Hacettepe Journal of Mathematics and Statistics
Volume 37 (2) (2008), 129 – 133

ON THE INVARIANTS OF TIME-LIKE DUAL CURVES

Melih Turgut*

Received 10 : 09 : 2008 : Accepted 21 : 11 : 2008

Abstract

In this work, for a time-like dual curve in D_1^3, a system of differential equation is established whose solution gives the components of the position vector on the dual Frenet axis. By means of some special solutions of this system, some characterizations are presented, such as the position vector of a time-like dual curve with constant dual curvature and torsion.

Keywords: Dual numbers, Time-like dual curve, Lorentzian space.

2000 AMS Classification: 53 C 50, 53 C 40.

1. Introduction and Preliminaries

W. K. Clifford, in [4], introduced *dual numbers* as the set

$$D = \{\hat{x} = x + \xi x^* : x, x^* \in R\}.$$

The symbol ξ designates the *dual unit* which has the property $\xi^2 = 0$ for $\xi \neq 0$. Thereafter, a good deal of research work has been done on dual numbers and dual functions, as well as dual curves [1], [6] and [8]. Then the *dual angle* was introduced. This is defined as $\widehat{\theta} = \theta + \xi\theta^*$, where θ is the projected angle between two spears and θ^* the shortest distance between them. In recent years, dual numbers have been applied to the study of the motion of a line in space; indeed they even seem to be the most appropriate way for doing this, and this has triggered the use of dual numbers in kinematical problems. There exists a vast literature on the subject, for instance [9], [10], [11] and [12].

The theory of relativity opened a door to the use of degenerate submanifolds, and researchers have treated some topics of classical differential geometry extended to Lorentz manifolds [9], [12]. In the light of the existing literature, we deal with dual curves in Lorentzian space.

*Dokuz Eylül University, Buca Educational Faculty, Department of Mathematics, 35160 Buca-Izmir, Turkey. E-mail: `melih.turgut@gmail.com`

The set D of dual numbers is a commutative ring with the operations $(+)$ and $(.)$. The set

$$\mathrm{D}^3 = \mathrm{D} \times \mathrm{D} \times \mathrm{D} = \left\{\widehat{\varphi} : \widehat{\varphi} = \varphi + \xi\varphi^*, \varphi \in \mathrm{E}^3, \varphi^* \in \mathrm{E}^3\right\}$$

is a module over the ring D, see [8].

Let us set $\hat{a} = a + \xi a^* = (a_1, a_2, a_3) + \xi(a_1^*, a_2^*, a_3^*)$, and likewise $\hat{b} = b + \xi b^* = (b_1, b_2, b_3) + \xi(b_1^*, b_2^*, b_3^*)$. The *Lorentzian inner product of* $\hat{a}$ *and* $\hat{b}$ is defined by

$$\langle \hat{a}, \hat{b} \rangle = \langle a, b \rangle + \xi(\langle a, b^* \rangle + \langle a^*, b \rangle).$$

We call the dual space D^3 together with the Lorentzian inner product the *dual Lorentzian space* and denote it by D_1^3. We call the elements of D_1^3 the *dual vectors*. For $\widehat{\varphi} \neq 0$, the norm $\|\widehat{\varphi}\|$ of $\widehat{\varphi}$ is defined by

$$\|\widehat{\varphi}\| = \sqrt{|\langle \widehat{\varphi}, \widehat{\varphi} \rangle|}.$$

A dual vector $\widehat{\varepsilon} = \varepsilon + \xi\varepsilon^*$ is called

- A *dual space-like vector* if $\langle \widehat{\varepsilon}, \widehat{\varepsilon} \rangle > 0$ or $\widehat{\varepsilon} = 0$,
- A *dual time-like vector* if $\langle \widehat{\varepsilon}, \widehat{\varepsilon} \rangle < 0$, and
- A *dual null (light-like) vector* if $\langle \widehat{\varepsilon}, \widehat{\varepsilon} \rangle = 0$ for $\widehat{\varepsilon} \neq 0$.

Therefore, an arbitrary dual curve, which is a differentiable mapping onto D_1^3, can locally be dual space-like, dual time-like or dual null, if its velocity vector is respectively, dual space-like, dual time-like or dual null. Besides, for the dual vectors $\hat{a}, \hat{b} \in \mathrm{D}_1^3$, the *Lorentzian vector product of dual vectors* is defined by

$$\hat{a} \times \hat{b} = a \times b + \xi(a^* \times b + a \times b^*),$$

where $a \times b$ is the classical Lorentzian cross product according to the signature $(+, +, -)$, (cf. [9]).

Let $\widehat{\varphi} : I \subset R \to \mathrm{D}^3$ be a C^4 time-like dual curve with the arc length parameter s. Then the unit tangent vector $\widehat{\varphi}' = \hat{t}$ is defined, and the principal normal is $\hat{n} = \frac{\hat{t}'}{\widehat{\kappa}}$, where $\widehat{\kappa}$ is never a pure-dual. The function $\widehat{\kappa} = \left\|\hat{t}'\right\| = \kappa + \xi\kappa^*$ is called the *dual curvature* of the dual curve $\widehat{\varphi}$. Then the binormal of $\widehat{\varphi}$ is given by the dual vector $\hat{b} = \hat{t} \times \hat{n}$. Hence, the triple $\left\{\hat{t}, \hat{n}, \hat{b}\right\}$ is called the *Frenet trihedra* at the point $\widehat{\varphi}$, and the *Frenet formulae* may be expressed as [12]

$$\begin{bmatrix} \hat{t}' \\ \hat{n}' \\ \hat{b}' \end{bmatrix} = \begin{bmatrix} 0 & \widehat{\kappa} & 0 \\ \widehat{\kappa} & 0 & \widehat{\tau} \\ 0 & -\widehat{\tau} & 0 \end{bmatrix} \begin{bmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{bmatrix}, \tag{1.1}$$

where $\widehat{\tau} = \tau + \xi\tau^*$ is the dual torsion of the time-like dual curve $\widehat{\varphi}$. Here, we suppose that the dual torsion $\widehat{\tau}$ is never pure-dual.

2. Main Results

Let $\widehat{\varphi}$ be a time-like dual curve of D_1^3. Then, the position vector of $\widehat{\varphi}$ may be expressed as

$$\widehat{\varphi} = \widehat{\gamma}\hat{t} + \widehat{\delta}\hat{n} + \widehat{\lambda}\hat{b}, \tag{2.1}$$

where $\widehat{\gamma}, \widehat{\delta}$ and $\widehat{\lambda}$ are arbitrary dual functions of s. Differentiating both sides of (2.1) and using (1.1), we have a system of ordinary differential equations as follows:

$$\begin{aligned} \widehat{\gamma}' + \widehat{\delta}\widehat{\kappa} - 1 &= 0 \\ \widehat{\delta}' + \widehat{\gamma}\widehat{\kappa} - \widehat{\lambda}\widehat{\tau} &= 0 \\ \widehat{\lambda}' + \widehat{\delta}\widehat{\tau} &= 0. \end{aligned} \tag{2.2}$$

Using system (2.2), we have a third-order differential equation with respect to $\widehat{\gamma}$,

$$\left\{\frac{1}{\widehat{\tau}}\left[\left(\frac{1}{\widehat{\kappa}} - \frac{\widehat{\gamma}'}{\widehat{\kappa}}\right)' + \widehat{\gamma}\widehat{\kappa}\right]\right\}' + \left(1 - \widehat{\gamma}'\right)\frac{\widehat{\tau}}{\widehat{\kappa}} = 0. \tag{2.3}$$

In view of (2.2) and (2.3), we immediately arrive at the following results.

2.1. Corollary. *For the system of equation* (2.2)*,*

i) *The components $\widehat{\gamma}$, $\widehat{\delta}$ and $\widehat{\lambda}$ cannot all be non-zero constants.*
ii) *We cannot have both $\widehat{\gamma} = 0$ and $\widehat{\lambda} = 0$.*
iii) *If $\widehat{\delta} = 0$ the following relation holds between the dual curvature and dual torsion,*

$$\frac{\widehat{\kappa}}{\widehat{\tau}} = \frac{\hat{c}}{s + \hat{c}}, \tag{2.4}$$

where $\hat{c} \in \mathrm{D}$ is constant.
iv) *Equation* (2.3) *is a characterization for the time-like dual curve $\widehat{\varphi}$. Via its solution, the position vector of $\widehat{\varphi}$ can be determined.*

For regular curves of the Euclidean space E^3 or the Minkowski space E^3_1, it is well-known that curves whose position vector always lies in their rectifying plane are called "Rectifying Curves" (cf. [2], [3] [5]) and for such curves in E^3 there is a relation between κ and τ of the form $\frac{\kappa}{\tau} = \frac{c}{s+c}$ for non-zero c [2]. In an analogous way, it is safe to report that statement iii) of the above corollary is a characterization for rectifying dual time-like curves in dual Lorentzian space.

However, a general solution of (2.3) has not yet been found. Due to this, we give some special values to the components and curvatures. Now, with the following definition, we adapt W-curves (cf. [7]) to time-like dual curves in D^3_1.

2.2. Definition. Let $\widehat{\varphi}$ be a time-like dual curve in D^3_1. If the dual curvature $\widehat{\kappa}$ and dual torsion $\widehat{\tau}$ are constant, then $\widehat{\varphi}$ is called a *time-like dual W-curve.*

Let us suppose $\widehat{\varphi}$ is a time-like dual W-curve. In this case, it can be deduced from (2.2) that,

$$\widehat{\delta}'' + \widehat{\delta}(\widehat{\tau}^2 - \widehat{\kappa}^2) + \widehat{\kappa} = 0. \tag{2.5}$$

We study the following cases for $\widehat{\kappa}$ and $\widehat{\tau}$.

Case I. $\tau = \tau^*$ and $\kappa = \kappa^*$. Then, (2.5) yields

$$\widehat{\delta} = -\frac{\widehat{\kappa}}{2}s^2 + \hat{c}_1 s + \hat{c}_2. \tag{2.6}$$

Using equation (2.6), the other components are as follows:

$$\begin{aligned} \widehat{\gamma} &= \frac{\widehat{\kappa}^2}{3}s^3 - \frac{\hat{c}_1\widehat{\kappa}^2}{2}s^2 - \hat{c}_2\widehat{\kappa}s + s + \hat{c}_3 \\ \widehat{\lambda} &= \frac{\widehat{\kappa}\widehat{\tau}}{3}s^3 - \frac{\hat{c}_1\widehat{\tau}}{2}s^2 - \hat{c}_2\widehat{\tau}s + \hat{c}_4. \end{aligned} \tag{2.7}$$

Here, $\hat{c}_i \in \mathrm{D}$.

Case II. $\tau > \tau^*$ and $\kappa > \kappa^*$. In this case, the solution of (2.5) has the form

$$\widehat{\delta} = \widehat{\alpha}_1 \cos\sqrt{\widehat{\tau}^2 - \widehat{\kappa}^2}s + \widehat{\alpha}_2 \sin\sqrt{\widehat{\tau}^2 - \widehat{\kappa}^2}s - \frac{\widehat{\kappa}}{\widehat{\tau}^2 - \widehat{\kappa}^2}. \tag{2.8}$$

Similarly, we may write

$$\begin{aligned} \widehat{\gamma} &= s - \frac{\widehat{\alpha}_1\widehat{\kappa}\sin\sqrt{\widehat{\tau}^2 - \widehat{\kappa}^2}s}{\sqrt{\widehat{\tau}^2 - \widehat{\kappa}^2}} + \frac{\widehat{\alpha}_2\widehat{\kappa}\cos\sqrt{\widehat{\tau}^2 - \widehat{\kappa}^2}s}{\sqrt{\widehat{\tau}^2 - \widehat{\kappa}^2}} + \frac{\widehat{\kappa}^2}{\widehat{\tau}^2 - \widehat{\kappa}^2}s \\ \widehat{\lambda} &= -\frac{\widehat{\alpha}_1\widehat{\tau}\sin\sqrt{\widehat{\tau}^2 - \widehat{\kappa}^2}s}{\sqrt{\widehat{\tau}^2 - \widehat{\kappa}^2}} + \frac{\widehat{\alpha}_2\widehat{\tau}\cos\sqrt{\widehat{\tau}^2 - \widehat{\kappa}^2}s}{\sqrt{\widehat{\tau}^2 - \widehat{\kappa}^2}} + \frac{\widehat{\kappa}\widehat{\tau}}{\widehat{\tau}^2 - \widehat{\kappa}^2}s. \end{aligned} \tag{2.9}$$

for $\widehat{\alpha}_1,\ \widehat{\alpha}_2 \in \mathrm{D}$.

Case III. $\tau < \tau^*$ and $\kappa < \kappa^*$. Then, using hyperbolic functions, we have the components

$$\begin{aligned} \widehat{\gamma} &= s - \frac{\widehat{\beta}_1\widehat{\kappa}\sinh\sqrt{\widehat{\kappa}^2 - \widehat{\tau}^2}s}{\sqrt{\widehat{\kappa}^2 - \widehat{\tau}^2}} - \frac{\widehat{\beta}_2\widehat{\kappa}\cosh\sqrt{\widehat{\kappa}^2 - \widehat{\tau}^2}s}{\sqrt{\widehat{\kappa}^2 - \widehat{\tau}^2}} - \frac{\widehat{\kappa}^2}{\widehat{\tau}^2 - \widehat{\kappa}^2}s, \\ \widehat{\delta} &= \widehat{\beta}_1\cosh\sqrt{\widehat{\kappa}^2 - \widehat{\tau}^2}s + \widehat{\beta}_2\sinh\sqrt{\widehat{\kappa}^2 - \widehat{\tau}^2}s + \frac{\widehat{\kappa}}{\widehat{\kappa}^2 - \widehat{\tau}^2}, \\ \widehat{\lambda} &= -\frac{\widehat{\beta}_1\widehat{\tau}\sinh\sqrt{\widehat{\kappa}^2 - \widehat{\tau}^2}s}{\sqrt{\widehat{\kappa}^2 - \widehat{\tau}^2}} - \frac{\widehat{\beta}_2\widehat{\tau}\cosh\sqrt{\widehat{\kappa}^2 - \widehat{\tau}^2}s}{\sqrt{\widehat{\kappa}^2 - \widehat{\tau}^2}} - \frac{\widehat{\kappa}\widehat{\tau}}{\widehat{\kappa}^2 - \widehat{\tau}^2}s, \end{aligned} \tag{2.10}$$

respectively, for constants $\widehat{\beta}_1,\ \widehat{\beta}_2 \in \mathrm{D}$.

In view of the equations obtained above, we may give the following theorem.

2.3. Theorem. *Let $\widehat{\varphi}$ be a time-like dual $W-$curve in D_1^3 with dual curvature $\widehat{\kappa} = \kappa + \xi\kappa^*$ and dual torsion $\widehat{\tau} = \tau + \xi\tau^*$. Then:*

i) *If $\tau = \tau^*$ and $\kappa = \kappa^*$, then the position vector of $\widehat{\varphi}$ with respect to the dual Frenet frame can be obtained using the equations* (2.6), $(2.7)_1$ *and* $(2.7)_2$.
ii) *If $\tau > \tau^*$ and $\kappa > \kappa^*$ then the position vector of $\widehat{\varphi}$ with respect to the dual Frenet frame can be obtained using the equations* (2.8), $(2.9)_1$ *and* $(2.9)_2$.
iii) *If $\tau < \tau^*$ and $\kappa < \kappa^*$, then the position vector of $\widehat{\varphi}$ with respect to the dual Frenet frame can be obtained using the equations* $(2.10)_1, (2.10)_2$ *and* $(2.10)_3$.

Acknowledgment The author would like to thank Tübitak-Bideb for their financial supports during his PhD. studies.

References

[1] Baky, R. A. A. *An explicit characterization of dual spherical curves*, Commun. Fac. Sci. Univ. Ank. Series A. **51** (2), 1–9, 2002.
[2] Chen, B. Y. *When does the position vector of a space curve always lie in its rectifying plane?*, Amer. Math. Mounthly **110** (2), 147–152, 2003.
[3] Chen, B. Y. and Dillen, F. *Rectifying curves as centrodes and extremal curves*, Bull. Inst. Math. Acad. Sinica **22**, 77–90, 2005.
[4] Clifford, W. K. *Preliminary sketch of biquaternions*, Proceedings of London Math. Soc. **4**, 361–395, 1873.
[5] Ilarslan, K., Nesovic, E. and Petrovic-Torgasev, M. *Some characterizations of rectifying curves in the Minkowski 3-Space*, Novi Sad J. Math. **33**, 23–32, 2003.
[6] Köse, Ö., Nizamoğlu, Ş. and Sezer, M. *An explicit characterization of dual spherical curve*, Doğa Mat. **12** (3), 105–113, 1988.
[7] Petrovic-Torgasev, M. and Sucurovic, E. *W-curves in Minkowski space-time*, Novi Sad J. Math. **32** (2), 55–65, 2002.
[8] Sezer, M., Köse, Ö. and Nizamoğlu, Ş. *A criterion for a ruled surface to be closed*, Doğa Mat. **14** (1), 39–47, 1990.

[9] Uğurlu, H. and Çalışkan, A. *The study mapping for directed space-like and time-like lines in Minkowski 3-Space* R_1^3, Math. Comput. Appl. **1** (2), 142–148, 1996.

[10] Veldkamp, G. R. *On the use of dual numbers, vectors, and matrices in instantaneous spatial kinematics*, Mech. Mach. Theory **11**, 141–156, 1976.

[11] Yang, A. T. *Application of Quaternion Algebra and Dual Numbers to the Analysis of Spatial Mechanisms*, (Doctoral Dissertation, Columbia University, 1963).

[12] Yucesan, A., Coken, A. C. and Ayyildiz, N. *On the dual Darboux rotation axis of the time-like dual space curve*, Balkan J. Geom. Appl. **7** (2), 137–142, 2002.