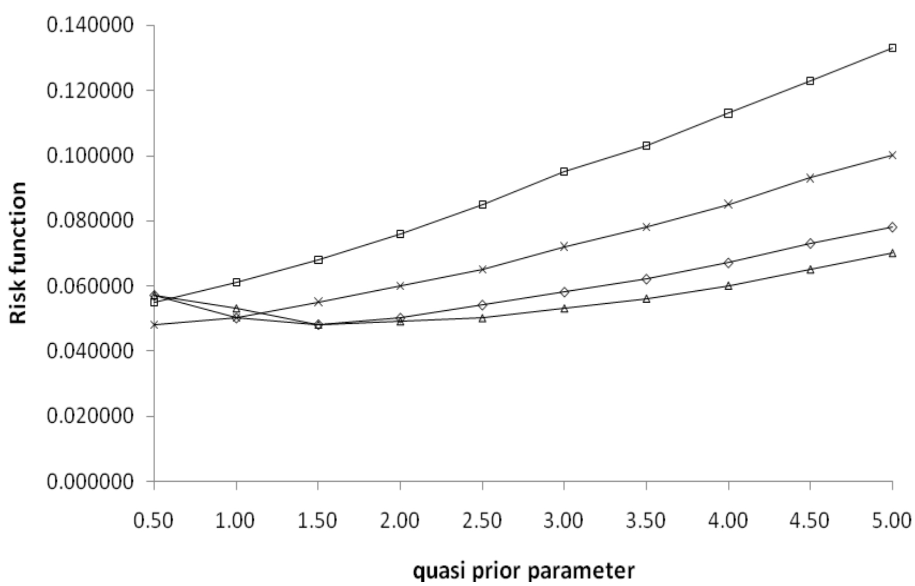


Figure 3 (d). Ratio of the risk functions to θ for r = 20

Legend: D_1 ◊ D_2 □ D_3 △ D_4 ×



From Figures 1, 2 and 3 it is clear that no one of the estimators uniformly dominates any other. We therefore recommend that the estimators be chosen according to the value of

d when quasi-density is used as the prior distribution, and this choice in turn depends on the situation at hand.

The risk functions under the inverted gamma prior are dependent on the population parameter θ, which is not separable. Therefore, a comparison could only be made by using numerical techniques.

References

[1] Basu, A. and Ebrahimi, N. *Bayesian approach to life testing and reliability estimation using asymmetric loss function*, J. Statist. Plan. Inf. **29**, 21–31, 1991.

[2] Berger, J. O. Statistical Decision Theory. Foundation, Concepts and Method (Springer-Verlag, New York, 1985).

[3] Calabria, R. and Pulcini, G. *An engineering approach to Bayes estimation for the Weibull distribution*, Micro-electron. Reliab. **34** (5), 789–802, 1994.

[4] Canfield, R. V. *A Bayesian approach to reliability estimation using a loss function*, IEEE Trans. Rel. **R-19**, 13–16, 1970.

[5] Castillo, E. and Hadi, A. S *Fitting the generalized Pareto distribution to data*, J. Amer. Statist. Assoc. **92** (440), 1609–1620, 1977.

[6] Dey, D. K., Ghosh, M. and Srinivasan, C. *Simultaneous estimation of parameters under entropy loss*, J. Statist. Plan. and Infer., 347–363, 1987.

[7] Dey, D. K. and Liu, Pei-San L. *On comparison of estimators in a generalized life model*, Micro-electron. Reliab. **32** (1), 207–221, 1992.

[8] Norstrom, J. G. *The use of precautionary loss functions in risk analysis*, IEEE Trans. Reliab. **45** (3), 400–403, 1996.

[9] Srivastava, R. S., Kumar, V. and Rao, A. K. *Bayesian estimation of shape parameter and reliability function of generalized Pareto distribution using linex loss function with censoring*, J. Nat. Acad. Math. **14**, 83–95, 2000.

[10] Varian, H. R. *A Bayesian approach to real state assessment*, in *Studies in Bayesian Econometrics and Statistics in honour of L.J. Savage*, Eds. Feinberg, S. E. and Zellner, A. (North Holland, Amsterdam, 1975), 195–208.

[11] Zellner, A. *Bayes estimation and prediction using asymmetric loss functions*, Jour. Amer. Statist. Assoc. **81**, 446–451, 1986.