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FOURIER METHOD FOR A QUASILINEAR PARABOLIC EQUATION WITH PERIODIC BOUNDARY CONDITION

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Abstract

A multidimensional mixed problem with Neuman type periodic boundary condition is studied for the quasilinear parabolic equation $\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(t, x, u)$. The existence, uniqueness and also continuity of the weak generalized solution is proved.

Keywords: Quasilinear parabolic equation, Mixed problem, Fourier method, Periodic boundary condition, Generalized solutions.

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1. Introduction

In this study we consider the following mixed problem

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(t, x, u), \quad (t, x) \in D := \{0 < t < T,\ 0 < x < \pi\} \tag{1}$$

$$u(t, 0) = u(t, \pi), \quad t \in [0, T] \tag{2}$$

$$u_x(t, 0) = u_x(t, \pi), \quad t \in [0, T] \tag{3}$$

$$u(0, x) = \varphi(x), \quad x \in [0, \pi] \tag{4}$$

for a quasilinear parabolic equation with nonlinear source term $f = f(t, x, u)$. Here $a^2 = \frac{k}{c\rho}$, where k denotes the heat conduction coefficient, ρ denotes density and c specific heat.

The functions $\varphi(x)$ and $f(t, x, u)$ are given functions on $[0, \pi]$ and $\overline{D} \times (-\infty, \infty)$, respectively.

Denote by $u = u(t, x)$ a solution of problem (1)-(4).

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In this study we consider the initial-boundary value problem (1)-(4) with periodic Dirichlet and Neumann conditions (2)-(3), respectively.

We will use the weak solution approach from [5] for the considered problem (1)-(4).

We assume the following definitions as in [2, 9].

1.1. Definition. The function $v(t,x) \in C^2(\overline{D})$ is called a *test function* if it satisfies the following conditions:

$$v(T,x)=0,\ v(t,0)=v(t,\pi),\ v_x(t,0)=v_x(t,\pi),\quad \forall\, t\in[0,T] \text{ and } \forall\, x\in[0,\pi].$$

1.2. Definition. A function $u(t,x) \in C(\overline{D})$ satisfying the integral identity

$$\int_0^T\int_0^\pi \left[\left(\frac{\partial v}{\partial t}+a^2\frac{\partial^2 v}{\partial x^2}\right)u+f(t,x,u)v\right]dx\,dt+\int_0^\pi \varphi(x)v(0,x)\,dx=0 \tag{5}$$

for an arbitrary test function $v=v(t,x)$, is called a *generalized* (*weak*) solution of the problem (1)-(4)

We will use the Fourier series representation of the weak solution to transform the initial-boundary value problem to an infinite set of nonlinear integral equations. For this aim we introduce an appropriate norm.

1.3. Definition. We denote by B the set of continuous functions $\bar{u}$ on $[0,T]$ whose Fourier coefficients

$$\left\{\frac{u_0(t)}{2}, u_{c1}(t), u_{s1}(t),\dots,u_{cn}(t),u_{sn}(t),\dots\right\}$$

satisfy the condition

$$\max_{0\le t\le T}\frac{|u_0(t)|}{2}+\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|u_{ck}(t)|+\max_{0\le t\le T}|u_{sk}(t)|\right)<\infty.$$

These functions are denoted by $\bar{u}=\left\{\frac{u_0(t)}{2}, u_{c1}(t), u_{s1}(t),\dots,u_{cn}(t),u_{sn}(t),\dots\right\}$ for short.

The norm on B is given by

$$\|\bar{u}(t)\|=\max_{0\le t\le T}\frac{|u_0(t)|}{2}+\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|u_{ck}(t)|+\max_{0\le t\le T}|u_{sk}(t)|\right).$$

It can be shown that B is a Banach space [10].

2. Reducing the problem to a countable system of integral equations

Let us look for a generalized solution of (1)-(4) in the form

$$u(t,x)=\frac{u_0(t)}{2}+\sum_{k=1}^{\infty}\left(u_{ck}(t)\cos 2kx+u_{sk}(t)\sin 2kx\right), \tag{6}$$

where $\varphi(x)$ in (4) has to be of the type

$$\varphi(x)=\frac{\varphi_0}{2}+\sum_{k=1}^{\infty}(\varphi_{ck}\cos 2kx+\varphi_{sk}\sin 2kx),\quad x\in[0,\pi].$$

First taking the derivative of (6) with respect to t once and with respect to x twice, and substituting this in equation (1), we obtain

$$\frac{u_0'(t)}{2}+\sum_{k=1}^{\infty}[u_{ck}'(t)\cos 2kx+u_{sk}'(t)\sin 2kx]$$
$$+(2a)^2\sum_{k=1}^{\infty}k^2[u_{ck}(t)\cos 2kx+u_{sk}(t)\sin 2kx]$$
$$=f\Big[t,x,\frac{u_0(t)}{2}+\sum_{k=1}^{\infty}\left(u_{ck}(t)\cos 2kx+u_{sk}(t)\sin 2kx\right)\Big].$$

Now integrating the last equation over the closed interval $[0,\pi]$ we obtain the following

$$\frac{1}{2}\int_0^{\pi}u_0'(t)d\xi=\int_0^{\pi}f(\tau,\xi,\frac{u_0(\tau)}{2}+\sum_{k=1}^{\infty}\left(u_{ck}(\tau)\cos 2k\xi+u_{sk}(\tau)\sin 2k\xi\right)\,d\xi.$$

Integrating the above equation once over the closed interval $[0,T]$ we have

$$u_0(t)=\varphi_0+\frac{2}{\pi}\int_0^t\int_0^{\pi}f(\tau,\xi,\frac{u_0(\tau)}{2}+\sum_{k=1}^{\infty}\left(u_{ck}(\tau)\cos 2k\xi+u_{sk}(\tau)\sin 2k\xi\right)\,d\xi\,d\tau.$$

In a similar way, we can obtain $u_{ck}(t)$ and $u_{sk}(t)$. Hence, we get the following infinite system of integral equations for the unknown functions $u_0(t),\ u_{ck}(t),\ u_{sk}(t),\ (k=\overline{1,\infty})$.

$$u_0(t)=\varphi_0+\frac{2}{\pi}\int_0^t\int_0^{\pi}f(\tau,\xi,(\frac{u_0(\tau)}{2}+\sum_{k=1}^{\infty}\left(u_{ck}(\tau)\cos 2k\xi+u_{sk}(\tau)\sin 2k\xi\right))\,d\xi\,d\tau$$

$$u_{ck}(t)=\varphi_{ck}e^{-(2ak)^2t}+\frac{2}{\pi}\int_0^t\int_0^{\pi}e^{-(2ak)^2(t-\tau)}$$

(7) $$\times f(\tau,\xi,(\frac{u_0(\tau)}{2}+\sum_{k=1}^{\infty}\left(u_{ck}(\tau)\cos 2k\xi+u_{sk}(\tau)\sin 2k\xi\right)\cos 2k\xi d\xi d\tau$$

$$u_{sk}(t)=\varphi_{sk}e^{-(2ak)^2t}+\frac{2}{\pi}\int_0^t\int_0^{\pi}e^{-(2ak)^2(t-\tau)}$$
$$\times f(\tau,\xi,(\frac{u_0(\tau)}{2}+\sum_{k=1}^{\infty}\left(u_{ck}(\tau)\cos 2k\xi+u_{sk}(\tau)\sin 2k\xi\right))\sin 2k\xi\,d\xi\,d\tau.$$

We denote the solution of the nonlinear system (7) by

$$\bar{u}(t)=\big\{\frac{u_0(t)}{2},\ u_{c1}(t),\ u_{s1}(t),\ldots,u_{cn}(t),\ u_{sn}(t),\ldots\big\}.$$

2.1. Theorem.

a) *Let the function* $f(t,x,u)$ *be continous with respect to all arguments in* $\overline{D}\times(-\infty,\infty)$ *and satisfy the following condition:*

(8) $$|f(t,x,u)-f(t,x,\tilde{u})|\le b(t,x)\,|u-\tilde{u}|\,,$$

where $b(t,x)\in L_2(D),\ b(t,x)\ge 0,$

b) $f(t,x,0)\in L_2(D),$

c) $\varphi(x)\in C([0,\pi])$ *is of the type* $\varphi(x)=\frac{\varphi_0}{2}+\sum\limits_{k=1}^{\infty}(\varphi_{ck}\cos 2kx+\varphi_{sk}\sin 2kx)$*, with* $\sum\limits_{k=1}^{\infty}(|\varphi_{ck}|+|\varphi_{sk}|)<+\infty.$

Then the system (7) *has a unique solution in* B.

Proof. Let us define an iteration $N = 0, 1, \ldots$ of the system (7) by the equalities:

$$
\begin{aligned}
u_0^{(N+1)}(t) &= u_0^{(0)}(t) + \tfrac{2}{\pi}\int_0^t\int_0^\pi f\Big(\tau,\xi,\Big(\tfrac{u_0^{(N)}(\tau)}{2} \\
&\qquad + \sum_{k=1}^{\infty}\big(u_{ck}^{(N)}(\tau)\cos 2k\xi + u_{sk}^{(N)}(\tau)\sin 2k\xi\big)\Big)\Big)\,d\xi\,d\tau, \\
u_{ck}^{(N+1)}(t) &= u_{ck}^{(0)}(t) + \tfrac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)} f\Big(\tau,\xi,\Big(\tfrac{u_0^{(N)}(\tau)}{2} \\
&\qquad + \sum_{k=1}^{\infty}\big(u_{ck}^{(N)}(\tau)\cos 2k\xi + u_{sk}^{(N)}(\tau)\sin 2k\xi\big)\Big)\Big)\cos 2k\xi\,d\xi\,d\tau, \\
u_{sk}^{(N+1)}(t) &= u_{sk}^{(0)}(t) + \tfrac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)} f\Big(\tau,\xi,\Big(\tfrac{u_0^{(N)}(\tau)}{2} \\
&\qquad + \sum_{k=1}^{\infty}\big(u_{ck}^{(N)}(\tau)\cos 2k\xi + u_{sk}^{(N)}(\tau)\sin 2k\xi\big)\Big)\Big)\sin 2k\xi\,d\xi\,d\tau.
\end{aligned}
\tag{9}
$$

For simplicity, letting

$$Au^{(N)}(\tau,\xi) = \frac{u_0^{(N)}(\tau)}{2} + \sum_{k=1}^{\infty}\Big(u_{ck}^{(N)}(\tau)\cos 2k\xi + u_{sk}^{(N)}(\tau)\sin 2k\xi\Big)$$

we obtain

$$
\begin{aligned}
u_0^{(N+1)}(t) &= u_0^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi f(\tau,\xi,Au^{(N)}(\tau,\xi))\,d\xi\,d\tau \\
u_{ck}^{(N+1)}(t) &= u_{ck}^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)} f\big(\tau,\xi,Au^{(N)}(\tau,\xi)\big)\cos 2k\xi\,d\xi\,d\tau \\
u_{sk}^{(N+1)}(t) &= u_{sk}^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)} f\big(\tau,\xi,Au^{(N)}(\tau,\xi)\big)\sin 2k\xi\,d\xi\,d\tau,
\end{aligned}
\tag{10}
$$

where

$$u_0^{(0)}(t) = \varphi_0,\ u_{ck}^{(0)}(t) = \varphi_{ck}e^{-(2ak)^2 t},\ \text{and}\ u_{sk}^{(0)}(t) = \varphi_{sk}e^{-(2ak)^2 t}.$$

From the condition of the theorem we have $\bar{u}^{(0)}(t) \in B$. We prove that the other approximations in the sequence satisfy this condition.

Let $N = 0$ in equality (9). Then:

$$u_0^{(1)}(t) = u_0^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi f(\tau,\xi,Au^{(0)}(\tau,\xi))\,d\xi\,d\tau.$$

Adding and subtracting $\frac{2}{\pi}\int\limits_0^t\int\limits_0^\pi f(\tau,\xi,0)\,d\xi\,d\tau$ to both sides of the last equation, we obtain

$$u_0^{(1)}(t)=u_0^{(0)}(t)+\frac{2}{\pi}\int\limits_0^t\int\limits_0^\pi\left[f(\tau,\xi,Au^{(0)}(\tau,\xi))-f(\tau,\xi,0)\right]d\xi\,d\tau + \frac{2}{\pi}\int\limits_0^t\int\limits_0^\pi f(\tau,\xi,0)\,d\xi\,d\tau.$$

Applying Cauchy's inequality to the last equation, we have

$$\left|u_0^{(1)}(t)\right|\le|\varphi_0|+\left(\int\limits_0^t d\tau\right)^{\frac{1}{2}}\left(\int\limits_0^t\left\{\frac{2}{\pi}\int\limits_0^\pi\left[f(\tau,\xi,Au^{(0)}(\tau,\xi))-f(\tau,\xi,0)\right]d\xi\right\}^2d\tau\right)^{\frac{1}{2}} + \left(\int\limits_0^t d\tau\right)^{\frac{1}{2}}\left(\int\limits_0^t\left\{\frac{2}{\pi}\int\limits_0^\pi f(\tau,\xi,0)\,d\xi\right\}^2d\tau\right)^{\frac{1}{2}}.$$

Applying Lipschitzs' condition to the last equation, we have

$$\left|u_0^{(1)}(t)\right|\le|\varphi_0|+\sqrt{t}\left(\int\limits_0^t\left\{\frac{2}{\pi}\int\limits_0^\pi b(\tau,\xi)\left|Au^{(0)}(\tau,\xi)\right|d\xi\right\}^2d\tau\right)^{\frac{1}{2}} + \sqrt{t}\left(\int\limits_0^t\left\{\frac{2}{\pi}\int\limits_0^\pi f(\tau,\xi,0)\,d\xi\right\}^2d\tau\right)^{\frac{1}{2}}.$$

Let $\left|Au^{(0)}(\tau,\xi)\right|\le\left|u^{(0)}(\tau)\right|$. Then taking the maximum of both side of the last inequality yields the following:

$$\max_{0\le t\le T}\left|u_0^{(1)}(t)\right|\le|\varphi_0|+2\sqrt{\frac{T}{\pi}}\left\|b(t,x)\right\|_{L_2(D)}\left\|\bar{u}^{(0)}(t)\right\|+2\sqrt{\frac{T}{\pi}}\left\|f(t,x,0)\right\|_{L_2(D)}.$$

Now,

$$u_{ck}(t)=\varphi_{ck}e^{-(2ak)^2t}+\frac{2}{\pi}\int\limits_0^t\int\limits_0^\pi e^{-(2ak)^2(t-\tau)} \times f\Big(\tau,\xi,\Big(\frac{u_0(\tau)}{2}+\sum_{k=1}^\infty\big(u_{ck}(\tau)\cos 2k\xi+u_{sk}(\tau)\sin 2k\xi\big)\Big)\Big)\cos 2k\xi\,d\xi\,d\tau.$$

Adding and subtracting $\frac{2}{\pi}\int\limits_0^t\int\limits_0^\pi e^{-(2ak)^2(t-\tau)}f(\tau,\xi,0)\cos 2k\xi\,d\xi\,d\tau$ to both sides of the last equation, we obtain

$$u_{ck}^{(1)}(t)=\varphi_{ck}e^{-(2ak)^2t}+\frac{2}{\pi}\int\limits_0^t\int\limits_0^\pi e^{-(2ak)^2(t-\tau)}\big[f(\tau,\xi,Au^{(0)}(\tau,\xi)) - f(\tau,\xi,0)\big]\cos 2k\xi\,d\xi\,d\tau+\frac{2}{\pi}\int\limits_0^t\int\limits_0^\pi e^{-(2ak)^2(t-\tau)}f(\tau,\xi,0)\cos 2k\xi\,d\xi\,d\tau.$$

Applying Cauchy's inequality to the last equation, we have

$$\left|u_{ck}^{(1)}(t)\right| \le |\varphi_{ck}| + \left(\int\limits_0^t e^{-(2ak)^2(t-\tau)}\,d\tau\right)^{\frac{1}{2}}$$
$$\times\left(\int\limits_0^t \left\{\tfrac{2}{\pi}\int\limits_0^\pi \left[f(\tau,\xi,Au^{(0)}(\tau,\xi)) - f(\tau,\xi,0)\right]\cos 2k\xi\,d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$
$$+\left(\int\limits_0^t e^{-(2ak)^2(t-\tau)}\,d\tau\right)^{\frac{1}{2}}\left(\int\limits_0^t \left\{\tfrac{2}{\pi}\int\limits_0^\pi f(\tau,\xi,0)\cos 2k\xi\,d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}.$$

Taking the sum of both sides with respect to k and using Hölder's inequality yields the following:

$$\sum_{k=1}^\infty \left|u_{ck}^{(1)}(t)\right| \le \sum_{k=1}^\infty |\varphi_{ck}| + \tfrac{1}{2}\sqrt{2}a\left(\sum_{k=1}^\infty \frac{1}{k^2}\right)^{\frac{1}{2}}\left(\tfrac{2}{\pi}\int\limits_0^t \sum_{k=1}^\infty\left\{\int\limits_0^\pi \left[f(\tau,\xi,Au^{(0)}(\tau,\xi))\right.\right.\right.$$
$$\left.\left.\left. - f(\tau,\xi,0)\right]\cos 2k\xi\,d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$
$$+\tfrac{1}{2\sqrt{2}a}\left(\sum_{k=1}^\infty \frac{1}{k^2}\right)^{\frac{1}{2}}\left(\tfrac{2}{\pi}\int\limits_0^t \sum_{k=1}^\infty\left\{\int\limits_0^\pi f(\tau,\xi,0)\cos 2k\xi\,d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}.$$

Applying Bessel's inequality to the last inequality, we obtain

$$\sum_{k=1}^\infty \left|u_{ck}^{(1)}(t)\right| \le \sum_{k=1}^\infty |\varphi_{ck}| + \tfrac{\pi}{4\sqrt{3}a}\left(\int\limits_0^t \frac{2}{\pi}\sum_{k=1}^\infty\left\{\int\limits_0^\pi \left[f(\tau,\xi,Au^{(0)}(\tau,\xi))\right.\right.\right.$$
$$\left.\left.\left. - f(\tau,\xi,0)\right]\cos 2k\xi\,d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$
$$+\frac{\pi}{4\sqrt{3}a}\left(\int\limits_0^t \tfrac{2}{\pi}\sum_{k=1}^\infty\left\{\int\limits_0^\pi f(\tau,\xi,0)\cos 2k\xi\,d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}.$$

Applying Lipschitz's condition to the last equation, and taking the maximum of both side of the resulting inequality yields the following:

$$\sum_{k=1}^\infty \max_{0\le t\le T}\left|u_{ck}^{(1)}(t)\right| \le \sum_{k=1}^\infty |\varphi_{ck}| + \tfrac{\sqrt{2\pi}}{4\sqrt{3}a}\left\|b(t,x)\right\|_{L_2(D)}\left\|\bar{u}^{(0)}(t)\right\| + \tfrac{\sqrt{2\pi}}{4\sqrt{3}a}\left\|f(t,x,0)\right\|_{L_2(D)}.$$

In a similar way, we can obtain

$$\sum_{k=1}^\infty \max_{0\le t\le T}\left|u_{sk}^{(1)}(t)\right| \le \sum_{k=1}^\infty |\varphi_{sk}| + \tfrac{\sqrt{2\pi}}{4\sqrt{3}a}\left\|b(t,x)\right\|_{L_2(D)}\left\|\bar{u}^{(0)}(t)\right\| + \tfrac{\sqrt{2\pi}}{4\sqrt{3}a}\left\|f(t,x,0)\right\|_{L_2(D)}.$$

Finally we have the following:

$$\left\|\bar{u}^{(1)}(t)\right\|_B = \max_{0\le t\le T}\tfrac{|u_0^{(1)}(t)|}{2} + \sum_{k=1}^\infty\left(\max_{0\le t\le T}\left|u_{ck}^{(1)}(t)\right| + \max_{0\le t\le T}\left|u_{sk}^{(1)}(t)\right|\right)$$
$$\le \tfrac{|\varphi_0|}{2} + \sum_{k=1}^\infty\left(|\varphi_{ck}| + |\varphi_{sk}|\right) + \left(\sqrt{\tfrac{T}{\pi}} + \tfrac{\sqrt{2\pi}}{2\sqrt{3}a}\right)\left(\left\|b(t,x)\right\|_{L_2(D)}\left\|\bar{u}^{(0)}(t)\right\|_B\right.$$
$$\left. + \left\|f(t,x,0)\right\|_{L_2(D)}\right).$$

Hence $\bar{u}^{(1)}(t) \in B$.

In the same way, for a general value of N we have

$$\left\|\bar{u}^{(N)}(t)\right\|_B = \max_{0\le t\le T} \frac{|u_0^{(N)}(t)|}{2} + \sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{ck}^{(N)}(t)\right| + \max_{0\le t\le T}\left|u_{sk}^{(N)}(t)\right|\right)$$
$$\le \frac{|\varphi_0|}{2} + \sum_{k=1}^{\infty}\left(\left|\varphi_{ck}\right| + \left|\varphi_{sk}\right|\right) + \left(\sqrt{\tfrac{T}{\pi}} + \tfrac{\sqrt{2\pi}}{2\sqrt{3}a}\right)\left(\left\|b(t,x)\right\|_{L_2(D)}\left\|\bar{u}^{(N-1)}(t)\right\|_B + \left\|f(t,x,0)\right\|_{L_2(D)}\right).$$

Making the induction hypothesis that $\bar{u}^{(N-1)} \in B$ we deduce that $\bar{u}^{(N)} \in B$, so by the principle of mathematical induction we obtain

$$\bar{u}^{(N)}(t) = \left\{\frac{u_0^{(N)}(t)}{2}, u_{c1}^{(N)}(t), u_{s1}^{(N)}(t), \dots, u_{cn}^{(N)}(t), u_{sn}^{(N)}(t), \dots\right\} \in B$$

Now we prove that the iterations $\bar{u}^{(N+1)}(t)$ converge in B, as $N \to \infty$.

Applying Cauchy inequality, Hölder Inequality, Lipshitz's condition and Bessel's inequality, respectively, to the right side of (9) we obtain after some calculations:

$$\left|\bar{u}^{(1)}(t) - \bar{u}^{(0)}(t)\right| = \frac{|u_0^{(1)}(t) - u_0^{(0)}(t)|}{2} + \sum_{k=1}^{\infty}\left(\left|u_{ck}^{(1)}(t) - u_{ck}^{(0)}(t)\right| + \left|u_{sk}^{(1)}(t) - u_{sk}^{(0)}(t)\right|\right)$$
$$\le \left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)\left[\left(\int_0^T\int_0^\pi b^2(\tau,\xi)\,d\xi\,d\tau\right)^{\frac{1}{2}}\left|\bar{u}^{(0)}(t)\right| + \left(\int_0^T\int_0^\pi f^2(\tau,\xi,0)d\xi d\tau\right)^{\frac{1}{2}}\right]$$
$$= A_T,$$
$$\left|\bar{u}^{(2)}(t) - \bar{u}^{(1)}(t)\right| = \frac{|u_0^{(2)}(t) - u_0^{(1)}(t)|}{2} + \sum_{k=1}^{\infty}\left(|u_{ck}^{(2)}(t) - u_{ck}^{(1)}(t)| + |u_{sk}^{(2)}(t) - u_{sk}^{(1)}(t)|\right)$$
$$\le \left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)A_T\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\,d\xi\,d\tau\right)^{\frac{1}{2}}.$$

Proceeding in the same way, for general N we obtain:

$$\left|\bar{u}^{(N+1)}(t) - \bar{u}^{(N)}(t)\right| = \frac{|u_0^{(N+1)}(t) - u_0^{(N)}(t)|}{2} + \sum_{k=1}^{\infty}\left(\left|u_{ck}^{(N+1)}(t) - u_{ck}^{(N)}(t)\right| + \left|u_{sk}^{(N+1)}(t) - u_{sk}^{(N)}(t)\right|\right)$$
$$\le \frac{1}{\sqrt{N!}}\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^N A_T\left\|b(t,x)\right\|_{L_2(D)}^{(N)},$$

or,

$$\left\|\bar{u}^{(N+1)}(t) - \bar{u}^{(N)}(t)\right\|_B \le \frac{1}{\sqrt{N!}}\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^N A_T\left(\int_0^T\int_0^\pi b^2(\tau,\xi)\,d\xi\,d\tau\right)^{\frac{N}{2}}. \tag{11}$$

By the comparison test we deduce from (11) that the series $\sum_{N=0}^{\infty}\left[\bar{u}^{(N+1)}(t)-\bar{u}^{(N)}(t)\right]$ is uniformly convergent to an element of B. However, the general term of the sequence $\{\bar{u}^{(N+1)}(t)\}$ may be written as

$$\bar{u}^{(N+1)}(t)=\bar{u}^{(0)}+\sum_{n=0}^{N}\left[\bar{u}^{(n+1)}(t)-\bar{u}^{(n)}(t)\right],$$

so the sequence $\{\bar{u}^{(N+1)}(t)\}$ is uniformly convergent to an element of B because the sum on the right is the Nth partial sum of the aforementioned uniformly convergent series.

Let $\lim\limits_{N\to\infty}\bar{u}^{(N+1)}(t)=\bar{u}(t)$. Noting that

$$\begin{aligned}
&\frac{1}{\pi}\left|\int_0^t\int_0^\pi\left\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}\,d\xi\,d\tau\right|\\
&\quad+\left|\sum_{k=1}^{\infty}\int_0^t\frac{2}{\pi}\int_0^\pi e^{-(2ak)^2(t-\tau)}\left\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}\cos 2k\xi\,d\xi\,d\tau\right|\\
&\quad+\left|\sum_{k=1}^{\infty}\int_0^t\frac{2}{\pi}\int_0^\pi e^{-(2ak)^2(t-\tau)}\left\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}\sin 2k\xi\,d\xi\,d\tau\right|\\
&\le\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\,\|b(t,x)\|_{L_2(D)}\left\|\bar{u}(\tau)-\bar{u}^{(N)}(\tau)\right\|_B,
\end{aligned}$$

it follows that if we prove $\lim\limits_{N\to\infty}\left\|\bar{u}(\tau)-\bar{u}^{(N)}(\tau)\right\|_B=0$ then we may deduce that $\bar{u}(t)$ satisfies (7).

With this aim we estimate the difference $\left\|\bar{u}(t)-\bar{u}^{(N+1)}(t)\right\|_B$. After some transformations we obtain:

$$\begin{aligned}
&\left|\bar{u}(t)-\bar{u}^{(N+1)}(t)\right|\\
&=\frac{|u_0(t)-u_0^{(N+1)}(t)|}{2}+\sum_{k=1}^{\infty}\left(\left|u_{ck}(t)-u_{ck}^{(N+1)}(t)\right|+\left|u_{sk}(t)-u_{sk}^{(N+1)}(t)\right|\right)\\
&\le\frac{1}{\pi}\left|\int_0^t\int_0^\pi\left\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}\,d\xi\,d\tau\right|\\
&\quad+\left|\sum_{k=1}^{\infty}\int_0^t\frac{2}{\pi}\int_0^\pi e^{-(2ak)^2(t-\tau)}\left\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}\cos 2k\xi\,d\xi\,d\tau\right|\\
&\quad+\left|\sum_{k=1}^{\infty}\int_0^t\frac{2}{\pi}\int_0^\pi e^{-(2ak)^2(t-\tau)}\left\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}\sin 2k\xi\,d\xi\,d\tau\right|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\pi}\left|\int_0^t\int_0^\pi \left\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N+1)}(\tau,\xi)]\right\}\,d\xi\,d\tau\right| \\
&\quad+\left|\sum_{k=1}^{\infty}\int_0^t\frac{2}{\pi}\int_0^\pi e^{-(2ak)^2(t-\tau)}\left\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N+1)}(\tau,\xi)]\right\}\cos 2k\xi\,d\xi\,d\tau\right| \\
&\quad+\left|\sum_{k=1}^{\infty}\int_0^t\frac{2}{\pi}\int_0^\pi e^{-(2ak)^2(t-\tau)}\left\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}\sin 2k\xi\,d\xi\,d\tau\right| \\
&\quad+\frac{1}{\pi}\left|\int_0^t\int_0^\pi \left\{f[\tau,\xi,Au^{(N+1)}(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}\,d\xi\,d\tau\right| \\
&\quad+\left|\sum_{k=1}^{\infty}\int_0^t\frac{2}{\pi}\int_0^\pi e^{-(2ak)^2(t-\tau)}\left\{f[\tau,\xi,Au^{(N+1)}(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}\cos 2k\xi\,d\xi\,d\tau\right| \\
&\quad+\left|\sum_{k=1}^{\infty}\int_0^t\frac{2}{\pi}\int_0^\pi e^{-(2ak)^2(t-\tau)}\left\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}\sin 2k\xi\,d\xi\,d\tau\right| \\
&\le \frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\left\{\int_0^t\int_0^\pi b^2(\tau,\xi)\left|\bar{u}(\tau)-\bar{u}^{(N+1)}(\tau)\right|^2\,d\xi\,d\tau\right\}^{\frac{1}{2}} \\
&\quad+\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\left\{\int_0^t\int_0^\pi b^2(\tau,\xi)\,d\xi\,d\tau\right\}^{\frac{1}{2}}\left\|\bar{u}^{(N+1)}(t)-\bar{u}^{(N)}(t)\right\|_B .
\end{aligned}
$$

Applying Gronwall's inequality to the last inequality and using the inequality (11) we have:

$$
\begin{aligned}
\left\|\overline{u}(t)-\overline{u}^{(N+1)}(t)\right\|_B &\le \sqrt{\frac{2}{N!}}A_T\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^{(N+1)}\left\|b(t,x)\right\|_{L_2(D)}^{(N+1)} \\
&\qquad\times\exp\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^2\left\|b(t,x)\right\|_{L_2(D)}^2 .
\end{aligned}
\tag{12}
$$

This completes the proof that $\bar{u}(t)$ satisfies (7).

For the uniqueness, we assume that the problem (1)-(4) has two solutions. Applying Cauchy's inequality, Hölder's Inequality, Lipshitzs' condition and Bessel's inequality to $|\bar{u}(t)-\bar{v}(t)|$ on the right side, after some calculations we obtain

$$
|\bar{u}(t)-\bar{v}(t)|^2 \le \left(\sqrt{\frac{t}{\pi}}+\frac{\sqrt{2\pi}}{2\sqrt{3}a}\right)^2\int_0^t\int_0^\pi b^2(\tau,\xi)\,|\bar{u}(\tau)-\bar{v}(\tau)|^2\,d\xi\,d\tau.
$$

Applying Gronwall's inequality to the last inequality we have $\bar{u}(t)=\bar{v}(t)$. The theorem is thus proved. □

3. Solution of Problem (1)–(4)

Using the solution of the system (7) we form the series

$$\frac{u_0(t)}{2}+\sum_{k=1}^{\infty}\left(u_{ck}(t)\cos 2kx+u_{sk}(t)\sin 2kx\right).$$

It is evident that this series convergence uniformly on D. Therefore the sum

$$u(\tau,\xi)=\frac{u_0(\tau)}{2}+\sum_{k=1}^{\infty}\left(u_{ck}(\tau)\cos 2k\xi+u_{sk}(\tau)\sin 2k\xi\right)$$

is continuous on D. Let

$$u_l(\tau,\xi)=\frac{u_0(\tau)}{2}+\sum_{k=1}^{l}\left(u_{ck}(\tau)\cos 2k\xi+u_{sk}(\tau)\sin 2k\xi\right). \tag{13}$$

From the conditions of Theorem 2.1 and from $\lim_{l\to\infty}u_l(\tau,\xi)=u(\tau,\xi)$ it follows that

$$\lim_{l\to\infty}f(\tau,\xi,u_l(\tau,\xi))=f(\tau,\xi,u(\tau,\xi)).$$

We denote by J_l the result of substituting for $u_l(\tau,\xi)$ and $\varphi_l(x)=\frac{\varphi_0}{2}+\sum_{k=1}^{l}(\varphi_{ck}\cos 2kx+\varphi_{sk}\sin 2kx)$ on the left hand of (5). Hence,

$$\begin{aligned} J_l=\int_0^T\int_0^\pi\left[\left(\frac{\partial v}{\partial t}+a^2\frac{\partial^2 v}{\partial x^2}\right)u_{(l)}(t,x)+f(t,x,u_{(l)}(t,x))v(t,x)\right]dx\,dt \\ +\int_0^\pi\varphi_{(l)}(x)v(0,x)\,dx. \end{aligned} \tag{14}$$

Applying the integration by part formula to the right hand side of the last equation and using the conditions of Theorem 2.1, we can show that

$$\lim_{l\to\infty}J_l=0.$$

This shows that the function $u(t,x)$ is a generalized (weak) solution of the problem (1)-(4).

The following existence and uniqueness result for generalized solutions to Problem (1)-(4) is thus achieved.

3.1. Theorem. *Under the assumptions of Theorem 2.1, Problem* (1)-(4) *possesses a unique generalized solution* $u=u(t,x)\in C(\overline{D})$ *of type* $u(t,x)=\frac{u_0(t)}{2}+\sum_{k=1}^{\infty}(u_{ck}(t)\cos 2kx+u_{sk}(t)\sin 2kx)$.

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