RANDOM FIXED POINTS AND INVARIANT RANDOM APPROXIMATION IN NON-CONVEX DOMAINS

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Abstract

Random fixed point results in the framework of a compact and weakly compact domain of a q-normed spaces which is not necessary star-shaped have been obtained in the present work. Invariant random approximation results have also been determined as an application. In this way, a random version of invariant approximation results due to R. N. Mukherjee and T. Som (*A note on an application of a fixed point theorem in approximation theory*, Indian J. Pure Appl. Math. **16** (3), 243–244, 1985) and S. P. Singh (*An application of a fixed point theorem to approximation theory*, J. Approx. Theory **25**, 89–90, 1979) have been given.

Keywords: Contractive jointly continuous family, Random best approximation, Random fixed point, Random operator.

2000 AMS Classification: 41 A 50, 41 A 65, 47 H 10, 60 H 25.

1. Introduction

Probabilistic functional analysis is an important mathematical discipline because of its applications to probabilistic models in applied problems. Random operator theory is needed for the study of various classes of random equations. The theory of random fixed point theorems was initiated by the Prague probabilistic school in the 1950s. The interest in this subject was enhanced after the publication of the survey paper by Bharucha Reid [5]. Random fixed point theory has received much attention in recent years (see, e.g. [2, 18, 16, 17, 21]).

Interesting and valuable results obtained by applying various random fixed point theorems have appeared in the literature of approximation theory. In this direction, some

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of the authors are Beg and Shahzad [1, 3, 4], Khan *et al.* [8], Lin [12], Tan and Yuan [21], and Papageorgion [16, 17]. In the subject of best approximation, one often wishes to know whether some useful property of the function being approximated is inherited by the approximating function.

In fact, Meinardus [13] was the first to state this as a general principle, and employ a fixed point theorem to established the existence of an invariant approximation. Later on, a number of results were developed in this direction under different conditions (see, e.g. [9, 13, 20]).

The aim of this paper is to establish the existence of random fixed points as random best approximations for compact and weakly compact domains of a q-normed spaces that are not necessary starshaped. To achieve this goal, the contractive jointly continuous family property given by Dotson [6] has been used. By doing so, a random version of certain invariant approximation theorems obtained by Mukherjee and Som [14] and Singh [20] have been obtained.

2. Preliminaries

The following definitions have been used to prove our results in the material presented here.

Let $\mathfrak{X}$ be a linear space. A q-norm on $\mathfrak{X}$ is a real-valued function $\|.\|_q$ on $\mathfrak{X}$ with $0 < q \leq 1$, satisfying the following conditions:

    (a) $\|x\|_q \geq 0$ and $\|x\|_q = 0$ iff $x = 0$,
    (b) $\|\lambda x\|_q = \mid \lambda \mid^q \|x\|_q$,
    (c) $\|x + y\|_q \leq \|x\|_q + \|y\|_q$,

for all $x, y \in \mathfrak{X}$ and all scalars λ. The pair $(\mathfrak{X}, \|.\|_q)$ is called a q-*normed space*. It is a metric space under the translation invariant metric $d_q(x, y) = \|x - y\|_q$ for all $x, y \in \mathfrak{X}$. If $q = 1$, we obtain the concept of a normed linear space. It is well-known that the topology of every Hausdorff locally bounded topological linear space is given by some q-norm, $0 < q \leq 1$. The spaces l_q and $\mathcal{L}_q[0, 1]$, $0 < q \leq 1$, are q-normed spaces. A q-normed space is not necessarily a locally convex space.

We recall that if $\mathfrak{X}$ is a topological linear space, then its continuous dual space $\mathfrak{X}^*$ is said to *separate the points of* $\mathfrak{X}$, if for each $x \neq 0$ in $\mathfrak{X}$, there exists a $g \in \mathfrak{X}^*$ such that $gx \neq 0$. In this case the weak topology on $\mathfrak{X}$ is well-defined. We mention that, if $\mathfrak{X}$ is not locally convex, then $\mathfrak{X}^*$ need not separate the points of $\mathfrak{X}$. For example, if $\mathfrak{X} = \mathcal{L}_q[0, 1]$, $0 < q < 1$, then $\mathfrak{X}^* = \{0\}$ [19, page 36-37]. However, there are some non-locally convex spaces (such as the q-normed space l_q, $0 < q < 1$) whose dual does separate the points [10].

2.1. Definition. [18] Let $(\Omega, \mathcal{A})$ be a measurable space and $\mathfrak{X}$ a metric space. Let $2^{\mathfrak{X}}$ be the family of all nonempty subsets of $\mathfrak{X}$ and let $\mathcal{C}(\mathfrak{X})$ denote the family of all nonempty compact subsets of $\mathfrak{X}$. Now, we call a mapping $\mathcal{F} : \Omega \to 2^{\mathfrak{X}}$ *measurable* (respectively, *weakly measurable*) if, for any closed (respectively, open) subset $\mathcal{B}$ of $\mathfrak{X}$,

$$\mathcal{F}^{-1}(\mathcal{B}) = \{\omega \in \Omega : \mathcal{F}(\omega) \cap \mathcal{B} \neq \phi\} \in \mathcal{A}.$$

Note that, if $\mathcal{F}(\omega) \in \mathcal{C}(\mathfrak{X})$ for every $\omega \in \Omega$, then $\mathcal{F}$ is weakly measurable if and only if measurable.

A mapping $\xi : \Omega \to \mathfrak{X}$ is called a *measurable selector* of a measurable mapping $\mathcal{F} : \Omega \to 2^{\mathfrak{X}}$, if ξ is measurable and, for any $\omega \in \Omega$, $\xi(\omega) \in \mathcal{F}(\omega)$. A mapping $f : \Omega \times \mathfrak{X} \to \mathfrak{X}$ is called a *random operator* if $f(., x)$ is measurable for any $x \in \mathfrak{X}$. A measurable mapping $\xi : \Omega \to \mathfrak{X}$ is called a *random fixed point* of a random operator $f : \Omega \times \mathfrak{X} \to \mathfrak{X}$ if

$\xi(\omega) = f(\omega, \xi(\omega))$ for very $\omega \in \Omega$. A random operator $f : \Omega \times \mathcal{X} \to \mathcal{X}$ is *continuous* if $f(\omega, .)$ is continuous for each $\omega \in \Omega$.

2.2. Definition. Let $\mathcal{M}$ be a nonempty subset of a Banach space $\mathcal{X}$. For $x_0 \in \mathcal{X}$, define

$$d(x_0, \mathcal{M}) = \inf_{y \in \mathcal{M}} \|x_0 - y\|$$

and

$$\mathcal{P}_{\mathcal{M}}(x_0) = \{y \in \mathcal{M} : \|x_0 - y\| = d(x_0, \mathcal{M})\}.$$

Then an element $y \in \mathcal{P}_{\mathcal{M}}(x_0)$ is called a *best approximant* of x_0 in $\mathcal{M}$. The set $\mathcal{P}_{\mathcal{M}}(x_0)$ is the set of all best approximants of x_0 in $\mathcal{M}$.

Further, the notion of contractive jointly continuous family introduced by Dotson [6] is given below:

2.3. Definition. [6] Let $\mathcal{M}$ be a subset of the metric space $(\mathcal{X}, d)$ and $\Delta = \{f_\alpha\}_{\alpha \in \mathcal{M}}$ a family of functions from $[0,1]$ into $\mathcal{M}$ such that $f_\alpha(1) = \alpha$ for each $\alpha \in \mathcal{M}$. The family Δ is said to be *contractive* if there exists a function $\phi : (0,1) \to (0,1)$ such that for all $\alpha, \beta \in \mathcal{M}$ and all $t \in (0,1)$ we have

$$d(f_\alpha(t), f_\beta(t)) \le \phi(t) d(\alpha, \beta).$$

The family is said to be *jointly continuous* if $t \to t_0$ in $[0,1]$ and $\alpha \to \alpha_0$ in $\mathcal{M}$ imply that $f_\alpha(t) \to f_{\alpha_0}(t_0)$ in $\mathcal{X}$.

2.4. Definition. [6] Let $\mathcal{M}$ be a subset of the metric space $(\mathcal{X}, d)$ and Δ a family as in Definition 2.3, then Δ is said to be *jointly weakly continuous* if $t \to t_0$ in $[0,1]$ and $\alpha \to^w \alpha_0$ in $\mathcal{M}$ imply that $f_\alpha(t) \to^w f_{\alpha_0}(t_0))$ in $\mathcal{M}$.

The following lemma helps to clarify when a subset of a metric space admits a contractive and jointly continuous family of functions, and names such a subset as having the *contractive and joint continuity property*. In particular it implies that in Euclidean n-space, such a set must be connected.

2.5. Lemma. [15] *Let $(\mathcal{X}, d)$ be a metric space and $\mathcal{M}$ a nonempty subset which, as a subspace, is not connected. Suppose that $\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1$, $\mathcal{M}_0 \cap \mathcal{M}_1 = \phi$ where $\mathcal{M}_0$ and $\mathcal{M}_1$ are both open and closed, and suppose that there exist $x \in \mathcal{M}_0$ and $y \in \mathcal{M}_1$ such that $d(x, y) = d(\mathcal{M}_0, \mathcal{M}_1)$. Then $\mathcal{M}$ does not admit a jointly continuous contractive family $\Delta = \{f_\alpha\}_{\alpha \in \mathcal{M}}$; i.e. $\mathcal{M}$ does not have the property of contractiveness and joint continuity.*

A consequence of this lemma is that, in a finite-dimensional Banach space, every bounded subset (considered as a metric space) that has the property of contractiveness and joint continuity must be connected. For closed bounded sets are compact, and the conditions of the lemma are satisfied in this case.

The following result would also be used in the sequel:

2.6. Theorem. [18] *Let $(\mathcal{X}, d)$ be a Polish space and $\mathcal{T} : \Omega \times \mathcal{X} \to \mathcal{X}$ a continuous random operator. Suppose there is some $h \in (0,1)$ such that for $x, y \in \mathcal{X}$ and $\omega \in \Omega$, we have*

$$d(\mathcal{T}(\omega, x), \mathcal{T}(\omega, y)) \le h \max \big\{ d(x,y), d(x, \mathcal{T}(\omega, x)), d(y, \mathcal{T}(\omega, y)),$$
$$\tfrac{1}{2}\big[d(x, \mathcal{T}(\omega, y)) + d(y, \mathcal{T}(\omega, x))\big]\big\}.$$

Then $\mathcal{T}$ has a random fixed point.

3. Main Results

We first prove a random fixed point result for compact subsets of a q-normed space which are not necessary starshaped.

3.1. Theorem. *Let $\mathcal{X}$ be a separable q-normed space and $\mathcal{M}$ a subset of $\mathcal{X}$. Let $\mathcal{T} : \Omega \times \mathcal{M} \to \mathcal{M}$ be a continuous random operator. Suppose that $\mathcal{M}$ is nonempty, compact and admits a contractive and jointly continuous family Δ. If $\mathcal{T}$ satisfies*

$$
(1) \quad \|\mathcal{T}(\omega,x) - \mathcal{T}(\omega,y)\|_q \leq \max\big\{\|x-y\|_q, \operatorname{dist}(x, f_{\mathcal{T}(\omega,x)}(t(\omega))), \operatorname{dist}(y, f_{\mathcal{T}(\omega,y)}(t(\omega))),
$$
$$
\tfrac{1}{2}\big[\operatorname{dist}(x, f_{\mathcal{T}(\omega,y)}(t(\omega))) + \operatorname{dist}(y, f_{\mathcal{T}(\omega,x)}(t(\omega)))\big]\big\},
$$

for $x,y \in \mathcal{M}$, $\omega \in \Omega$ and $t(\omega) \in (0,1)$, then there exists a measurable map $\xi : \Omega \to \mathcal{M}$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ for each $\omega \in \Omega$.

Proof. Choose a fixed sequence of measurable mappings $k_n : \Omega \to (0,1)$ such that $k_n(\omega) \to 1$ as $n \to \infty$. For $n \geq 1$, define a sequence of random operators $\mathcal{T}_n : \Omega \times \mathcal{M} \to \mathcal{M}$ by

$$
(2) \quad \mathcal{T}_n(\omega, x) = f_{\mathcal{T}(\omega,x)}(k_n(\omega)).
$$

Then $\mathcal{T}_n$ is a well-defined map from $\mathcal{M}$ into $\mathcal{M}$, and $\mathcal{T}_n$ is continuous because of the joint continuity of $f_x(t(\omega))$ ($x \in \mathcal{M}$, $t(\omega) \in (0,1)$). It follows from (1) and (2) that

$$
\|\mathcal{T}_n(\omega,x) - \mathcal{T}_n(\omega,y)\|_q = \|f_{\mathcal{T}(\omega,x)}(k_n(\omega)) - f_{\mathcal{T}(\omega,y)}(k_n(\omega))\|_q
$$
$$
\leq [\phi(k_n(\omega))]^q \|\mathcal{T}(\omega,x) - \mathcal{T}(\omega,y)\|_q
$$
$$
\leq [\phi(k_n(\omega))]^q \max\big\{\|x-y\|_q, \operatorname{dist}(x, f_{\mathcal{T}(\omega,x)}(k_n(\omega))),
$$
$$
\operatorname{dist}(x, f_{\mathcal{T}(\omega,y)}(k_n(\omega))), \tfrac{1}{2}\big[\operatorname{dist}(y, f_{\mathcal{T}(\omega,y)}(k_n(\omega))) +
$$
$$
\operatorname{dist}(y, f_{\mathcal{T}(\omega,x)}(k_n(\omega)))\big]\big\}
$$
$$
\leq [\phi(k_n(\omega))]^q \max\big\{\|x-y\|_q, \|x - \mathcal{T}_n(\omega,x)\|_q, \|y - \mathcal{T}_n(\omega,y)\|_q,
$$
$$
\tfrac{1}{2}\big[\|x - \mathcal{T}_n(\omega,y)\|_q + \|y - \mathcal{T}_n(\omega,x)\|_q\big]\big\}
$$

i.e.,

$$
\|\mathcal{T}_n(\omega,x) - \mathcal{T}_n(\omega,y)\|_q \leq [\phi(k_n(\omega))]^q \max\big\{\|x-y\|_q, \|x - \mathcal{T}_n(\omega,x)\|_q, \|y - \mathcal{T}_n(\omega,y)\|_q,
$$
$$
\tfrac{1}{2}\big[\|x - \mathcal{T}_n(\omega,y)\|_q + \|y - \mathcal{T}_n(\omega,x)\|_q\big]\big\}
$$

for all $x,y \in \mathcal{M}$, $\omega \in \Omega$ and $\phi(k_n(\omega)) \in (0,1)$. By the continuity of $\mathcal{T}_n(.,x)$, $(x \in \mathcal{M})$, the inverse image of any open subset $\mathcal{K}$ of $\mathcal{M}$ is open in $\Omega = [0,1]$ and hence Lebsegue measurable. Thus each $\mathcal{T}_n(.,x)$ is a random operator. By Theorem 2.6, $\mathcal{T}_n$ has a random fixed point ξ_n in $\mathcal{T}_n$ such that $\xi_n(\omega) = \mathcal{T}_n(\omega, \xi_n(\omega))$ for all $n \in \mathbb{N}$.

For each n, define $\mathcal{G}_n : \Omega \to \mathcal{C}(\mathcal{M})$ by

$$
\mathcal{G}_n = \operatorname{cl}\{\xi_i(\omega) : i \geq n\}
$$

where $\mathcal{C}(\mathcal{M})$ is the set of all nonempty compact subset of $\mathcal{M}$.

Let $\mathcal{G} : \Omega \to \mathcal{C}(\mathcal{M})$ be the mapping defined by

$$
\mathcal{G}(\omega) = \cap_{n=1}^{\infty} \mathcal{G}_n(\omega).
$$

Then, by a result of Himmelberg [7, Theorem 4.1] we see that $\mathcal{G}$ is measurable. The Kuratowski and Ryll-Nardzewski Selection Theorem [11] further implies that $\mathcal{G}$ has a measurable selector $\xi : \Omega \to \mathcal{M}$. We now show that ξ is a random fixed point of $\mathcal{T}$. We first fix $\omega \in \Omega$. Since $\xi(\omega) \in \mathcal{G}(\omega)$, there exists a subsequence $\{\xi_m(\omega)\}$ of $\{\xi_n(\omega)\}$

that converges to $\xi(\omega)$; that is $\xi_m(\omega) \to \xi(\omega)$. Since $\mathcal{T}_m(\omega, \xi_m(\omega)) = \xi_m(\omega)$, we have $\mathcal{T}_m(\omega, \xi_m(\omega)) \to \xi(\omega)$.

Proceeding to the limit as $m \to \infty$, $k_m(\omega) \to 1$ and using joint continuity,

$$\mathcal{T}_m(\omega, \xi_m(\omega)) = f_{\mathcal{T}(\omega, \xi_m(\omega))}(k_m(\omega)) \to f_{\mathcal{T}(\omega, \xi(\omega))}(1) = \mathcal{T}(\omega, \xi(\omega)).$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

An immediately consequence of Theorem 3.1 is as follows:

3.2. Corollary. *Let $\mathcal{X}$ be a separable q-normed space and $\mathcal{M}$ a subset of $\mathcal{X}$. Let $\mathcal{T} : \Omega \times \mathcal{M} \to \mathcal{M}$ be a continuous random operator. Suppose $\mathcal{M}$ is a nonempty compact set that admits a contractive and jointly continuous family Δ. If $\mathcal{T}$ satisfies*

$$(3) \qquad \begin{aligned} \|\mathcal{T}(\omega, x) - \mathcal{T}(\omega, y)\|_q \leq \max \big\{ &\|x - y\|_q, \operatorname{dist}(x, f_{\mathcal{T}(\omega, x)}(t(\omega))), \operatorname{dist}(y, f_{\mathcal{T}(\omega, y)}(t(\omega))), \\ &\tfrac{1}{2}\operatorname{dist}(x, f_{\mathcal{T}(\omega, y)}(t(\omega))), \tfrac{1}{2}\operatorname{dist}(y, f_{\mathcal{T}(\omega, x)}(t(\omega))) \big\}, \end{aligned}$$

for $x, y \in \mathcal{M}$, $\omega \in \Omega$ and $t(\omega) \in (0, 1)$, then there exists a measurable map $\xi : \Omega \to \mathcal{M}$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ for each $\omega \in \Omega$.

As an application of Theorem 3.1, we have following result on invariant approximations:

3.3. Theorem. *Let $\mathcal{X}$ be a separable q-normed space and $\mathcal{T} : \Omega \times \mathcal{M} \to \mathcal{M}$ a continuous random operator. Let $\mathcal{M} \subseteq \mathcal{X}$ be such that $\mathcal{T}(\omega, .) : \partial\mathcal{M} \to \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$. Let $x_0 \in \mathcal{X}$ and $x_0 = \mathcal{T}(\omega, x_0)$. Suppose $\mathcal{D} = \mathcal{P}_\mathcal{M}(x_0)$ is nonempty, compact and admits a contractive and jointly continuous family Δ. If $\mathcal{T}$ satisfies*

$$(4) \qquad \|\mathcal{T}(\omega, x) - \mathcal{T}(\omega, y)\|_q \leq \begin{cases} \|x - x_0\|_q, & \text{if } y = x_0, \\[4pt] \begin{aligned}\max\big\{&\|x - y\|_q, \operatorname{dist}(x, f_{\mathcal{T}(\omega, x)}(t(\omega))), \\ &\operatorname{dist}(y, f_{\mathcal{T}(\omega, y)}(t(\omega))), \\ &\tfrac{1}{2}\big[\operatorname{dist}(x, f_{\mathcal{T}(\omega, y)}(t(\omega))) \\ &\quad + \operatorname{dist}(y, f_{\mathcal{T}(\omega, x)}(t(\omega)))\big]\big\}, \end{aligned} & \text{if } y \in \mathcal{D}, \end{cases}$$

for $x, y \in \mathcal{D} \cup \{x_0\}$, $\omega \in \Omega$ and $t(\omega) \in (0, 1)$, then there exists a measurable map $\xi : \Omega \to \mathcal{D}$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ for each $\omega \in \Omega$.

Proof. Let $y \in \mathcal{D}$. Also, if $y \in \partial\mathcal{M}$ then $\mathcal{T}(\omega, y) \in \mathcal{M}$, because $\mathcal{T}(\omega, \partial\mathcal{M}) \subseteq \mathcal{M}$ for each $\omega \in \Omega$. Now since $x_0 = \mathcal{T}(\omega, x_0)$,

$$\|\mathcal{T}(\omega, y) - x_0\|_q = \|\mathcal{T}(\omega, y) - \mathcal{T}(\omega, x_0)\|_q \leq \|x - x_0\|_q,$$

yielding thereby $\mathcal{T}(\omega, y) \in \mathcal{D}$; consequently $\mathcal{D}$ is $\mathcal{T}(\omega, .)$-invariant, that is, $\mathcal{T}(\omega, .) \subseteq \mathcal{D}$. Now, Theorem 3.1 guarantees that there exists a measurable map $\xi : \Omega \to \mathcal{D}$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ for each $\omega \in \Omega$. $\qquad\qquad\qquad\square$

Next, an immediate consequence of the Theorem 3.3 is as follows:

3.4. Corollary. *Let $\mathcal{X}$ be a separable q-normed space and $\mathcal{T} : \Omega \times \mathcal{M} \to \mathcal{M}$ a continuous random operator. Let $\mathcal{M} \subseteq \mathcal{X}$ be such that $\mathcal{T}(\omega, .) : \partial\mathcal{M} \to \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$. Let $x_0 \in \mathcal{X}$ and $x_0 = \mathcal{T}(\omega, x_0)$. Suppose that $\mathcal{D} = \mathcal{P}_\mathcal{M}(x_0)$ is nonempty, compact and admits a contractive and jointly continuous family Δ. If $\mathcal{T}$ satisfies*

$$(5) \qquad \|\mathcal{T}(\omega, x) - \mathcal{T}(\omega, y)\|_q \leq \begin{cases} \|x - x_0\|_q, & \text{if } y = x_0, \\[4pt] \begin{aligned}\max\big\{&\|x - y\|_q, \operatorname{dist}(x, f_{\mathcal{T}(\omega, x)}(t(\omega))), \\ &\operatorname{dist}(y, f_{\mathcal{T}(\omega, y)}(t(\omega))), \\ &\tfrac{1}{2}\operatorname{dist}(x, f_{\mathcal{T}(\omega, y)}(t(\omega))), \\ &\tfrac{1}{2}\operatorname{dist}(y, f_{\mathcal{T}(\omega, x)}(t(\omega)))\big\}, \end{aligned} & \text{if } y \in \mathcal{D}, \end{cases}$$

for $x, y \in \mathcal{D} \cup \{x_0\}$, $\omega \in \Omega$ *and* $t(\omega) \in (0, 1)$, *then there exists a measurable map* $\xi : \Omega \to \mathcal{D}$ *such that* $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ *for each* $\omega \in \Omega$.

An analogue of Theorem 3.1 for weakly compact subsets is as follows:

3.5. Theorem. *Let* $\mathcal{X}$ *be a separable q-normed space and* $\mathcal{M}$ *a subset of* $\mathcal{X}$. *Let* $\mathcal{T}$: $\Omega \times \mathcal{M} \to \mathcal{M}$ *be a weakly continuous random operator. Suppose that* $\mathcal{M}$ *is nonempty, weakly compact and admits a contractive and jointly weakly continuous family* Δ. *If* $\mathcal{T}$ *satisfies* (1) *for* $x, y \in \mathcal{M}$, $\omega \in \Omega$ *and* $t(\omega) \in (0, 1)$, *then there exists a measurable map* $\xi : \Omega \to \mathcal{M}$ *such that* $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ *for each* $\omega \in \Omega$.

Proof. For each $n \in \mathbb{N}$, define $\{k_n(\omega)\}$, $\{\mathcal{T}_n\}$ as in the proof of the Theorem 3.1. Also, we have

$$\|\mathcal{T}_n(\omega, x) - \mathcal{T}_n(\omega, y)\|_q \le [\phi(k_n(\omega))]^q \max\{\|x - y\|_q, \|x - \mathcal{T}_n(\omega, x)\|_q, \|y - \mathcal{T}_n(\omega, y)\|_q,$$
$$\tfrac{1}{2}[\|x - \mathcal{T}_n(\omega, y)\|_q + \|y - \mathcal{T}_n(\omega, x)\|_q]\}$$

for all $x, y \in \mathcal{M}$, $\omega \in \Omega$ and $\phi(k_n(\omega)) \in (0, 1)$. Since the weak topology is Hausdorff and $\mathcal{M}$ is weakly compact, it follows that $\mathcal{M}$ is strongly closed and a complete metric space. Thus, the weak continuity of $\mathcal{T}$, the jointly weakly continuous family Δ and Theorem 2.6 guarantee that there exists a random fixed point ξ_n of $\mathcal{T}_n$ such that $\xi_n(\omega) = \mathcal{T}_n(\omega, \xi_n(\omega))$ for each $\omega \in \Omega$.

For each n, define $\mathcal{G}_n : \Omega \to \mathcal{WC}(\mathcal{M})$ by $\mathcal{G}_n = \text{w–cl}\,\{\xi_i(\omega) : i \ge n\}$, where $\mathcal{WC}(\mathcal{M})$ is the set of all nonempty weakly compact subsets of $\mathcal{M}$, and w–cl denotes weak closure. Define a mapping $\mathcal{G} : \Omega \to \mathcal{WC}(\mathcal{M})$ by $\mathcal{G}(\omega) = \cap_{n=1}^{\infty} \mathcal{G}_n(\omega)$. Because $\mathcal{M}$ is weakly compact and separable, the weak topology on $\mathcal{M}$ is a metric topology. Then a result of Himmelberg [7, Theorem 4.1] implies that $\mathcal{G}$ is w-measurable. The Kuratowski and Ryll-Nardzewski Selection Theorem [11] further implies that $\mathcal{G}$ has a measurable selector $\xi : \Omega \to \mathcal{M}$. We now show that ξ is a random fixed point of $\mathcal{T}$.

First fix $\omega \in \Omega$. Since $\xi(\omega) \in \mathcal{G}(\omega)$ there exists a subsequence $\{\xi_m(\omega)\}$ of $\{\xi_n(\omega)\}$ that converges weakly to $\xi(\omega)$; that is $\xi_m(\omega) \xrightarrow{w} \xi(\omega)$. Since $\mathcal{T}_m(\omega, \xi_m(\omega)) = \xi_m(\omega)$, we have $\mathcal{T}_m(\omega, \xi_m(\omega)) \xrightarrow{w} \xi(\omega)$. The weak continuity of $\mathcal{T}(\omega, .)$ implies that $\mathcal{T}(\omega, \xi_m(\omega)) \xrightarrow{w} \mathcal{T}(\omega, \xi(\omega))$. Proceeding to the limit as $m \to \infty$, $k_m(\omega) \to 1$ and by using joint weak continuity,

$$\mathcal{T}_m(\omega, \xi_m(\omega)) = f_{\mathcal{T}(\omega, \xi_m(\omega))}(k_m(\omega)) \xrightarrow{w} f_{\mathcal{T}(\omega, \xi(\omega))}(1) = \mathcal{T}(\omega, \xi(\omega)).$$

By the Hausdorff property of the weak topology, we get the required result $\mathcal{T}(\omega, \xi(\omega)) = \xi(\omega)$. $\qquad\qquad\square$

An immediate consequence of the Theorem 3.5 is as follows:

3.6. Corollary. *Let* $\mathcal{X}$ *be a separable q-normed space and* $\mathcal{M}$ *a subset of* $\mathcal{X}$. *Let* $\mathcal{T}$: $\Omega \times \mathcal{M} \to \mathcal{M}$ *be a weakly continuous random operator. Suppose that* $\mathcal{M}$ *is nonempty, weakly compact and admits a contractive and jointly weakly continuous family* Δ. *If* $\mathcal{T}$ *satisfies* (3) *for* $x, y \in \mathcal{M}$, $\omega \in \Omega$ *and* $t(\omega) \in (0, 1)$, *then there exists a measurable map* $\xi : \Omega \to \mathcal{M}$ *such that* $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ *for each* $\omega \in \Omega$.

As an application of Theorem 3.5, we have following results on invariant approximations:

3.7. Theorem. *Let* $\mathcal{X}$ *be a separable q-normed space and* $\mathcal{T} : \Omega \times \mathcal{X} \to \mathcal{X}$ *a weakly continuous random operator. Let* $\mathcal{M} \subseteq \mathcal{X}$ *be such that* $\mathcal{T}(\omega, .) : \partial\mathcal{M} \to \mathcal{M}$, *where* $\partial\mathcal{M}$ *stands for the boundary of* $\mathcal{M}$. *Let* $x_0 \in \mathcal{X}$ *and* $x_0 = \mathcal{T}(\omega, x_0)$. *Suppose that* $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(x_0)$ *is nonempty, weakly compact and admits a contractive and jointly weakly continuous family* Δ. *Further, Suppose* $\mathcal{T}$ *satisfies the condition* (4) *for* $x, y \in \mathcal{D} \cup \{x_0\}$, $\omega \in \Omega$ *and*

$t(\omega) \in (0,1)$. *Then there exists a measurable map* $\xi : \Omega \to \mathcal{D}$ *such that* $\xi(\omega) = \mathfrak{T}(\omega, \xi(\omega))$ *for each* $\omega \in \Omega$.

Proof. This follows from the proof of Theorem 3.3. $\hfill\square$

Next, an immediate consequence of Theorem 3.7 is as follows:

3.8. Corollary. *Let* $\mathfrak{X}$ *be a separable q-normed space and* $\mathfrak{T} : \Omega \times \mathfrak{X} \to \mathfrak{X}$ *a weakly continuous random operator. Let* $\mathcal{M} \subseteq \mathfrak{X}$ *be such that* $\mathfrak{T}(\omega, .) : \partial\mathcal{M} \to \mathcal{M}$, *where* $\partial\mathcal{M}$ *stands for the boundary of* $\mathcal{M}$. *Let* $x_0 \in \mathfrak{X}$ *and* $x_0 = \mathfrak{T}(\omega, x_0)$. *Suppose that* $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(x_0)$ *is nonempty, weakly compact and admits a contractive and jointly weakly continuous family* Δ. *Further, Suppose that* $\mathfrak{T}$ *satisfies the condition (5) for* $x, y \in \mathcal{D} \cup \{x_0\}$, $\omega \in \Omega$ *and* $t(\omega) \in (0,1)$. *Then there exists a measurable map* $\xi : \Omega \to \mathcal{D}$ *such that* $\xi(\omega) = \mathfrak{T}(\omega, \xi(\omega))$ *for each* $\omega \in \Omega$.

3.9. Remark. In the light of the comment made by Dotson [6] and Khan *et al.* [9], if $\mathcal{M} \subseteq \mathfrak{X}$ is p–starshaped and $f_\alpha(t) = (1-t)p + t\alpha$, $(\alpha \in \mathcal{M}, t \in [0,1])$, then $\{f_\alpha\}_{\alpha \in \mathcal{M}}$ is a contractive jointly continuous family with $\phi(t) = t$. Thus the class of subsets of $\mathfrak{X}$ with the property of contractiveness and joint continuity contains the class of starshaped sets, which in turn contain the class of convex sets.

Suppose that $\mathcal{H} = \{f_\alpha\}_{\alpha \in \mathcal{M}}$ is a family of functions from $[0,1]$ into $\mathcal{M}$ having the property that for each sequence (λ_n) in $(0,1]$ with $\lambda_n \to 1$ as $n \to \infty$, we have

$$f_\alpha(\lambda_n) = \lambda_n \alpha. \qquad\qquad (*)$$

It is observed that $\mathcal{H} \subseteq \Delta$ and it has additional properties, that is it is contractive, jointly continuous and weakly jointly continuous [8].

3.10. Example. [8] A subspace, a convex set containing 0, a star-shaped subset with center 0 and a cone of a normed space all have a family of functions associated with them that satisfy condition $(*)$.

If we restrict to the family $\mathcal{H}$, then the operators $\mathfrak{T}_n(\omega, x) = f_{\mathfrak{T}(\omega, x)}(k_n(\omega)) = k_n(\omega)\mathfrak{T}(\omega, x)$ are random operators because of the randomness of $\mathfrak{T} : \Omega \times \mathcal{M}$, where $(\Omega, \mathcal{A})$ is a measurable space. Hence all the above theorems remain valid for the family $\mathcal{H}$ in the context of an arbitrary measurable space $(\Omega, \mathcal{A})$. The following theorem would be a reformulation of Theorem 3.1 or of Theorem 3.5 in this setting. In fact, it removes the conditions of convexity and fixed point property of $\mathcal{M}$ and the strict convexity of $\mathfrak{X}$ required in Xu [22, Theorem 1].

3.11. Theorem. *Let* $\mathfrak{X}$ *be a separable q-normed space and* $\mathcal{M}$ *a subset of* $\mathfrak{X}$. *Let* $\mathfrak{T} : \Omega \times \mathcal{M} \to \mathcal{M}$ *be a continuous random operator. Suppose that* $\mathcal{M}$ *is nonempty and has a family* $\mathcal{H}$ *satisfying condition* $(*)$. *If* $\mathfrak{T}$ *satisfies (1) or (3) for* $x, y \in \mathcal{M}$, $\omega \in \Omega$ *and* $t(\omega) \in (0,1)$, *then there exists a measurable map* $\xi : \Omega \to \mathcal{M}$ *such that* $\xi(\omega) = \mathfrak{T}(\omega, \xi(\omega))$ *for each* $\omega \in \Omega$, *if one of the following conditions is satisfied:*

    (1) $\mathcal{M}$ *is compact and* $\mathfrak{T}$ *is continuous;*
    (2) $\mathcal{M}$ *is weakly compact and* $\mathfrak{T}$ *is weakly continuous.*

Following is an application of Theorem 3.11 to invariant approximations:

3.12. Theorem. *Let* $\mathfrak{X}$ *be a separable q-normed space and* $\mathfrak{T} : \Omega \times \mathcal{M} \to \mathcal{M}$ *a random operator. Let* $\mathcal{M} \subseteq \mathfrak{X}$ *be such that* $\mathfrak{T}(\omega, .) : \partial\mathcal{M} \to \mathcal{M}$, *where* $\partial\mathcal{M}$ *stands for the boundary of* $\mathcal{M}$. *Let* $x_0 \in \mathfrak{X}$ *and* $x_0 = \mathfrak{T}(\omega, x_0)$. *Suppose that* $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(x_0)$ *is nonempty and has a family* $\mathcal{H}$ *satisfying condition* $(*)$. *If* $\mathfrak{T}$ *satisfies (4) or (5) for* $x, y \in \mathcal{D} \cup \{x_0\}$, $\omega \in \Omega$ *and* $t(\omega) \in (0,1)$, *then there exists a measurable map* $\xi : \Omega \to \mathcal{D}$ *such that* $\xi(\omega) = \mathfrak{T}(\omega, \xi(\omega))$ *for ach* $\omega \in \Omega$, *if one of the following conditions is satisfied:*

(1) $\mathcal{P}_{\mathcal{M}}(x_0)$ *is compact and* $\mathcal{T}$ *is continuous;*

(2) $\mathcal{P}_{\mathcal{M}}(x_0)$ *is weakly compact and* $\mathcal{T}$ *is weakly continuous.*

3.13. Remark. Theorem 3.3, Corollary 3.4, Theorem 3.7 and Corollary 3.8 generalize and give a random version of the result due to Mukherjee and Som [14].

3.14. Remark. In view of Remark 3.9 and the remark concerning the family $\mathcal{H}$, Theorem 3.3 – Theorem 3.12 generalize and give random versions of the result of Singh [20] without the condition that the domain be starshaped.

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