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FUZZY SETS OVER THE POSET $\mathbb{I}$

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Abstract

The author studies fuzzy sets over the poset $\mathbb{I} = [0, 1]$ with the usual order. These form a canonical example of fuzzy sets over a poset discussed in (Tiryaki, İ. U. and Brown, L. M. *Plain textures and fuzzy sets via posets*, preprint). Characterizations of these so called "soft fuzzy sets" are obtained, and soft fuzzy sets are shown to have a richer mathematical theory than classical $\mathbb{I}$-fuzzy sets. In particular soft fuzzy points behave like the points of crisp set theory with respect to join, and moreover there exists a Lowen type functor from **Top** to the construct **SF-Top** that preserves both separation and compactness.

Keywords: Texture, Unit interval texture, Hutton algebra, Fuzzy subset, Soft fuzzy subset, Point, Copoint, Construct, SF-topology, Ditopology, Separation, Compactness, Generalized Lowen functor, Rotund soft fuzzy set, Lowen rotund functor, Preservation of topological properties.

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1. Introduction

In [17] the author and L. M. Brown used the characterization of plain textures in terms of posets given in [8] to present several new results in the theory of plain ditopological texture spaces. As part of this investigation they considered fuzzy sets over a poset and mentioned a canonical example of such fuzzy sets that coincides with the notion of "soft fuzzy set" introduced by the author in his PhD thesis [15] from a different view-point. This paper presents an updated account of the theory of soft fuzzy sets based on the discussion in [17] and placed within a more suitable categorical framework than that given in [15]. As mentioned in [17], fuzzy sets over a poset have properties that make them potentially useful in applications. Naturally, soft fuzzy sets share these properties and it is anticipated that they will find useful applications in various areas.

If $(N, \leq)$ is a partially ordered set (poset, for short) we denote by $\mathcal{L}_N$ the set of lower sets of N as in [17]. Hence $(N, \mathcal{L}_N)$ is a plain texture, and all plain textures can

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be given in this form [8]. If $(X, \leq)$ is also a poset and we take $X \times N$ together with the product order $(x_1, n_1) \leq_{\text{prod}} (x_2, n_2) \iff x_1 \leq x_2$ and $n_1 \leq n_2$, then $\mathcal{L}^{\text{prod}}_{X\times N}$ denotes the corresponding texturing of $X \times N$ and by [17, Proposition 4.1] the texture $(X \times N,\ \mathcal{L}^{\text{prod}}_{X\times N})$ is the product of the textures $(X, \mathcal{L}_X)$ and $(N, \mathcal{L}_N)$.

The elements of $\mathcal{L}^{\text{prod}}_{X\times N}$ may be regarded as $\mathcal{L}_N$-fuzzy subsets of X of a special kind. Indeed if we set

(1.1) $$F_N(X) = \{\mu : X \to \mathcal{L}_N \mid x \leq y \implies \mu(y) \subseteq \mu(x)\}$$

then by [17, Proposition 4.2] the mapping

(1.2) $$\mu \mapsto A_\mu,\ A_\mu = \{(x, n) \mid n \in \mu(x)\}$$

is an isomorphism from $F_N(X)$ to the texturing $\mathcal{L}^{\text{prod}}_{X\times N}$ of $X \times N$ with inverse

(1.3) $$A \mapsto \mu_A,\ \mu_A(x) = \{n \mid (x, n) \in A\}.$$

Using this isomorphism to transfer the lattice structure of $\mathcal{L}^{\text{prod}}_{X\times N}$ to $F_N(X)$ by setting

(1.4) $$\bigvee_{j\in J} \mu_j = \mu \iff A_\mu = \bigcup_{j\in J} A_{\mu_j} \text{ and } \bigwedge_{j\in J} \mu_j = \mu \iff A_\mu = \bigcap_{j\in J} A_{\mu_j}$$

for any $\mu_j \in F_N(X)$, $j \in J$, the mappings $\mu \mapsto A_\mu$ and $A \mapsto \mu_A$ become isomorphisms between the complete, completely distributive lattices $F_N(X)$ and $\mathcal{L}^{\text{prod}}_{X\times N}$.

In [17, Theorem 2.10] it is shown that every complementation on a plain texture is grounded. More specifically, every order-reversing involution $n \mapsto n'$ on N leads to a complementation σ satisfying $\sigma(P_n) = Q_{n'}$, $n \in N$, and conversely. Hence order-reversing involutions $x \mapsto x'$, $n \mapsto n'$ on $(X, \leq)$, $(N, \leq)$ give rise to complementations σ_X, σ_N, respectively. Also $(x, n) \mapsto (x, n)' = (x', n')$ is an order-reversing involution on $(X \times N, \leq_{\text{prod}})$, and by [17, Proposition 4.3] the corresponding complementation $\sigma_{X\times N}$ on $(X \times N,\ \mathcal{L}^{\text{prod}}_{X\times N})$ is the product [3] $\sigma_X \otimes \sigma_N$ of σ_X and σ_N.

For $\mu \in F_N(X)$ and A_μ defined as in (1.2) we have

(1.5) $$\sigma_{X\times N}(A_\mu) = \{(x, n) \mid x \in X,\ n \in N,\ n' \notin \mu(x')\}$$

by [17, Lemma 4.4]. We also recall the following corollary:

1.1. Corollary. [17] *For $\mu \in F_N(X)$ define $\mu'(x) = \{n \in N \mid n' \notin \mu(x')\}$, $x \in X$. Then*

(i) *$\mu' \in F_N(X)$ for all $\mu \in F_N(X)$.*
(ii) *The mapping $\mu \mapsto \mu'$ is an order-reversing involution on $F_N(X)$.*
(iii) *$A_{\mu'} = \sigma_{X\times N}(A_\mu)$ for all $\mu \in F_N(X)$.*

As a consequence of Corollary 1.1 it is clear that under the given hypotheses, the mapping $\mu \to A_\mu$ becomes an isomorphism between the Hutton algebras $F_N(X)$ and $\mathcal{L}^{\text{prod}}_{X\times N}$ with inverse $A \mapsto \mu_A$.

In this paper we shall make the following special choices. Firstly X will just be a set, which, to conform to the above analysis we take with the discrete ordering $x_1 \leq x_2 \iff x_1 = x_2$ and the trivial involution $x \mapsto x$, that is $x' = x$ for all $x \in X$, which is certainly order-reversing and leads to the standard complementation π_X, $\pi_X(Y) = X \setminus Y$, $Y \subseteq X$ by [17, Examples 2.11 (1)]. Secondly, we take $N = \mathbb{I} = [0, 1]$ with the standard ordering and the order-reversing involution $r \mapsto 1 - r$, $r \in \mathbb{I}$. Hence, by [17, Examples 2.2 (2) and Examples 2.11 (2)], the corresponding texture is the unit interval texture $(\mathbb{I}, \mathcal{I}, \iota)$, where $\mathcal{I} = \{[0, r), [0, r] \mid r \in \mathbb{I}\}$ and $\iota([0, r)) = [0, 1 - r]$, $\iota([0, r]) = (0, 1 - r)$, $r \in \mathbb{I}$. This gives us the family $F_{\mathbb{I}}$ of $\mathcal{L}_{\mathbb{I}}$-fuzzy subsets of X, and the textural representation $(X, \mathcal{P}(X), \pi_X) \otimes (\mathbb{I}, \mathcal{I}, \iota)$. It is instructive to compare this with the textural representation of the Hutton algebra $F(X)$ of classical Zadeh fuzzy sets [18] (that is $\mathbb{I}$-fuzzy subsets of X), which by [3] is known to be $(X, \mathcal{P}(X), \pi_X) \otimes (L, \mathcal{L}, \lambda)$, where $L = (0, 1]$, $\mathcal{L} = \{(0, r] \mid r \in \mathbb{I}\}$

and $\lambda((0,r]) = (0,1-r]$. The unit interval texture $(\mathbb{I}, \mathcal{I}, \iota)$ has a much richer mathematical structure than does $(L, \mathcal{L}, \lambda)$, particularly when endowed with the natural ditopology $(\tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ defined by $\tau_{\mathbb{I}} = \{[0,s) \mid s \in \mathbb{I}\} \cup \{\mathbb{I}\}$, $\kappa_{\mathbb{I}} = \{[0,s] \mid s \in \mathbb{I}\} \cup \{\emptyset\}$. The unit interval texture is also plain, whereas $(L, \mathcal{L}, \lambda)$ is not, a fact that has far reaching consequences.

This paper is devoted to a consideration of the changes to the theory of $\mathbb{I}$-fuzzy subsets of X that result from replacing $(L, \mathcal{L}, \lambda)$ by the texture $(\mathbb{I}, \mathcal{I}, \iota)$. We will show that the elements of $F_{\mathbb{I}}(X)$ may be represented as pairs (μ, M), where $\mu \in F(X)$ and $M \in \mathcal{P}(X)$. For $x \in X$ it will follow that there are two possible states, corresponding to $x \in M$ and $x \notin M$, associated with the degree of membership $\mu(x)$. In case $x \in M$ we may think of $\mu(x)$ as a *realized* or *hard* value, otherwise it will be *soft* or *unrealized*. For this reason we shall refer to the pairs (μ, M) as *soft fuzzy sets*. Although we will be concerned solely with the mathematical properties of soft fuzzy sets in this paper, and not consider applications at all, it is clear that one way of making use of this extra degree of freedom would be to regard a transition from a soft to a hard value as representing a *potential* increase in the degree of membership, that is one for which it is not possible to give a numerical value at the current stage. It is anticipated that significant applications along these or similar lines will be found to parallel the richer mathematical theory.

For terms from the theory of ditopological texture spaces not explained here, and for additional results and motivation, the reader is referred to [2–8], [11] and [16]. A useful reference to lattice theory is [9], and we will generally follow the notation of [1] for concepts from category theory. In particular Ob $\mathbf{A}$ will denote the class of objects and Mor $\mathbf{A}$ the class of morphisms for a category $\mathbf{A}$. Sometimes $\mathbf{A}(A_1, A_2)$ will be used to denote the set of $\mathbf{A}$–morphisms from A_1 to A_2.

2. Lattice of Soft Fuzzy Subsets

We begin by associating a fuzzy subset and a crisp subset of X with a given element of $F_{\mathbb{I}}(X)$. Since X has the discrete ordering we just have $F_{\mathbb{I}}(X) = \{\eta \mid \eta : X \to \mathcal{I}\}$. Hence for $x \in X$ we have $\eta(x) = P_r$ or $\eta(x) = Q_r$ for some $r \in \mathbb{I}$. In either case we may associate the number $r = \sup \eta(x)$ with x to give a function $\eta_1 \in F(X)$, and we may define $\eta_2 \in \mathcal{P}(X)$ by $x \in \eta_2 \iff \eta(x) = P_x \iff \eta_1(x) \in \eta(x)$. That is

2.1. Definition. For $\eta \in F_{\mathbb{I}}(X)$ we denote by η_1, η_2 respectively the fuzzy subset of X and the crisp subset of X given by

$$\eta_1(x) = \sup \eta(x),\ x \in X,\ \text{and}\ \eta_2 = \{x \in X \mid \eta_1(x) \in \eta(x)\}.$$

This focuses our attention on pairs consisting of an $\mathbb{I}$-fuzzy subset and a crisp subset of X. It is these pairs that will occupy our attention throughout this paper, and we make the following definition.

2.2. Definition. Let X be a set, μ an $\mathbb{I}$-fuzzy subset of X and $M \subseteq X$. Then the pair (μ, M) will be called a *soft fuzzy subset of* X. The set of all soft fuzzy subsets of X will be denoted by $SF(X)$.

We now see $\eta \mapsto (\eta_1, \eta_2)$ as setting up a mapping from $F_{\mathbb{I}}(X)$ to $SF(X)$. Conversely if $(\mu, M) \in SF(X)$ then we may set $\xi(\mu, M) = \eta$ where

$$\eta(x) = \begin{cases} P_{\mu(x)} & x \in M, \\ Q_{\mu(x)} & x \notin M. \end{cases}$$

It is clear from the definitions that

2.3. Lemma. *The mapping $\xi : SF(X) \to F_{\mathbb{I}}(X)$ defined above is a bijection with inverse ξ^{-1} given by $\xi^{-1}(\eta) = (\eta_1, \eta_2)$.* □

Composing this mapping with the bijection $\eta \mapsto A_\eta$ given in (1.2) produces the bijection $(\mu, M) \mapsto A_{\xi(\mu,M)}$ between $SF(X)$ and $\mathcal{L}^{\mathrm{prod}}_{X\times\mathbb{I}}$ considered in [15]. A simple calculation shows that

$$(2.1) \qquad A_{\xi(\mu,M)} = \{(x,s) \mid s < \mu(x) \text{ or } (s = \mu(x) \text{ and } x \in M)\}$$

2.4. Example. By the definition of product texture [2] the elements of $\mathcal{P}(X) \otimes \mathcal{I}$ are arbitrary intersections of sets of the form $(Y \times \mathbb{I}) \cup (X \times [0, s])$ and $(Y \times \mathbb{I}) \cup (X \times [0, s))$ for $Y \subseteq X$ and $s \in \mathbb{I}$. It will be interesting to find the soft fuzzy sets corresponding to these basic elements of $\mathcal{P}(X) \otimes \mathcal{I}$. For this purpose consider $\mu : X \to \mathbb{I}$ defined by

$$\mu(x) = \begin{cases} 1 & x \in Y \\ s & x \in X \setminus Y \end{cases}.$$

It is straightforward to verify that

$$A_{\xi(\mu,X)} = (Y \times \mathbb{I}) \cup (X \times [0, s]),$$
$$A_{\xi(\mu,Y)} = (Y \times \mathbb{I}) \cup (X \times [0, s)).$$

It is significant that these soft fuzzy sets differ only in the crisp set M.

Our next step is to define an order relation on $SF(X)$ which reflects the ordering of $\mathcal{P}(X) \otimes \mathcal{I}$ by inclusion, or equivalently the corresponding order on $F_{\mathbb{I}}(X)$.

2.5. Lemma. *For all (μ, M), $(\nu, N) \in SF(X)$ we have $A_{\xi(\mu,M)} \subseteq A_{\xi(\nu,N)}$ if and only if*

$$(2.2) \qquad \mu(x) < \nu(x) \text{ or } (\mu(x) = \nu(x) \text{ and } x \notin M \setminus N)\ \forall\, x \in X.$$

Proof. First suppose that (2.2) holds but that $A_{\xi(\mu,M)} \not\subseteq A_{\xi(\nu,N)}$. Take $(x, s) \in A_{\xi(\mu,M)}$ with $(x, s) \notin A_{\xi(\nu,N)}$. We have the following two cases:

(1) $s > \nu(x)$. From (2.1) we have $\mu(x) \le \nu(x)$ and so $s > \mu(x)$, which contradicts $(x, s) \in A_{\xi(\mu,M)}$.

(2) $s = \nu(x)$ and $x \notin N$. Now $\mu(x) < s$ will contradict $(x, s) \in A_{\xi(\mu,M)}$ so $\mu(x) = s$ and $x \in M$. This gives $\mu(x) = \nu(x)$ and $x \in M \setminus N$, which contradicts (2.1).

Conversely, assume that $A_{\xi(\mu,M)} \subseteq A_{\xi(\nu,N)}$, but that (2.1) does not hold. Then for some $x \in X$ we have $\mu(x) \ge \nu(x)$ and $(\mu(x) \ne \nu(x)$ or $x \in M \setminus N)$, so we may distinguish the two cases $\mu(x) > \nu(x)$ and $\mu(x) = \nu(x)$, $x \in M \setminus N$. Both give an immediate contradiction to $A_{\xi(\mu,M)} \subseteq A_{\xi(\nu,N)}$. □

This leads to the following definition of a relation $\sqsubseteq$ on $SF(X)$.

2.6. Definition. The relation $\sqsubseteq$ on $SF(X)$ is given by

$$(\mu, M) \sqsubseteq (\nu, N) \iff (\mu(x) < \nu(x)) \text{ or } (\mu(x) = \nu(x) \text{ and } x \notin M \setminus N)\ \forall\, x \in X$$

for all (μ, M), $(\nu, N) \in SF(X)$.

By Lemma 2.5, $(\mu, M) \sqsubseteq (\nu, N) \iff A_{\xi(\mu,M)} \subseteq A_{\xi(\nu,N)}$, and since inclusion is a partial order on $\mathcal{P}(X) \otimes \mathcal{I}$ we obtain an order-preserving bijection between $(SF(X), \sqsubseteq)$ and $(\mathcal{P}(X) \otimes \mathcal{I}, \subseteq)$.

It is known that $(\mathcal{P}(X) \otimes \mathcal{I}, \subseteq)$ is a complete lattice. We establish the same result for $(SF(X), \sqsubseteq)$, at the same time giving formulae for calculating arbitrary meets and joins.

2.7. Proposition. *If $(\mu_j, M_j) \in SF(X)$, $j \in J$, then the family $\{(\mu_j, M_j) \mid j \in J\}$ has a meet, that is greatest lower bound, in $(SF(X), \sqsubseteq)$, denoted by $\prod_{j\in J}(\mu_j, M_j)$ and given by*

$$\prod_{j\in J}(\mu_j, M_j) = (\mu, M)$$

where $\mu(x) = \bigwedge_{j\in J} \mu_j(x)\ \forall\, x \in X$ and $M = \{x \in X \mid \forall\, j \in J,\ x \in M_j \text{ or } \mu(x) < \mu_j(x)\}$.

Proof. Take $j \in J$. Clearly $\mu(x) \leq \mu_j(x)$ for all $x \in X$. If $\mu(x) = \mu_j(x)$ and $x \in M$ then $x \in M_j$ and so $x \notin M \setminus M_j$. Thus $(\mu, M) \sqsubseteq (\mu_j, M_j)$ for all $j \in J$.

Now take $(\nu, N) \in SF(X)$ with $(\nu, N) \sqsubseteq (\mu_j, M_j)$ for all $j \in J$. Suppose that $(\nu, N) \not\sqsubseteq (\mu, M)$. Again we may distinguish the following two cases for some $x \in X$

(1) $\nu(x) > \mu(x)$. In this case $\inf_{j\in J} \mu_j(x) < \nu(x)$ and so there exists $j \in J$ satisfying $\mu(x) \leq \mu_j(x) < \nu(x)$, which contradicts $(\nu, N) \sqsubseteq (\mu_j, M_j)$.

(2) $\nu(x) = \mu(x)$ and $x \in N \setminus M$. Since $x \notin M$ there exists $j \in J$ satisfying $x \notin M_j$ and $\mu(x) = \mu_j(x)$. This gives $\nu(x) = \mu_J(x)$ and $x \in N \setminus M_j$, which contradicts $(\nu, N) \sqsubseteq (\mu_j, M_j)$.

This establishes that (μ, M) is indeed the greatest lower bound of the elements (μ_j, M_j), $j \in J$. □

2.8. Corollary. *For all $(\mu_j, M_j) \in SF(X)$, $j \in J$, we have*

$$A_{\xi\left(\bigsqcap_{j\in J}(\mu_j, M_j)\right)} = \bigcap_{j\in J} A_{\xi(\mu_j, M_j)}.$$

Proof. Denote $\bigsqcap_{j\in J}(\mu_j, M_j)$ by (μ, M) as in Proposition 2.7. By Lemma 2.5 we clearly have $A_{\xi(\mu,M)} \subseteq \bigcap_{j\in J} A_{\xi(\mu_j,M_j)}$.

Assume the opposite inclusion is false and take $(x, s) \in \bigcap_{j\in J} A_{\xi(\mu_j,M_j)}$ with $(x, s) \notin A_{\xi(\mu,M)}$. Then we have the following two cases:

(1) $s > \mu(x)$. This leads to a contradiction since $(x, s) \in \bigcap_{j\in J} A_{\xi(\mu_j,M_j)}$ implies $s \leq \mu_j(x)$ for all $j \in J$.

(2) $s = \mu(x)$ and $x \notin M$. In this case there exists $j \in J$ with $x \notin M_j$ and $\mu(x) = \mu_j(x)$. Hence $(x, s) \notin A_{\xi(\mu_j,M_j)}$, which again is a contradiction.

This establishes the stated equality. □

2.9. Proposition. *If $(\mu_j, M_j) \in SF(X)$, $j \in J$, then the family $\{(\mu_j, M_j) \mid j \in J\}$ has a join, that is least upper bound, in $(SF(X), \sqsubseteq)$, denoted by $\bigsqcup_{j\in J}(\mu_j, M_j)$ and given by*

$$\bigsqcup_{j\in J}(\mu_j, M_j) = (\mu, M)$$

where $\mu(x) = \bigvee_{j\in J} \mu_j(x)\ \forall x \in X$ and $M = \{x \in X \mid \exists j \in J$ with $x \in M_j$ and $\mu(x) = \mu_j(x)\}$.

Proof. Dual to the proof of Proposition 2.10, and is omitted. □

2.10. Corollary. *For all $(\mu_j, M_j) \in SF(X)$, $j \in J$, we have*

$$A_{\xi\left(\bigsqcup_{j\in J}(\mu_j, M_j)\right)} = \bigcup_{j\in J} A_{\xi(\mu_j, M_j)}.$$

Proof. Dual to the proof of Corollary 2.11, and is omitted. □

We now see that $\eta \mapsto A_{\xi(\eta)}$ is an isomorphism between the complete lattice $(SF(x), \sqsubseteq)$ and $(\mathcal{P}(X)\otimes\mathcal{I}, \subseteq)$. Since the latter is known to be completely distributive the same is true for $(SF(X), \sqsubseteq)$. In particular we deduce that ξ is an isomorphism between $(SF(x), \sqsubseteq)$ and $F_{\mathbb{I}}(X)$ with the lattice operations given by (1.4).

To define an appropriate complementation on $SF(X)$ we recall from Corollary 1.1 that for $\eta \in F_{\mathbb{I}}(X)$ we have $\eta'(x) = \{s \in \mathbb{I} \mid s' \notin \eta(x')\}$, and since we have $s' = 1 - s$ and

$x' = x$ this gives $\eta'(x) = \{s \in \mathbb{I} \mid 1 - s \notin \eta(x)\}$. Hence, for $\mu \in F(X)$ and $M \in \mathcal{P}(X)$ we have

$$\begin{aligned}\xi(\mu, M)'(x) &= \{s \in \mathbb{I} \mid 1 - s \notin \xi(\mu, M)\}\\ &= \{s \in \mathbb{I} \mid \mu(x) < 1 - s \text{ or } (\mu(x) = 1 - s \text{ and } x \notin M)\}\\ &= \{s \in \mathbb{I} \mid s < 1 - \mu(x) \text{ or } (s = 1 - \mu(x) \text{ and } x \in X \setminus M)\}\\ &= \xi(1 - \mu, X \setminus M)(x).\end{aligned}$$

This justifies the following:

2.11. Definition. For $(\mu, M) \in SF(X)$ the soft fuzzy set $(\mu, M)' = (1 - \mu, X \setminus M)$ is called the *complement* of (μ, M).

We deduce that with this definition $(SF(X), \sqsubseteq)$ is a Hutton algebra isomorphic to $F_{\mathbb{I}}(X)$, and hence to $\mathcal{L}^{\text{prod}}_{X \times \mathbb{I}} = \mathcal{P}(X) \otimes \mathcal{I}$. In particular,

(i) $((\mu, M)')' = (\mu, M)$, and
(ii) $(\mu, M) \sqsubseteq (\nu, N) \iff (\nu, N)' \sqsubseteq (\mu, M)'$.

In order to be able to set up a relationship between the set $X \times \mathbb{I}$ itself and $SF(X)$ we define a notion of "point" in $SF(X)$. It will be useful also to define a dual notion of "copoint".

2.12. Definition. Take $x \in X$ and $s \in \mathbb{I}$.

(1) Define $x_s : X \to \mathbb{I}$ by $x_s(z) = \begin{cases} s & \text{if } z = x \\ 0 & \text{otherwise} \end{cases}$. Then the soft fuzzy set $(x_s, \{x\})$ is called the *point of $SF(X)$ with base x and value s.*

(2) Define $x^s : X \to \mathbb{I}$ by $x^s(z) = \begin{cases} s & \text{if } z = x \\ 1 & \text{otherwise} \end{cases}$. Then the soft fuzzy set $(x^s, X \setminus \{x\})$ is called the *copoint of $SF(X)$ with base x and value s.*

Note that, contrary to the situation with classical fuzzy subsets, it is meaningful to consider the point $(x_s, \{x\})$ for $s = 0$. This is because, although x_s is again the zero function, x is distinguished as being the only point with a hard value. Dually, for the copoint $(x^s, X \setminus \{x\})$ with $s = 1$, x^s is the constant function with value 1 but x is distinguished as the only point with a soft value.

2.13. Definition. We denote $(x_r, \{x\}) \sqsubseteq (\mu, M)$ by $(x_r, \{x\}) \in (\mu, M)$, and refer to $(x_r, \{x\})$ as an *element* of (μ, M).

It should be stressed that this definition merely provides a suggestive notation which is appropriate to the notion of point.

Now we relate the points and copoints of $SF(X)$ with $(X \times \mathbb{I}, \mathcal{P}(X) \otimes \mathcal{I})$.

2.14. Proposition. *For $x \in X$, $s \in \mathbb{I}$ we have*

$$A_{\xi(x_s, \{x\})} = P_{(x,s)} \text{ and } A_{\xi(x^s, X \setminus \{x\})} = Q_{(x,s)}.$$

Proof. We prove the first equality, leaving the dual proof of the second equality to the interested reader. Take $(x, s) \in X \times \mathbb{I}$. Then for $(z, r) \in X \times \mathbb{I}$,

$$\begin{aligned}(z, r) \in A_{\xi(x_s, \{x\})} &\iff r < x_s(z) \text{ or } (r = x_s(z) \text{ and } z \in \{x\})\\ &\iff z = x \text{ and } r \leq s,\end{aligned}$$

whence $A_{\xi(x_s, \{x\})} = \{x\} \times [0, s] = P_{(x,s)}$. □

It is an immediate corollary of this result that ξ is a bijection between the points and copoints of $SF(X)$, and the corresponding points x_s, $x_s(u) = \begin{cases} P_s & \text{if } u = x \\ \emptyset & \text{otherwise} \end{cases}$, $u \in X$ and copoints x^s, $x^s(u) = \begin{cases} Q_s & \text{if } x = u \\ \mathbb{I} & \text{otherwise} \end{cases}$, $u \in X$ of $F_{\mathbb{I}}(X)$ in the sense of [17].

The following important results will be proved by using known properties of the texturing $(X \times \mathbb{I}, \mathcal{P}(X) \otimes \mathcal{I})$. The interested reader could easily supply direct proofs based on the definitions given above, see also [17, Lemma 4.7].

2.15. Theorem. *For $(\mu, M) \in SF(X)$, $(\mu_j, M_j) \in SF(X)$, $j \in J$, and $(x, s) \in X \times \mathbb{I}$ we have:*

(1) $(\mu, M) = \bigsqcup\{(x_s, \{x\}) \mid (x_s, \{x\}) \in (\mu, M)\}$.
(2) $(\mu, M) = \bigsqcap\{(x^s, X \setminus \{x\}) \mid (\mu, M) \sqsubseteq (x_s, X \setminus \{x\})\}$.
(3) $(\mu, M) \not\sqsubseteq (x^s, X \setminus \{x\}) \iff (x_s, \{x\}) \in (\mu, M)$.
(4) $(x_s, \{x\}) \notin (x^s, X \setminus \{x\})$.
(5) $(x_s, \{x\}) \in \bigsqcup_{j\in J}(\mu_j, M_j) \implies \exists j \in J$ *with* $(x_s, \{x\}) \in (\mu_j, M_j)$.

Proof. (1). From ([5], Theorem 1.2 (7)) we have $A_{\xi(\mu,M)} = \bigvee\{P_{(x,s)} \mid \xi(\mu, M) \not\subseteq Q_{(x,s)}\} = \bigcup\{P_{(x,s)} \mid P_{(x,s)} \subseteq \xi(\mu, M)\}$ as $(X \times \mathbb{I}, \mathcal{P}(X) \otimes \mathcal{I})$ is a plain texture. Hence

$$\begin{aligned} A_{\xi(\mu,M)} &= \bigcup\big\{A_{\xi(x_s,\{x\})} \mid \xi(x_s, \{x\}) \subseteq \xi(\mu, M)\big\} \\ &= \xi\big(\bigsqcup\big\{(x_s, \{x\}) \mid (x_s, \{x\}) \in (\mu, M)\big\}\big) \end{aligned}$$

by Proposition 2.16, Corollary 2.13 and Corollary 2.9. The result now follows since ξ is injective.

(2). Similar to (1) using ([5], Theorem 1.2 (6)).

(3). By Corollary 2.9 and Proposition 2.16 we have

$$\begin{aligned} (\mu, M) \not\sqsubseteq (x^s, X \setminus \{x\}) &\iff A_{\xi(\mu,M)} \not\subseteq A_{\xi(x^s,X\setminus\{x\})} \\ &\iff A_{\xi(\mu,M)} \not\subseteq Q_{(x,s)} \\ &\iff P_{(x,s)} \subseteq A_{\xi(\mu,M)} \text{ by plainness} \\ &\iff A_{\xi(x_s,\{x\})} \subseteq A_{\xi(\mu,M)} \\ &\iff (x_s, \{x\}) \sqsubseteq (\mu, M). \end{aligned}$$

(4). Immediate from (3) on taking $(\mu, M) = (x_s, \{x\})$.

(5). From $(x_s, \{x\}) \in \bigsqcup_{j\in J}(\mu_j, M_j)$ we have $P_{(x,s)} \subseteq \bigcup_{j\in J} A_{\xi(\mu_j,M_j)}$, which is equivalent to $(x, s) \in \bigcup_{j\in J} A_{\xi(\mu_j,M_j)}$. Hence there exists $j \in J$ with $(x, s) \in A_{\xi(\mu_j,M_j)}$, whence $A_{\xi(x_s,\{x\})} = P_{(x,s)} \subseteq A_{\xi(\mu_j,M_j)}$ and so $(x_s, \{x\}) \in (\mu_j, M_j)$ for this j. □

Equalities (1) and (2) above show the way in which the soft fuzzy subsets of X may be generated by the points or copoints of $SF(X)$. Results (3) and (4) are technical but extremely powerful results which reflect the fact that the texture $(X \times \mathbb{I}, \mathcal{P}(X) \otimes \mathcal{I})$ is plain. Property (5), which is also a consequence of plainness, shows that the points in $SF(X)$ act like the points in classical set theory with respect to join, in contrast to the fuzzy points in classical fuzzy set theory.

The effect of the complement on the points and copoints of $SF(X)$ is given below.

(iii) $(x_s, \{x\})' = (x^{1-s}, X \setminus \{x\})$, and
(iv) $(x^s, X \setminus \{x\})' = (x_{1-s}, \{x\})$.

The proofs are straightforward, and are omitted.

Let us recall that if $(S, \mathcal{S})$, $(T, \mathcal{T})$ are textures and $\psi : S \to T$ a point function, then ψ is called *ω-preserving* if $P_{s_1} \not\subseteq Q_{s_2} \implies P_{\psi(x_1)} \not\subseteq Q_{\psi(x_2)}$. We denote by **ifTex** the construct of textures and ω-preserving functions between the base sets.

Now let X, Y be sets with the discrete order and $\varphi : X \to Y$ a point function. If we denote the identity on $\mathbb{I}$ by id then $\langle \varphi, id \rangle : X \times \mathbb{I} \to Y \times \mathbb{I}$ defined by $\langle \varphi, id \rangle(x, s) = (\varphi(x), s)$ is order-preserving and hence ω-preserving regarded as a mapping from $(X \times \mathbb{I}, \mathcal{L}^{\mathrm{prod}}_{X\times\mathbb{I}})$ to $(Y \times \mathbb{I}, \mathcal{L}^{\mathrm{prod}}_{Y\times\mathbb{I}})$. If we denote by **SF-Set** the construct whose objects are pairs $(X, SF(X))$ and morphisms from **Set** this gives us a (non-full) embedding of **SF-Set** in **ifTex**. Specifically we define $\mathfrak{E} :$ **SF-Set** $\to$ **ifTex** by setting

$$\mathfrak{E}((X, SF(X)) \xrightarrow{\varphi} (Y, SF(Y))) = (X \times \mathbb{I}, \mathcal{L}^{\mathrm{prod}}_{X\times\mathbb{I}}) \xrightarrow{\langle \varphi, id \rangle} (Y \times \mathbb{I}, \mathcal{L}^{\mathrm{prod}}_{Y\times\mathbb{I}}).$$

It clear that $\mathfrak{E}$ is indeed an embedding. Also, since an **ifTex**-isomorphism preserves plainness it is easy to see that **SF-Set** is embedded as an isomorphism-closed subconstruct of **ifTex**.

The point function φ may be used to define mappings between $SF(X)$ and $SF(Y)$. To this end we look at the difunction (f, F) corresponding to $\langle \varphi, id \rangle(x, s) = (\varphi(x), s)$ as in [5, Lemma 3.4]. Since we are dealing with plain textures this mapping automatically satisfies conditions (b) and (c) of [6, Lemma 3.8] and therefore we have

$$\begin{aligned} f &= \bigcup \{ \overline{P}_{((x,s),\,(\varphi(x),s))} \mid (x, s) \in X \times \mathbb{I} \}, \text{ and} \\ F &= \bigcap \{ \overline{Q}_{((x,s),\,(\varphi(s),s))} \mid (x, s) \in X \times \mathbb{I} \}. \end{aligned} \tag{2.3}$$

The image and co-image operators now map from $\mathcal{L}^{\mathrm{prod}}_{X\times\mathbb{I}}$ to $\mathcal{L}^{\mathrm{prod}}_{Y\times\mathbb{I}}$, and the inverse image and inverse co-image operators, which are equal, map from $\mathcal{L}^{\mathrm{prod}}_{Y\times\mathbb{I}}$ to $\mathcal{L}^{\mathrm{prod}}_{X\times\mathbb{I}}$. In view of the isomorphism between $SF(X)$ and $\mathcal{L}^{\mathrm{prod}}_{X\times\mathbb{I}}$, and that between $SF(Y)$ and $\mathcal{L}^{\mathrm{prod}}_{Y\times\mathbb{I}}$, these lead to the required mappings, as detailed below.

2.16. Proposition. *Let $\varphi : X \to Y$ be a point function.*

(1) *The mapping $\varphi^{\to}$ from $SF(X)$ to $SF(Y)$ corresponding to the image operator of the difunction (f, F) is given by*

$$\varphi^{\to}(\mu, M) = (\nu, N) \text{ where } \nu(y) = \sup\{\mu(x) \mid y = \varphi(x)\}, \text{ and}$$
$$N = \{\varphi(x) \mid x \in M \text{ and } \nu(\varphi(x)) = \mu(x)\}.$$

(2) *The mapping $\varphi^{\rightharpoondown}$ from $SF(X)$ to $SF(Y)$ corresponding to the co-image operator of the difunction (f, F) is given by*

$$\varphi^{\rightharpoondown}(\mu, M) = (\nu, N) \text{ where } \nu(y) = \inf\{\mu(x) \mid y = \varphi(x)\}, \text{ and}$$
$$N = Y \setminus \{\varphi(x) \mid x \in X \setminus M \text{ and } \nu(\varphi(x)) = \mu(x)\}.$$

(3) *The mapping $\varphi^{\leftarrow}$ from $SF(Y)$ to $SF(X)$ corresponding to the inverse image and inverse co-image of the difunction (f, F) is given by*

$$\varphi^{\leftarrow}(\nu, N) = (\nu \circ \varphi,\ \varphi^{-1}[N]).$$

Proof. (1) We are required to show that $f^{\to} A_{\xi(\mu,M)} = A_{\xi(\nu,N)}$. By the above formulae for (f, F) and [5, Definition 2.5] and it is straightforward to verify that for $(\mu, M) \in SF(X)$ we have

$$f^{\to} A_{\xi(\mu,M)} = \{(y, r) \in Y \times \mathbb{I} \mid \exists\, (x, s) \in A_{\xi(\mu,M)} \text{ with } (y, r) \leq_{\mathrm{prod}} (\varphi(x), s)\}.$$

Alternatively, this follows immediately from [17, Lemma 2.5 (3 i)].

Taking $(y,r) \in f^{\to}A_{\xi(\mu,M)}$ gives $(x,s) \in A_{\xi(\mu,M)}$ with $(y,r) \leq_{\text{prod}} (\varphi(x),s)$, whence $y = \varphi(x)$ and $r \leq s$. But $s < \mu(x)$ or $(s = \mu(x)$ and $x \in M)$ by (2.1), so in either case $r \leq s \leq \mu(x) \leq \nu(y)$ and we have $r < \nu(y)$ or $r = \nu(y)$. In the second case we deduce $s = \mu(x)$, whence $x \in M$, and $\nu(\varphi(x)) = \mu(x)$ whence $y = \varphi(x) \in N$. This gives $(y,r) \in A_{\xi(\nu,N)}$, so $f^{\to}A_{\xi(\mu,M)} \subseteq A_{\xi(\nu,N)}$.

Conversely, take $(y,r) \in A_{\xi(\nu,N)}$. There are two cases to consider.

Case (i). $r < \nu(y)$. Since $\nu(y) = \sup\{\mu(x) \mid y = \varphi(x)\}$ there exists $x \in X$ with $y = \varphi(x)$ and $r < \mu(x) \leq \nu(y)$. Take $s = r$. Then $s < \mu(x)$ gives $(x,s) \in A_{\xi(\mu,M)}$ by (2.1), and clearly $(y,r) \leq_{\text{prod}} (\varphi(x),s)$ whence $(y,r) \in f^{\to}A_{\xi(\mu,M)}$.

Case (ii). $r = \nu(y)$ and $y \in N$. By the definition of N there exists $x \in M$ with $y = \varphi(x)$ and $\nu(y) = \nu(\varphi(x)) = \mu(x)$. Taking $s = r$ gives $s = \mu(x)$, and $x \in M$ so $(x,s) \in A_{\xi(\mu,M)}$ and $(y,r) \leq_{\text{prod}} (\varphi(x),s)$ so again $(y,r) \in f^{\to}A_{\xi(\mu,M)}$. Thus $A_{\xi(\nu,N)} \subseteq f^{\to}A_{\xi(\mu,M)}$ and the proof is complete.

(2) Dual to the proof of (1), and is omitted.

(3) Straightforward. □

The following inclusions may easily be obtained from [5, Theorem 2.24].

$$\varphi^{\leftarrow}(\varphi^{\rightharpoondown}(\mu,M)) \sqsubseteq (\mu,M) \sqsubseteq \varphi^{\leftarrow}(\varphi^{\rightharpoonup}(\mu,M)),\ \forall\,(\mu,M) \in SF(X),$$
$$\varphi^{\rightharpoonup}(\varphi^{\leftarrow}(\nu,N)) \sqsubseteq (\nu,N) \sqsubseteq \varphi^{\rightharpoondown}(\varphi^{\leftarrow}(\nu,N)),\ \forall\,(\nu,N) \in SF(Y).$$

There are many more results that can be deduced from the properties of the (co) image and inverse image operators. We mention just a few of these in the following notes.

2.17. Note. The mappings $\varphi^{\leftarrow} : SF(Y) \to SF(X)$ preserve arbitrary intersections and unions by [5, Corollary 2.12]. They also preserve complementation. Indeed using (2.3) and [5, Definition 2.18 (2)] it is not difficult to show that $F' = f$, so (f,F) is a complemented difunction. Hence

$$\begin{aligned} f^{\leftarrow}A_{\xi(\mu,M)'} &= f^{\leftarrow}\sigma_{X\times\mathbb{I}}(A_{\xi(\mu,M)}) = F^{\leftarrow}\sigma_{X\times\mathbb{I}}(A_{\xi(\mu,M)}) \\ &= \sigma_{Y\times\mathbb{I}}((F')^{\leftarrow}A_{\xi(\mu,M)}) = \sigma_{Y\times\mathbb{I}}(f^{\leftarrow}A_{\xi(\mu,M)}) \end{aligned}$$

by [5, Lemma 2.20], which gives $\varphi^{\leftarrow}(\mu,M)' = (\varphi^{\leftarrow}(\mu,M))'$ as required. We also have

2.18. Lemma. (1) *For* $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ *we have* $(\psi \circ \varphi)^{\leftarrow} = \varphi^{\leftarrow} \circ \psi^{\leftarrow}$.

(2) $(\iota_X)^{\leftarrow} = \iota_{SF(X)}$, *where* ι_X *is the identity on* X *and* $\iota_{SF(X)}$ *that on* $SF(X)$.

Proof. Left to the interested reader. □

For general complemented textures $(S_1,\mathcal{S}_1,\sigma_1)$, $(S_2,\mathcal{S}_2,\sigma_2)$ mappings from $\mathcal{S}_2$ to $\mathcal{S}_1$ that preserve complementation, arbitrary intersections and joins are the morphisms for a category named **ctmTex**$^{\text{op}}$, which is isomorphic to the category **cdfTex** of complemented textures and complemented difunctions [6]. It is clear that **SF-Set** may be embedded as a non-full subcategory of **ctmTex**$^{\text{op}}$, and hence of **cdfTex**. The details are given in [15], and are not repeated here. It must be stressed that this embedding involves a loss of information because the morphisms of these categories do not preserve the point structure, that is the embedding functor is a forgetful functor. For example, if we replace $\mathcal{I}$ by its Hutton texture [6, Example 2.14] then we obtain an isomorphic object in **ctmTex**$^{\text{op}}$ or **cdfTex** since the texturings are isomorphic, but we do not obtain an isomorphic object in **ifTex** or **cifTex** because the point structures are different, and indeed the Hutton texture is not even plain. Since the point structure is an important aspect of soft fuzzy sets we prefer therefore the embedding in **ifTex** or **cifTex**, which gives an isomorphism closed subconstruct.

2.19. Note. The mappings $\varphi^{\rightharpoonup} : SF(X) \to SF(Y)$ preserve arbitrary unions by [5, Corollary 2.12]. They also preserve the points of $SF(X)$ strongly in the sense that $\varphi^{\rightharpoonup}(x_s, \{x\}) = (y_s, \{y\})$ for some $y \in Y$. Indeed it is clear from Proposition 2.16 (1) that this equality holds for $y = \varphi(x)$. Conversely, a mapping $\theta : SF(X) \to SF(Y)$ that preserves arbitrary unions and is strongly point preserving defines a function $\varphi : X \to Y$ for which $\theta = \varphi^{\rightharpoonup}$. It is clear that such mappings could also be used as morphisms is a category isomorphic to **SF-Set**, and be generalized to a wider context. We refer the reader for a discussion along these lines, and do not take up this topic in greater detail here.

Dually, the mappings $\varphi^{\leftharpoondown} : SF(X) \to SF(Y)$ preserve arbitrary intersections and are strongly co-point preserving. Similar comments to the naturally apply to these mappings also.

We conclude this section by presenting an alternative description of soft fuzzy sets. As mentioned earlier, for a soft fuzzy subset (μ, M) of X and for $x \in X$, one of two states may be associated with the value $\mu(x)$ according as $x \in M$ or $x \notin M$. If we denote the first state by 1 and the second by 0 we may associate with (μ, M) the function $\langle \mu, \chi_M \rangle : X \to \mathbb{D} = \mathbb{I} \times \{0, 1\}$ defined by $\langle \mu, \chi_M \rangle(x) = (\mu(x), \chi_M(x))$, where as usual χ_M denotes the characteristic function $\chi_M(x) = \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{if } x \notin M \end{cases}$, $x \in X$, of M. Now we have:

2.20. Proposition. *Denote by $\leq$ the lexical ordering on $\mathbb{D} = \mathbb{I} \times \{0, 1\}$. That is, for $(r, k), (s, l) \in \mathbb{D}$, $(r, k) \leq (s, l) \iff (r < s)$ or $(r = s$ and $k \leq l)$, where $\mathbb{I}$ and $\{0, 1\}$ have their usual orderings. Then for $(\mu, M), (\nu, N) \in SF(X)$,*

$$\langle \mu, \chi_M \rangle \leq \langle \nu, \chi_N \rangle \iff (\mu, M) \sqsubseteq (\nu, N)$$

where $\langle \mu, \chi_M \rangle \leq \langle \nu, \chi_N \rangle$ is defined pointwise.

Proof. If $\langle \mu, \chi_M \rangle \leq \langle \nu, \chi_N \rangle$ then given $x \in X$ either $\mu(x) < \nu(x)$ or $(\mu(x) = \nu(x)$ and $\chi_M(x) \leq \chi_N(x))$. Clearly $\chi_M(x) \leq \chi_N(x)$ is equivalent to $x \notin M \setminus N$, whence $(\mu, M) \sqsubseteq (\nu, N)$ by Definition 2.8. The reverse implication is proved in the same way. □

If for $(s, k) \in \mathbb{D}$ we define $(s, k)' = (1 - s, 1 - k)$ it is clear that the mapping $' : \mathbb{D} \to \mathbb{D}$, $(s, k) \mapsto (s, k)'$, is an order-reversing involution. Moreover, if $\langle \mu, \chi_M \rangle$ corresponds to (μ, M) then $\langle \mu, \chi_M \rangle'$ defined by $\langle \mu, \chi_M \rangle'(x) = (\langle \mu, \chi_M \rangle(x))'$ corresponds to the complement $(\mu, M)'$ of (μ, M). Hence:

2.21. Proposition. *The Hutton algebra $(SF(X), \sqsubseteq, ')$ is isomorphic to the Hutton algebra $(\mathbb{D}^X, \leq, ')$ of $\mathbb{D}$-fuzzy subsets of X.*

2.22. Corollary. *$(\mathbb{D}, \leq, ')$ is a Hutton algebra isomorphic to $(\mathfrak{I}, \subseteq, \iota)$.*

Proof. If X is chosen to be a singleton then it is straightforward to show that $SF(X)$ is isomorphic to $\mathfrak{I}$. On the other hand $SF(X)$ is then isomorphic to $\mathbb{D}$ by Proposition 2.21. Hence $\mathbb{D}$ is isomorphic to $\mathfrak{I}$. □

Finally, the points are distinguished as the $\mathbb{D}$-fuzzy subsets $\langle x_s, \chi_{\{x\}} \rangle$, and the copoints $\langle x^s, \chi_{X \setminus \{x\}} \rangle$, $x \in X, s \in \mathbb{I}$. Hence all aspects of the theory of soft fuzzy sets may be equally well expressed using this new representation.

3. SF-topologies

In this section we specialize the notion of $\mathbb{L}$-topologies on X to the case of SF-topologies on X. As expected, we will have a considerable simplification arising from the very clean point structure.

3.1. Definition. Let S be a set. A subset $T \subseteq SF(X)$ is called an *SF-topology on X* if

SFT1 $(\mathbf{0}, \emptyset) \in T$ and $(\mathbf{1}, X) \in T$.
SFT2 $(\mu_j, M_j) \in T$, $j = 1, 2, \dots, n \implies \bigsqcap_{j=1}^{n} (\mu_j, M_j) \in T$.
SFT3 $(\mu_j, M_j) \in T$, $j \in J \implies \bigsqcup_{j \in J} (\mu_j, M_j) \in T$.

As usual, the elements of T are called *open*, and those of $T' = \{(\mu, M) \mid (\mu, M)' \in T\}$ *closed*.

If T is an SF-topology on X we call the pair (X, T) an *SF-topological space*.

The *closure* of a soft fuzzy set (μ, M) will be denoted by $\overline{(\mu, M)}$. It is given by

$$\overline{(\mu, M)} = \bigsqcap \{(\nu, N) \mid (\mu, M) \sqsubseteq (\nu, N) \in T'\}.$$

Likewise the *interior* is given by

$$(\mu, M)^o = \bigsqcup \{(\nu, N) \mid (\nu, N) \in T,\ (\nu, N) \sqsubseteq (\mu, M)\}.$$

Bases and subbases may be defined and characterized exactly as in classical topology.

3.2. Definition. Let T be an SF-topology on X.

(1) $B \subseteq T$ is called a *base for* T if each element of T can be written as a join of elements of B. Equivalently, B is a base of T if and only if given $(\mu, M) \in T$ and $(x_r, \{x\}) \in (\mu, M)$ there exists $(\nu, N) \in B$ with $(x_r, \{x\}) \in (\nu, N) \sqsubseteq (\mu, M)$.
(2) $S \subseteq T$ is called a *subbase of* T if the set of finite meets of elements of S is a base of T.

3.3. Proposition. *A subset $B \subseteq SF(X)$ is a base for some SF-topology on X if and only if it satisfies the following conditions*

SFB1 $\bigsqcup \{(\nu, N) \mid (\nu, N) \in B\} = (\mathbf{1}, X)$.
SFB2 *Given $(\nu_1, N_1), (\nu_2, N_2) \in B$ and $(x_r, \{x\}) \in (\nu_1, N_1) \sqcap (\nu_2, N_2)$, there exists $(\nu_3, N_3) \in B$ satisfying $(x_r, \{x\}) \in (\nu_3, N_3) \sqsubseteq (\nu_1, N_1) \sqcap (\nu_2, N_2)$.*

Proof. Immediate in view of Theorem 2.17 (5). □

We note that, as in classical topology, any non-empty subset S of $SF(X)$ is a subbase for some SF-topology on X since the set of finite meets of elements of S trivially satisfies SFB1 and SFB2.

Now let us consider continuity. Again our definition is a specialization of that used for $\mathbb{L}$-topologies.

3.4. Definition. Let T be an SF-topology on X and V an SF-topology on Y. Then a function $\varphi : X \to Y$ is called *T–V continuous* if $(\nu, N) \in V \implies \varphi^{\leftarrow}(\nu, N) \in T$.

By Lemma 2.18 we see that identity functions are continuous, and that the composition of two continuous functions is continuous. Hence SF-topological spaces and continuous functions between the base sets define a construct which we denote by **SF-Top**. It is straightforward to verify that **SF-Top** is topological over **SF-Set**.

The following result will be useful when discussing continuity.

3.5. Lemma. *Let $\varphi : X \to Y$ be a function, $x \in X$, $r \in \mathbb{I}$ and $(\nu, N) \in SF(Y)$. Then*

$$(\varphi(x)_r, \{\varphi(x)\}) \in (\nu, N) \iff (x_r, \{x\}) \in \varphi^{\leftarrow}(\nu, N).$$

Proof.

$$\begin{aligned}(x_r, \{x\}) \in \varphi^{\leftarrow}(\nu, N) &\iff (x_r, \{x\}) \in (\nu \circ \varphi, \varphi^{\leftarrow} N)\\ &\iff r < (\nu \circ \varphi)(x) \text{ or } r = (\nu \circ \varphi)(x) \text{ and } x \in \varphi^{\leftarrow} M\\ &\iff r < \nu(\varphi(x)) \text{ or } r = \nu(\varphi(x)) \text{ and } \varphi(x) \in N\\ &\iff (\varphi(x)_r, \{\varphi(x)\}) \in (\nu, N),\end{aligned}$$

whence the result. □

Now let us relate SF-topologies on X with ditopologies on $(X \times \mathbb{I}, \mathcal{P}(X) \otimes \mathcal{I}, \sigma_X)$. If T is an SF-topology on X, then $T \subseteq SF$ and we may apply the isomorphism $(\mu, M) \mapsto A_{\xi(\mu,M)}$ to give $A_{\xi[T]} \subseteq \mathcal{P}(X) \otimes \mathcal{I}$. Since $A_{\xi(\mathbf{0},\emptyset)} = \emptyset$, $A_{\xi(\mathbf{1},X)} = X \times \mathbb{I}$, and the isomorphism takes meet and join in $SF(X)$ to intersection and union respectively in $\mathcal{P}(X) \otimes \mathcal{I}$, it is immediate that $A_{\xi[T]}$ is a topology on the plain complemented texture $(X \times \mathbb{I}, \mathcal{P}(X) \otimes \mathcal{I}, \sigma_X)$. Also, by [5, Lemma 2.20], for $(\mu, M) \in T$ we have $A_{\xi((\mu,M)')} = \sigma_X(A_{\xi(\mu,M)})$, so $A_{\xi[T']} = \sigma_X(A_{\xi(T)})$ is a cotopology on $(X \times \mathbb{I}, \mathcal{P}(X) \otimes \mathcal{I}, \sigma_X)$. Hence $(A_{\xi[T]}, A_{\xi[T']})$ is the complemented ditopology on $(X \times \mathbb{I}, \mathcal{P}(X) \otimes \mathcal{I}, \sigma_X)$ corresponding to T.

Note that $A_{\xi(\overline{(\mu,M)})} = \bigcap \{A_{\xi(\nu,N)} \mid A_{\xi(\mu,M)} \subseteq A_{\xi(\nu,N)} \in A_{\xi[T']}\}$, which is just the closure of $A_{\xi(\mu,M)}$ with respect to the ditopology $(A_{\xi[T],\xi[T']})$. Likewise, $A_{\xi((\mu,M)^o)}$ is the interior of $A_{\xi(\mu,M)}$.

Next we recall the functor $\mathfrak{E}$, which we now regard as mapping from **SF-Set** to **cifPTex**. Let T be an SF-topology on X and V an SF-topology on Y. We wish to show that if $\varphi : X \to Y$ is T–V continuous then $\langle \varphi, id \rangle : X \times \mathbb{I} \to Y \times \mathbb{I}$ is $(A_{\xi[T]}, A_{\xi[T']})$–$(A_{\xi[V]}, A_{\xi[V']})$ bicontinuous. We will need the following result.

3.6. Lemma. *For $\varphi : X \to Y$ and $(\nu, N) \in SF(Y)$ we have*

$$\langle \varphi, id \rangle^{-1}[A_{\xi(\nu,N)}] = A_{\xi(\varphi^{\leftarrow}(\nu,N))}.$$

Proof. This equality follows at once from [5, Lemma 3.9] and the fact that the textures are plain. □

Now take $H \in A_{\xi[V]}$. Then we have $(\nu, N) \in V$ with $H = A_{\xi(\nu,N)}$, so $\langle \varphi, id \rangle^{-1}[H] = A_{\xi(\varphi^{\leftarrow}(\nu,N))}$ by Lemma 3.6. But if φ is T–V continuous then $\varphi^{\leftarrow}(\nu, N) \in T$ so $\langle \varphi, id \rangle^{-1}[H] \in A_{\xi[T]}$ as required. The cocontinuity of $\langle \varphi, id \rangle$ is proved likewise, and we see that $\mathfrak{E}$ may be regarded as a functor from **SF-Top** to **cifPDitop**. Restricting to complemented ditopologies on textures of the form $(X \times \mathbb{I}, \mathcal{P}(X) \otimes \mathcal{I}, \sigma_X)$ we may regard $\mathfrak{E}$ as a functor from **SF-Top** to the subcategory $\mathbf{cifPDitop_{SF}}$. To make $\mathfrak{E}$ into an isomorphism we will also need to restrict our attention to the morphisms in $\mathbf{cfPDitop_{SF}}$ of the form $\langle \varphi, id \rangle$. This leads to the following definition.

3.7. Definition. The category whose objects are complemented ditopological textures of the form $(X \times \mathbb{I},\ \mathcal{P}(X) \otimes \mathcal{I},\ \sigma_{X \times \mathbb{I}},\ \tau_X,\ \kappa_X)$, $X \in \mathrm{Ob}\,\mathbf{Set}$, and whose morphisms are the bicontinuous mappings $\langle \varphi, id \rangle$, $\varphi \in \mathbf{Set}(X, Y)$, will be denoted by **SF-Ditop**.

Clearly **SF-Ditop** is a non-full subcategory of $\mathbf{cfPDitop_{SF}}$, and $\mathfrak{E}$: **SF-Top** → **SF-Ditop** an isomorphism.

This isomorphism may be used to translate concepts and results for ditopological texture spaces to SF-topologies on a set X. Indeed, this will be the source of the material on separation axioms and compactness presented in the next section.

Since $(A_{\xi[T]}, A_{\xi[T']})$ is a ditopology on the product texture $(X, \mathcal{P}(X)) \otimes (\mathbb{I}, \mathcal{I})$ it is natural to ask when this is the product of a ditopology on $(X, \mathcal{P}(X))$ and a ditopology on $(\mathbb{I}, \mathcal{I})$. Given a ditopology (τ, κ) on $(X \times \mathbb{I}, \mathcal{P}(X) \otimes \mathcal{I})$ we let

$$\tau^1 = \{G \in \mathcal{P}(X) \mid G \times \mathbb{I} \in \tau\} \text{ and } \kappa^1 = \{K \in \mathcal{P}(X) \mid K \times \mathbb{I} \in \kappa\}.$$

Clearly, (τ^1, κ^1) is a ditopology on $(X, \mathcal{P}(X))$. Likewise, (τ^2, κ^2) defined by

$$\tau^2 = \{G \in \mathcal{I} \mid X \times G \in \tau\} \text{ and } \kappa^2 = \{K \in \mathcal{I} \mid X \times K \in \kappa\}$$

is a ditopology on $(\mathbb{I}, \mathcal{I})$. The product of (τ^1, κ^1) and (τ^2, κ^2) is a ditopology on $(X \times \mathbb{I}, \mathcal{P}(X) \otimes \mathcal{I})$ which is clearly coarser than (τ, κ). The following result gives necessary and sufficient conditions under which these ditopologies coincide.

3.8. Lemma. *The following are equivalent:*

(1) *The product of (τ^1, κ^1) and (τ^2, κ^2) coincides with (τ, κ).*
(2) *The following conditions hold:*
   (a) *Given $G \in \tau$, $(x, r) \in X \times \mathbb{I}$ with $P_{(x,r)} \subseteq G$, there exist $G_1 \in \tau^1$, $G_2 \in \tau^2$ satisfying $\overline{P}_{(x,r)} \subseteq (G_1 \times \mathbb{I}) \cap (X \times G_2) \subseteq G$.*
   (b) *Given $K \in \kappa$ and $(x, r) \in X \times \mathbb{I}$ with $K \subseteq Q_{(x,r)}$, there exist $K_1 \in \kappa^1$, $K_2 \in \kappa^2$ satisfying $K \subseteq (K_1 \times \mathbb{I}) \cup (X \times K_2) \subseteq Q_{(x,r)}$.*

Proof. Clear from the definition of product ditopology and the fact that the texture $(X \times \mathbb{I}, \mathcal{P}(X) \otimes \mathcal{I})$ is plain. $\square$

If (τ, κ) is complemented then (τ^1, κ^1), (τ^2, κ^2) are also complemented, so in particular τ^1 is a topology on X in the usual sense, and κ^1 is the set of closed sets under τ^1. This suggests that we may define a functor $\mathfrak{G} : \mathbf{SF\text{-}Ditop} \to \mathbf{Top}$ by setting $\mathfrak{G}(X \times \mathbb{I}, \mathcal{P}(X) \otimes \mathcal{I}, \sigma_X, \tau, \kappa) = (X, \tau^1)$ and $\mathfrak{G}(\langle \varphi, id\rangle) = \varphi$. We verify that if $\langle \varphi, id\rangle \in \operatorname{Mor} \mathbf{SF\text{-}Ditop}$ then $\varphi \in \operatorname{Mor} \mathbf{Top}$. Suppose that $\varphi : X \to Y$ and that the complemented ditopologies are respectively (τ_X, κ_X) and (τ_Y, κ_Y). Then for $G \in \tau_Y^1$ we have $G \times \mathbb{I} \in \tau_Y$ and so if $\langle \varphi, id\rangle$ is bicontinuous we have $\langle \varphi, id\rangle^{-1}[G \times \mathbb{I}] \in \tau_X$. However it is trivial to verify that $\langle \varphi, id\rangle^{-1}[G \times \mathbb{I}] = \varphi^{-1}[G] \times \mathbb{I}$, so $\varphi^{-1}[G] \in \tau_X^1$ and we have established that φ is τ_X^1–τ_Y^1 continuous.

The remaining conditions to be satisfied by $\mathfrak{G}$ are easily verified, and we deduce that $\mathfrak{G} : \mathbf{SF\text{-}Ditop} \to \mathbf{Top}$ is indeed a functor.

In the opposite direction we may define a functor $\mathfrak{F} : \mathbf{Set} \to \mathbf{cifPTex_{SF}}$ by

$$\mathfrak{F}(X \xrightarrow{\varphi} Y) = (X \times \mathbb{I}, \mathcal{P}(X) \otimes \mathcal{I}, \sigma_{X \times \mathbb{I}}) \xrightarrow{\langle \varphi, id\rangle} (Y \times \mathbb{I}, \mathcal{P}(Y) \otimes \mathcal{I}, \sigma_{Y \times \mathbb{I}})$$

and specialize this functor to produce a family of functors from $\mathbf{Top} \to \mathbf{SF\text{-}Ditop}$. To this end, let (τ_0, κ_0) be a fixed but arbitrary complemented ditopology on $(\mathbb{I}, \mathcal{I}, \iota)$. Then if $\mathcal{T}$ is a topology on X and $\mathcal{T}^c = \{X \setminus G \mid G \in \mathcal{T}\}$, $(\mathcal{T}, \mathcal{T}^c)$ is a complemented ditopology on $(X, \mathcal{P}(X), \pi_X)$ and we may define a complemented ditopology on $(X \times \mathbb{I}, \mathcal{P}(X) \otimes \mathcal{I}, \sigma_X)$ by taking the product of $(\mathcal{T}, \mathcal{T}^c)$ and (τ_0, κ_0). Now let $(X, \mathcal{T})$, $(Y, \mathcal{V})$ be topological spaces and $\varphi : (X, \mathcal{T}) \to (Y, \mathcal{V})$ continuous. The product topology $\mathcal{V} \otimes \tau_0$ has base $G \times H$, $G \in \mathcal{V}$, $H \in \tau_0$, and $\langle \varphi, id\rangle^{-1}[G \times H] = \varphi^{-1}[G] \times H \in \mathcal{T} \otimes \tau_0$, whence $\langle \varphi, id\rangle$ is continuous, and hence bicontinuous since the ditopologies are complemented.

This shows that the functor $\mathfrak{F}$ specializes to a functor $\mathfrak{F}_{(\tau_0, \kappa_0)} : \mathbf{Top} \to \mathbf{SF\text{-}Ditop}$ defined by setting $\mathfrak{F}_{(\tau_0,\kappa_0)}(X, \mathcal{T}) = (X \times \mathbb{I}, \mathcal{P}(X) \otimes \mathcal{I}, \sigma_X, \mathcal{T} \otimes \tau_0, \mathcal{T}^c \otimes \kappa_0)$ and $\mathfrak{F}_{(\tau_0,\kappa_0)}(f) = \langle f, \iota_{\mathbb{I}}\rangle$.

As for the classical case [6, Theorem 5.12] we have:

3.9. Theorem. *Choose $\tau_0 = \{\mathbb{I}, \emptyset\} = \kappa_0$. Then $\mathfrak{F}_{(\tau_0,\kappa_0)}$ is an adjoint of $\mathfrak{G}$.*

Proof. Take $B = (X \times \mathbb{I}, \mathcal{P}(X) \otimes \mathcal{I}, \sigma_X, \tau, \kappa) \in \operatorname{Ob} \mathbf{SF\text{-}Ditop}$. Then, recalling that $\mathfrak{G}(X \times \mathbb{I}, \mathcal{P}(X) \otimes \mathcal{I}, \sigma_X, \tau, \kappa) = (X, \tau^1)$, it will be sufficient to show that $(\langle id_X, id_{\mathbb{I}}\rangle,\ (X, \tau^1))$ is an $\mathfrak{F}_{(\tau_0,\kappa_0)}$ universal arrow with domain B.

Certainly $\langle id_X, id_{\mathbb{I}}\rangle : X \times \mathbb{I} \to X \times \mathbb{I}$ is τ–$\tau^1 \times \{\mathbb{I}, \emptyset\}$ continuous, and hence bicontinuous since the ditopologies are complemented. This verifies that $(\langle id_X, id_{\mathbb{I}}\rangle,\ (X, \tau^1))$ is an

$\mathfrak{F}_{(\tau_0,\kappa_0)}$ structured arrow with domain B. If $(\langle\varphi, id_{\mathbb{I}}\rangle, (Y,\mathcal{V}))$ is also an $\mathfrak{F}_{(\tau_0,\kappa_0)}$ structured arrow with domain B we must prove the existence of a unique continuous function $\overline{\varphi} : (X,\tau^1) \to (Y,\mathcal{V})$ making the following diagram commutative.

$$\begin{tikzcd}
B \arrow[r, "\langle id_X, id_{\mathbb{I}}\rangle"] \arrow[dr, "\langle\varphi, id_{\mathbb{I}}\rangle"'] & \mathfrak{F}_{(\tau_0,\kappa_0)}(X,\tau^1) \arrow[d, dashed, "\mathfrak{F}_{(\tau_0,\kappa_0)}(\overline{\varphi})"] \\
& \mathfrak{F}_{(\tau_0,\kappa_0)}(Y,\mathcal{V})
\end{tikzcd}$$

Clearly the only possible choice for $\overline{\varphi}$ is φ, so we must prove that $\varphi : X \to Y$ is τ^1–$\mathcal{V}$ continuous. However, $V \in \mathcal{V} \implies V \times \mathbb{I} \in \mathcal{V} \otimes \tau_0 \implies \varphi^{-1}[V] \times \mathbb{I} = \langle\varphi, id_{\mathbb{I}}\rangle^{-1}[V \times \mathbb{I}] \in \tau$ since $\langle\varphi, id_{\mathbb{I}}\rangle$ is a **SF-Ditop** morphism, whence $\varphi^{-1}[V] \in \tau^1$ as required. $\square$

There are other natural choices for the ditopology (τ_0,κ_0), and we will return to the family of functors $\mathfrak{F}_{(\tau_0,\kappa_0)}$ again later on.

The following results will be useful when working directly in terms of **SF-Top**.

3.10. Lemma.

(1) $\tau_T^1 = \{G \subseteq X \mid (\chi_G, G) \in T\}$ *and*
$\tau_T^2 = \{[0,r) \mid (\mathbf{r}, \emptyset) \in T\} \cup \{[0,s] \mid (\mathbf{s}, X) \in T\}$,

(2) *The following are equivalent:*

(i) $\tau_T = \tau_T^1 \otimes \tau_T^2$.

(ii) *For* $(h,H) \in T$ *and* $(x,r) \in X \times \mathbb{I}$ *satisfying* $(x_r, \{x\}) \in (h,H)$ *there exists* $Y \subseteq X$ *with* $(\chi_Y, Y) \in T$ *and* $s \in \mathbb{I}$ *so that*

$$(x_r, \{x\}) \in (\chi_Y \wedge \mathbf{s}, \emptyset) \sqsubseteq (h,H),\ (\mathbf{s}, \emptyset) \in T,\ \textit{or}$$

$$(x_r, \{x\}) \in (\chi_Y \wedge \mathbf{s}, Y) \sqsubseteq (h,H),\ (\mathbf{s}, X) \in T.$$

(iii) *There exists a subbase* B *of* T *so that for* $(h,H) \in B$ *and* $(x,r) \in X \times \mathbb{I}$ *with* $(x_r, \{x\}) \sqsubseteq (h,H)$ *there exist* $Y \subseteq X$ *and* $s \in \mathbb{I}$ *as in (ii).*

(iv) $(X,T) = \mathfrak{F}_{(\tau_T^2,\kappa_T^2)}(\mathfrak{G}(X,T))$.

Proof. (1) Clear since $A_{\xi(\chi_G,G))} = G \times \mathbb{I}$, $A_{\xi(\mathbf{r},\emptyset)} = X \times [0,r)$ and $A_{\xi(\mathbf{r},X)} = X \times [0,r]$.

(2) (i) $\Longrightarrow$ (ii). Since $(x_r, \{x\}) \in (h,H)$ we have $P_{(x,r)} \subseteq A_{\xi(h,H)}$. Also $A_{\xi(h,H)} \in \tau_T$, so we have $G_1 \in \tau_T^1$, $G_2 \in \tau_T^2$ with $P_{(x,r)} \subseteq (G_1 \times \mathbb{I}) \cap (X \times G_2) \subseteq A_{\xi(h,H)}$. By (1), $(\chi_{G_1}, G_1) \in T$ so we may take $Y = G_1$, whence $Y \subseteq X$ and $(\chi_Y, Y) \in T$. There are two cases to consider:

Case a. $G_2 = [0,s)$ for some $s \in \mathbb{I}$. Then $(\mathbf{s}, \emptyset) \in T$ by (1), $P_{(x,r)} \subseteq (Y \times \mathbb{I}) \cap (X \times [0,s)) \subseteq A_{\xi(h,H)}$. Since $A_{\xi(\chi_Y,Y)} = Y \times \mathbb{I}$ and $A_{\xi(\mathbf{s},\emptyset)} = X \times [0,s)$ we have $(Y \times \mathbb{I}) \cap (X \times [0,s)) = A_{\xi((\chi_Y,Y)\sqcap(\mathbf{s},\emptyset))} = A_{\xi(\chi_Y \wedge \mathbf{s},\emptyset)}$, whence $(x_r, \{x\}) \in (\chi_Y \wedge \mathbf{s}, \emptyset) \sqsubseteq (h,H)$, as required.

Case b. $G_2 = X \times [0,s]$ for some $s \in \mathbb{I}$. Then $(\mathbf{s}, X) \in T$, $P_{(x,r)} \subseteq (Y \times \mathbb{I}) \cap (X \times [0,s])A_{\subseteq\xi(h,H)}$. Since $A_{\xi(\chi_Y,Y)} = Y \times \mathbb{I}$ and $A_{\xi(\mathbf{s},X)} = X \times [0,s]$ we have $(Y \times \mathbb{I}) \cap (X \times [0,s]) = A_{\xi((\chi_Y,Y)\sqcap(\mathbf{s},X))} = A_{\xi(\chi_Y \wedge \mathbf{s},Y)}$, whence $(x_r, \{x\}) \in (\chi_Y \wedge \mathbf{s}, Y) \sqsubseteq (h,H)$, as required.

(ii) $\Longrightarrow$ (iii). Immediate.

(iii) $\Longrightarrow$ (i). Take $(h,H) \in T$ and $(x,r) \in X \times \mathbb{I}$ with $P_{(x,r)} \subseteq A_{\xi(h,H)}$. Suppose that B is a subbase of T for which (ii) holds. Then there exist $(h_i^j, H_i^j) \in B$, $i \in I_j$, I_j

finite, $j \in J$, for which

$$(h,H) = \bigsqcup_{j\in J}\Big(\bigsqcap_{i\in I_j}(h_i^j, H_i^j)\Big).$$

Now $A_{\xi(h,H)} = \bigcup_{j\in J}\big(\bigcap_{i\in I_j} A_{\xi(h_i^j,H_i^j)}\big)$, so $P_{(x,r)} \subseteq \bigcup_{j\in J}\big(\bigcap_{i\in I_j} A_{\xi(h_i^j,H_i^j)}\big)$, and there exists $j \in J$ for which $P_{(x,r)} \subseteq \bigcap_{i\in I_j} A_{\xi(h_i^j,H_i^j)}$. It follows that for this j, $P_{(x,r)} \subseteq A_{\xi(h_i^j,H_i^j)}$ for each $i \in I_j$. By (ii) we have $Y_i \subseteq X$ with $(\chi_{Y_i}, Y_i) \in T$, and $s_j \in \mathbb{I}$ having the stated properties. Let $Y = \bigcap_{i\in I_j} Y_i$. Then,

$$A_{\xi(\chi_Y,Y)} = Y \times \mathbb{I} = \bigcap_{i\in I_j}(Y_i \times \mathbb{I}) = A_{\xi\left(\bigsqcap_{i\in I_j}(\chi_{Y_i},Y_i)\right)},$$

so $(\chi_Y, Y) = \bigsqcap_{i\in I_j}(\chi_{Y_i}, Y_i) \in T$, since I_j is finite. Hence $Y \in \tau_T^1$ and clearly $x \in Y$. Let $s = \min\{s_i \mid i \in I_j\}$. There are two cases to consider.

Case 1. There exists $k \in I_j$ with $s = s_k$, $(\mathbf{s_k}, \emptyset) \in T$ and $(x_r, \{x\}) \in (\chi_{Y_k} \wedge \mathbf{s}, \emptyset)$. Now $r < s$, so since $x \in Y$ we have $(x_r, \{x\}) \in (\chi_Y \wedge \mathbf{s}, \emptyset)$. On the other hand, for $i \in I_j$ we have $Y \subseteq Y_i$, $s \le s_i$, so $(\chi_Y \wedge s, \emptyset) \subseteq (\chi_{Y_i} \wedge \mathbf{s_i}, \emptyset) \sqsubseteq (h_i^j, H_i^j)$ or $(\chi_Y \wedge s, \emptyset) \subseteq (\chi_{Y_i} \wedge \mathbf{s_i}, Y_i) \sqsubseteq (h_i^j, H_i^j)$. Hence $(\chi_Y \wedge \mathbf{s}, \emptyset) \sqsubseteq \bigsqcap_{i\in I_j}(h_i^j, H_i^j) \sqsubseteq (h,H)$, so $P_{(x,r)} \subseteq (Y \times \mathbb{I}) \cap (X \times [0,s)) \subseteq A_{\xi(h,H)}$. Here $Y \in \tau_T^1$, $[0,s) \in \tau_T^2$.

Case 2. For all $i \in I_j$ with $s = s_i$, $(\mathbf{s_i}, X) \in T$ and $(x_r, \{x\}) \sqsubseteq (\chi_{Y_i} \wedge \mathbf{s_i}, Y_i)$. Now $(x_r, \{x\}) \in (\chi_Y \wedge \mathbf{s}, Y)$, while if $(\mathbf{s_i}, \emptyset) \in T$ then by hypothesis $s < s_i$. Hence $(\chi_Y \wedge \mathbf{s}, Y) \sqsubseteq (\chi_{Y_i} \wedge \mathbf{s_i}, \emptyset) \sqsubseteq (h_i^j, H_i^j)$ or $(\chi_Y \wedge \mathbf{s}, Y) \sqsubseteq (\chi_{T_i} \wedge \mathbf{s_i}, Y_i) \sqsubseteq (h_i^j, H_i^j)$ for each $i \in I_j$. As in Case 1, $(x_r, \{x\}) \in (\chi_Y \wedge \mathbf{s}, Y) \sqsubseteq (h,H)$, so $P_{(x,r)} \subseteq (Y \times \mathbb{I}) \cap (X \times [0,s]) \subseteq A_{\xi(h,H)}$. Here $Y \in \tau_T^1$, $[0,s] \in \tau_T^2$.

This completes the proof that $\tau_T = \tau_T^1 \otimes \tau_T^2$.

(i) $\Longleftrightarrow$ (iv). Immediate from the definitions. $\square$

3.11. Definition. An SF-topology T is called *productive* if it satisfies one, and hence all, of the equivalent conditions of Lemma 3.10 (2).

4. Separation and compactness of SF-topological spaces

There are well-established separation axioms for ditopological texture spaces [7] which apply, in particular, to spaces in the subcategory $\mathbf{cifPDitop_{SF}}$. Hence we may use the isomorphism $\mathfrak{E} : \mathbf{SF\text{-}Top} \to \mathbf{cifPDitop_{SF}}$ to define corresponding axioms for SF-topologies. To illustrate this process we give the details for the R_0 and co-R_0 axioms.

4.1. Proposition. *Let T be an SF-topology on X. Then the ditopology $(A_{\xi[T]}, A_{\xi[T']})$ on $(X \times \mathbb{I}, \mathcal{P}(X) \otimes \mathcal{I}, \sigma_X)$ is R_0 if and only if*

$$(g,G) \in T,\ (x_s, \{x\}) \in (g,G) \implies \overline{(x_s, \{x\})} \sqsubseteq (g,G),$$

and co-R_0 if and only if

$$(k,K) \in T',\ (x_s, \{x\}) \notin (k,K) \implies (k,K) \sqsubseteq \{x^s, X \setminus \{x\})^\circ.$$

Proof. Under the isomorphism $\mathfrak{E}$, $(g,G) \in T$ corresponds to $A_{\xi(g,G)} \in A_{\xi[T]}$ and $(x_s, \{x\}) \in (g,G)$ to $P_{(x,s)} \subseteq A_{\xi(g,G)}$, and hence to $A_{\xi(g,G)} \not\subseteq Q_{(x,s)}$ since we are dealing with a plain texture. The R_0 axiom now gives $[P_{(x,s)}] \subseteq A_{\xi(g,G)}$, which corresponds to $\overline{(x_s, \{x\})} \sqsubseteq (g,G)$, as required. The result for the co-R_0 axiom follows in a similar way, and we omit the details. $\square$

It is shown in [7, Corollary 3.5] that for complemented ditopologies the notions of R_0 and co-R_0 coincide. Since the ditopology $(A_{\xi[T]}, A_{\xi[T']})$ is complemented this will be the case here, so we need only consider the R_0 axiom for SF-topologies, regarding the ditopological co-R_0 axiom as giving an alternative description of the R_0 axiom. Hence we may make the following definition:

4.2. Definition. An SF–topology T on X is said to be R_0 if it satisfies either, and hence both, of the following equivalent conditions,

(a) $(g,G) \in T$, $(x_s, \{x\}) \in (g,G) \implies \overline{(x_s,\{x\})} \sqsubseteq (g,G)$,
(b) $(k,K) \in T'$, $(x_s,\{x\}) \notin (k,K) \implies (k,K) \sqsubseteq \ \{x^s, X \setminus \{x\})^{\circ}$.

Further equivalent conditions for the ditopological R_0 and co-R_0 axioms are given in [7, Lemma 3.4], and these translate easily under the isomorphism $\mathfrak{E}_f$ to equivalent conditions for an SF-topology to be R_0:

4.3. Proposition. *Let T be an SF-topology on X. Then (X,T) is R_0 if and only if one, and hence all, of the following equivalent conditions hold.*

(i) *For $(g,G) \in T$ there are sets $(k_i, K_i) \in T'$, $i \in I$, with $(g,G) = \bigsqcup_{i\in I}(k_i,K_i)$.*
(ii) *Given $(g,G) \in T$, $s \in \mathbb{I}$ with $(x_s,\{x\}) \in (g,G)$ there exists $(k,K) \in T'$ with $(k,K) \sqsubseteq (g,G)$ and $(x_s,\{x\}) \in (k,K)$.*
(iii) *For $(k,K) \in T'$ there are sets $(g_i,G_i) \in T$, $i \in I$, with $(k,K) = \bigsqcap_{i\in I}(g_i,G_i)$.*
(iv) *Given $(k,K) \in T'$, $(x,s) \in X \times \mathbb{I}$ with $(x_s,\{x\}) \not\sqsubseteq (f,F)$ there exists $(g,G) \in T$ with $(k,K) \subseteq (g,G)$ and $(x_s,\{x\}) \not\sqsubseteq (g,G)$.*

Using the above treatment of the R_0 axiom as a guide, we now give the R_1 and regularity axioms without discussing the link with the corresponding ditopological axioms in detail.

4.4. Definition. An SF–topology T on X is said to be R_1 if it satisfies either, and hence both, of the following equivalent conditions,

(a) $(g,G) \in T$, $(x_s,\{x\}) \in (g,G)$, $\{y_t,\{y\}) \notin (g,G) \implies \exists\,(h,H) \in T$ with $(x_s,\{x\}) \in (h,H)$ and $\{y_t,\{y\}) \notin \overline{(h,H)}$.
(b) $(k,K) \in T'$, $(x_s,\{x\}) \notin (k,K)$, $\{y_t,\{y\}) \in (k,K) \implies \exists\,(f,F) \in T'$ with $(x_s,\{x\}) \notin (f,F)$ and $\{y_t,\{y\}) \in (f,F)^{\circ}$.

From [7, Lemma 3.7] we have the further equivalent conditions given below.

4.5. Proposition. *Let T be SF-topology on X. Then (X,T) is R_1 if and only if it satisfies any one of the following equivalent conditions.*

(i) *Given $(g,G) \in T$, $(x_s,\{x\}) \in (g,G)$ and $\{y_t,\{y\}) \notin (g,G)$ we have $(h,H) \in T$ with $(x_s,\{x\}) \in (h,H) \sqsubseteq \overline{(h,H)} \sqsubseteq (y^t, X\setminus\{y\})$.*
(ii) *For $(g,G) \in T$ we have $(h^i_j, H^i_j) \in T$, $j \in J_i$, $i \in I$, with*

$$(g,G) = \bigsqcup_{i\in I}\bigsqcap_{j\in J_i}(h^i_j,H^i_j) = \bigsqcup_{i\in I}\bigsqcap_{j\in J_i}\overline{(h^i_j,H^i_j)}.$$

(iii) *Given $(k,K) \in T'$, $(x_s,\{x\}) \notin (k,K)$ and $(y_t,\{y\}) \in (k,K)$ we have $(f,F) \in T'$ with $(y_t,\{y\}) \in (f,F)^{\circ} \sqsubseteq (k,K) \sqsubseteq (x^s, X\setminus\{x\})$.*
(iv) *For $(k,K) \in T'$ we have $(f^i_j,F^i_j) \in T'$, $j \in J_i$, $i \in I$ with*

$$(k,K) = \bigsqcap_{i\in I}\bigsqcup_{j\in J_i}(f^i_j,F^i_j) = \bigsqcap_{i\in I}\bigsqcup_{j\in J_i}(f^i_j,F^i_j)^{\circ}.$$

4.6. Definition. An SF–topology T on X is said to be *regular* if it satisfies either, and hence both, of the following equivalent conditions,

(a) $(g,G) \in T$, $(x_s,\{x\}) \in (g,G) \implies \exists (h,H) \in T$ with $(x_s,\{x\}) \in (h,H))$, $\overline{(h,H)} \sqsubseteq (g,G)$.
(b) $(k,K) \in T'$, $(x_s,\{x\}) \notin (k,K) \implies \exists (f,F) \in T'$ with $(x_s,\{x\}) \notin (f,F)$, $(f,F) \sqsubseteq (k,K)^\circ$.

According to [7, Lemma 3.10] we have:

4.7. Proposition. *Let T be a SF-topology on X. Then (X,T) is regular if and only if it satisfies either of the following equivalent conditions:*

(i) *For $(g,G) \in T$ we have $(h_i,H_i) \in T$, $i \in I$, with*
$$G = \bigsqcup_{i\in I}(h_i,H_i) = \bigsqcup_{i\in I}\overline{(h_i,H_i)}.$$
(ii) *For $(k,K) \in T'$ we have $(f_i,F_i) \in T'$, $i \in I$, with*
$$(k,K) = \bigsqcap_{i\in I}(f_i,F_i) = \bigsqcap_{i\in I}(f_i,F_i)^\circ.$$

It is clear from the definitions that
$$\text{regular} \implies R_1 \implies R_0.$$
We now turn to the T_0 axiom, which is a self-dual property of ditopological texture spaces. In [7, Theorem 4.7] several equivalent conditions for a ditopological texture space to be T_0 are given, one of which holds only for coseparated textures [7, Definition 4.2], that is for textures satisfying $Q_s \not\subseteq Q_t \iff P_s \not\subseteq P_t$. Since by [7, Lemma 4.3] every plain texture is coseparated, this condition is included in the following definition of the T_0 axiom for SF-topological spaces.

4.8. Definition. A SF-topology T on X is T_0 if it satisfies one, and hence all, of the following equivalent conditions.

(a) $(x_s,\{x\}) \notin (y_t,\{t\}) \implies \exists\,(b,B) \in T \cup T'$ with $(x_s,\{x\}) \notin (b,B)$ and $(y_t,\{y\}) \in (b,B)$.
(b) $(x_s,\{x\}) \notin (y_t,\{x\}) \implies \exists\,(b_i,B_i) \in T \cup T'$, $i \in I$, with $(y_t,\{y\}) \in \bigsqcup_{i\in I}(b_i,B_i) \sqsubseteq (x^s, X \setminus \{x\})$.
(c) $(x_s,\{x\}) \notin (y_t,\{y\})) \implies \exists\,(b_i,B_i) \in T \cup T'$, $i \in I$, with $(y_t,\{y\}) \in \bigsqcap_{i\in I}(b_i,B_i) \sqsubseteq (x^s, X \setminus \{x\})$.
(d) For $(\mu,M) \in SF(X)$ there exist $(b^i_j,B^i_j) \in T \cup T'$, $i \in I$, $j \in J_i$, with $(\mu,M) = \bigsqcup_{i\in I}\bigsqcap_{j\in J_i}(b^i_j,B^i_j)$.
(e) $\overline{(x_s,\{x\})} \sqsubseteq \overline{(y_t,\{y\})}$ and $(x^s, X\setminus\{x\})^\circ \sqsubseteq (y^t, X \setminus \{y\})^\circ \implies (x_s,\{x\}) \in (y_t,\{y\})$.
(f) For all copoints $(x^s, X\setminus\{x\})$ there exist $(b_i,B_i) \in T \cup T'$, $i \in I$, with $(x^s, X \setminus \{x\}) = \bigsqcup_{i\in I}(b_i,B_i)$.
(g) For all points $(x_s,\{x\})$ there exist $(b_i,B_i) \in T \cup T'$, $i \in I$, with $(x_s,\{x\}) = \bigsqcap_{i\in I}(b_i,B_i)$.

Now we may give:

4.9. Definition. An SF-topology is called:

(1) T_1 if it is T_0 and R_0.
(2) T_2 if it is T_0 and R_1.
(3) T_3 if it is T_0 and regular.

Clearly,
$$T_3 \implies T_2 \implies T_1 \implies T_0,$$

exactly as for classical topology. According to [7, Theorem 4.11] we have the following characterizations of the T_1 property.

4.10. Proposition. *An SF-topology T on X is T_1 if and only if it satisfies one, and hence all, of the equivalent conditions below:*

(i) *For any $(\mu, M) \in SF(X)$ we have $(f_i, F_i) \in T'$, $i \in I$, with $(\mu, M) = \bigsqcup_{i \in I}(f_i, F_i)$.*
(ii) $(x_s, \{x\}) \notin (y_t, \{y\}) \implies \exists\,(f, F) \in T'$ *satisfying* $(x_s, \{x\}) \notin (f, F) \not\sqsubseteq (y^t, X \setminus \{y\})$.
(iii) $(x_s, \{x\}) \in T'$ *for all points* $(x_s, \{x\})$.
(iv) *For any $(\mu, M) \in SF(X)$ we have $(g_i, G_i) \in T$, $i \in I$, with $(\mu, M) = \bigsqcap_{i \in I}(g_i, G_i)$.*
(v) $(x_s, \{x\})) \notin (y_t, \{y\})) \implies \exists\,(g, G) \in T$ *satisfying* $(x_s, \{x\}) \notin (g, G) \not\sqsubseteq (y^t, X \setminus \{y\})$.
(vi) $(x^s, X \setminus \{x\}) \in T$ *for all copoints* $(x^s, X \setminus \{x\})$.

Finally, [7, Theorem 4.17] leads to the following characterizations of the T_2 or Hausdorff property for SF-topological spaces.

4.11. Proposition. *An SF-topology T on X is T_2 if and only if it satisfies either, and hence both, of the following equivalent conditions.*

(a) $(x_s, \{x\}) \notin (y_t, \{y\}) \implies \exists\,(h, H) \in T$ *and* $(k, K) \in T'$ *with* $(h, H) \sqsubseteq (k, K)$, $(x_s, \{x\}) \notin (k, K)$ *and* $(y_t, \{y\}) \in (h, H)$.
(b) *For* $(\mu, M) \in SF(X)$ *there exist* $(h^i_j, H^i_j) \in T$, $(k^i_j, K^i_j) \in T'$, $i \in I$, $j \in J_i$, *with* $(h^i_j, H^i_j) \sqsubseteq (k^i_j, K^i_j)$ *for all* i, j *so that*
$$(\mu, M) = \bigsqcup_{i \in I} \bigsqcap_{j \in J_i} (h^i_j, H^i_j) = \bigsqcup_{i \in I} \bigsqcap_{j \in J_i} (k^i_j, K^i_j).$$

Finally we note the following:

4.12. Definition. The SF-topology T on X is *normal* if given $(k, K) \in T'$, $(g, G) \in T$ with $(k, K) \sqsubseteq (g, G)$ there exists $(h, H) \in T$ satisfying $(f, F) \sqsubseteq (h, H) \sqsubseteq \overline{(h, H)} \sqsubseteq (g, G)$.

It should be noted that [7] also gives a notion of complete regularity for ditopological texture spaces. However, we do not discuss the corresponding property for SF-topological spaces in the present paper.

We now turn to compactness. Notions of compactness, cocompactness; and of stability, costability for ditopological texture spaces are given in [2], see also [11], the terminology being adapted from that used by R. D. Kopperman in [10] for the corresponding bitopological properties. Again, compactness and cocompactness, stability and costability coincide for complemented ditopologies, so we need only define corresponding notions of compactness and stability for SF-topological spaces.

4.13. Definition. Let T be a SF-topology on X. Then (X, T) is called:

(1) *Compact* if it satisfies either, and hence both, of the following equivalent conditions:
  (a) Whenever $\bigsqcup_{i \in I}(g_i, G_i) = (\mathbf{1}, X)$, $(g_i, G_i) \in T$, $i \in I$, there is a finite subset J of I with $\bigsqcup_{j \in J}(g_j, G_j) = (\mathbf{1}, X)$.
  (b) Whenever $\bigsqcap_{i \in I}(f_i, F_i) = (\mathbf{0}, \emptyset)$, $(f_i, F_i) \in T'$, $i \in I$, there is a finite subset J of I with $\bigsqcap_{j \in J}(f_j, F_j) = (\mathbf{0}, \emptyset)$.
(2) *Stable* if it satisfies either, and hence both, of the following equivalent conditions:
  (a) For $(k, K) \in T' \setminus \{(\mathbf{1}, X)\}$, whenever $(k, K) \sqsubseteq \bigsqcup_{i \in I}(g_i, G_i)$, $(g_i, G_i) \in T$, $i \in I$, there is a finite subset J of I with $(k, K) \sqsubseteq \bigsqcup_{j \in J}(g_j, G_j)$.
  (b) For $(g, G) \in T \setminus \{(\mathbf{0}, \emptyset)\}$, whenever $\bigsqcap_{i \in I}(f_i, F_i) \sqsubseteq (g, G)$, $(f_i, F_i) \in T'$, $i \in I$, there is a finite subset J of I with $\bigsqcap_{j \in J}(f_j, F_j) \sqsubseteq (g, G)$.

4.14. Proposition.

(1) *A stable R_1 SF-topology is regular.*
(2) *A regular stable SF-topology is normal.*

Proof. This follows immediately from the corresponding results for ditopological texture spaces presented in [11]. □

A ditopological space that satisfies all four of the properties compact, cocompact, stable and costable is called *dicompact*. There is good evidence that dicompactness is an extremely important compactness notion in ditopological texture spaces, just as its counterpart is in bitopological spaces, and as mentioned previously one of our main motivations in replacing the texture $(L, \mathcal{L}, \lambda)$ by the unit interval texture $(\mathbb{I}, \mathcal{I}, \iota)$ as the foundation of a theory of fuzzy sets is that the natural ditopology $(\tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ on $(\mathbb{I}, \mathcal{I}, \iota)$ is dicompact. An important consequence of this fact will become apparent later in connection with a concept of Lowen functor for SF-topologies.

5. The Lowen Functors

In this section we define and study functors between **Top** and **SF-Top** analogous to the Lowen functors ω and ι between **Top** and $\mathbb{I}$**-Top** [12, 13, 14].

5.1. Definition. Let $\mathcal{T}$ be a topology on X. We denote by $\omega(\mathcal{T})$ the SF-topology on X with subbase $B_{\mathcal{T}} = \{(\mu, M) \in SF(X) \mid \{x \mid (x_r, \{x\}) \in (\mu, M)\} \in \mathcal{T}\ \forall\, r \in \mathbb{I}\}$, and speak of $\mathcal{T} \mapsto \omega(\mathcal{T})$ as the (*generalized*) *Lowen mapping* ω.

Suppose that $(X, \mathcal{T})$, $(Y, \mathcal{V})$ are topological spaces and that $\varphi : X \to Y$ is $\mathcal{T}$–$\mathcal{V}$ continuous. By Lemma 3.5 it is easy to see that for $(\nu, N) \in B_{\mathcal{V}}$, $r \in \mathbb{I}$, we have

$$\{x \mid (x_r, \{x\}) \in \varphi^{\leftarrow}(\nu, N)\} = \varphi^{-1}[\{y \mid (y_r, \{y\}) \in (\nu, N)\}].$$

On the other hand, $\{y \mid (y_r, \{y\}) \in (\nu, N)\} \in \mathcal{V}$, so $\{x \mid (x_r, \{x\}) \in \varphi^{\leftarrow}(\nu, N)\} \in \mathcal{T}$ for all $r \in \mathbb{I}$. This shows that $\varphi^{\leftarrow}(\nu, N) \in B_{\mathcal{T}} \subseteq \omega(\mathcal{T})$, whence φ is $\omega(\mathcal{T})$–$\omega(\mathcal{V})$ is continuous. Hence the Lowen mapping ω may be regarded as the functor from **Top** to **SF-Top** that takes $(X, \mathcal{T}) \in \mathrm{Ob}\,\mathbf{Top}$ to $(X, \omega(\mathcal{T})) \in \mathrm{Ob}\,\mathbf{SF\text{-}Top}$, and $\varphi \in \mathbf{Top}((X, \mathcal{T}), (Y, \mathcal{V}))$ to $\varphi \in \mathbf{SF\text{-}Top}((X, \omega(\mathcal{T})), (Y, \omega(\mathcal{V})))$.

5.2. Definition. Let T be an SF-topology on X. We denote by $\iota(T)$ the topology on X with subbase $\{\{x \mid (x_r, \{x\}) \in (\mu, M)\} \mid (\mu, M) \in T,\ r \in \mathbb{I}\}$, and speak of $T \mapsto \iota(T)$ as the (*generalized*) *Lowen mapping* ι.

Suppose that (X, T), (Y, V) are SF–topological spaces and that $\varphi : X \to Y$ is T–V continuous. Take $(\nu, N) \in V$, $r \in \mathbb{I}$, so that $\{y \mid (y_r, \{y\}) \in (\nu, N)\}$ is a sub-basic element of $\iota(V)$. By the above equality $\varphi^{-1}[\{y \mid (y_r, \{y\}) \in (\nu, N)\}] \in \iota(T)$ since $\varphi^{\leftarrow}(\nu, N) \in T$ and so $\{x \mid (x_r, \{x\}) \in \varphi^{\leftarrow}(\nu, N)\} \in \iota(T)$ for each $r \in \mathbb{I}$. This verifies that φ is $\iota(T)$– $\iota(V)$ continuous. Hence the Lowen mapping ι may be regarded as the functor from **SF-Top** to **Top** that takes $(X, T) \in \mathrm{Ob}\,\mathbf{SF\text{-}Top}$ to $(X, \iota(T)) \in \mathrm{Ob}\,\mathbf{Top}$, and $\varphi \in \mathbf{SF\text{-}Top}((X, T), (Y, V))$ to $\varphi \in \mathbf{Top}((X, \iota(T)), (Y, \iota(V)))$.

As for the classical Lowen functors (see [14]) we have:

5.3. Theorem. *The Lowen functor ι is an adjoint and the Lowen functor ω a co-adjoint.*

Proof. Take $(X, \mathcal{T}) \in \mathrm{Ob}\,\mathbf{Top}$. We claim that $(id_X, (X, \omega(\mathcal{T}))$ is a universal arrow with domain $(X, \mathcal{T})$. To prove this we need to first verify that the identity point function $id_X : X \to X$ is $\mathcal{T}$–$\iota(\omega(\mathcal{T}))$ continuous, and for this it is clearly sufficient to note that for $(\mu, M) \in B_{\mathcal{T}}$ we have $\{x \mid (x_r, \{x\}) \in (\mu, M)\} \in \mathcal{T}$ for all $r \in \mathbb{I}$. To show that ι is an adjoint it remains only to show that the structured arrow $(id_X, (X, \omega(\mathcal{T}))$ has the universal property. Hence, let $(\varphi, (Y, V))$ be any structured arrow with domain $(X, \mathfrak{T})$. We must prove the existence of a unique **SF-Top** morphism $\overline{\varphi} : X \to Y$ making the following diagram commutative.

$$\begin{tikzcd}
(X, \mathfrak{T}) \arrow[r, "id_X"] \arrow[dr, "f"'] & (X, \iota(\omega(\mathfrak{T}))) \arrow[d, dashed, "\iota(\overline{\varphi})"] \\
 & (Y, \iota(V))
\end{tikzcd}$$

Clearly the only possible choice for $\overline{\varphi}$ is φ, so we must show that φ is $\omega(\mathfrak{T})$–V continuous. Since φ is $\mathfrak{T}$–$\iota(V)$ continuous, for $(\nu, N) \in V$ and $r \in \mathbb{I}$ we have $\varphi^{-1}[\{y \mid (y_r, \{y\}) \in (\nu, N)\}] \in \mathfrak{T}$. But by Lemma 3.5 we deduce $\{x \mid (x_r, \{x\}) \in \varphi^{\leftarrow}(\nu, N)\} \in \mathfrak{T}$ for all $r \in \mathbb{I}$, whence $\varphi^{\leftarrow}(\nu, N) \in \omega(\mathfrak{T})$ for all $(\nu, N) \in V$, which establishes the $\omega(\mathfrak{T})$–V continuity of φ.

Hence, ι is an adjoint, and by [1, Theorem 19.1] we may deduce that ω is a co-adjoint. □

5.4. Proposition. *Let $\mathfrak{T}$ be a topology on X and ω the Lowen mapping. Then:*

(1) *For $s \in \mathbb{I}$ we have $(\mathbf{s}, \emptyset) \in \omega(\mathfrak{T})$ and $(\mathbf{s}, X) \in \omega(\mathfrak{T})$, where $\mathbf{s}$ is the constant function on X with value s.*
(2) *For $G \in \mathfrak{T}$ we have $(\chi_G, G) \in \omega(\mathfrak{T})$.*
(3) *The SF-topology $\omega(\mathfrak{T})$ is productive.*

Proof. (1) It is clear that

$$\{x \mid (x_r, \{x\}) \in (\mathbf{s}, \emptyset)\} = \begin{cases} X & \text{if } r < s, \\ \emptyset & \text{otherwise}, \end{cases},$$

$$\{x \mid (x_r, \{x\}) \in (\mathbf{s}, X)\} = \begin{cases} X & \text{if } r \leq s, \\ \emptyset & \text{otherwise}, \end{cases}$$

and so these soft fuzzy sets belong to $B_{\mathfrak{T}}$, and therefore to $\omega(\mathfrak{T})$.

(2). Clear since $\{x \mid (x_r, \{x\}) \in (\chi_G, G)\} = G \in \mathfrak{T}$ for all $r \in \mathbb{I}$.

(3) Take $(x, r) \in X \times \mathbb{I}$ and $(\mu, M) \in B_{\mathfrak{T}}$ with $(x_r, \{x\}) \in (\mu, M)$. Define $Y = \{y \in X \mid (y_r, \{y\}) \in (\mu, M)\}$. Then $x \in Y$, and $Y \in \mathfrak{T}$ by the definition of $B_{\mathfrak{T}}$. Since $\{u \in X \mid (u_s, \{u\}) \in (\chi_Y, Y)\} = Y \in \mathfrak{T}$ for all $s \in \mathbb{I}$ we deduce that $(\chi_Y, Y) \in B_{\mathfrak{T}} \subseteq \omega(\mathfrak{T})$. Finally, for $s = r$ we have $(\mathbf{s}, X) \in \omega(\mathfrak{T})$ by (1), and it is easy to verify that

$$(x_r, \{x\}) \in (\chi_Y \wedge \mathbf{s}, Y) \sqsubseteq (\mu, M),$$

so the conditions of Lemma 3.10 (2 iii) hold for the subbase $B_{\mathfrak{T}}$ of Definition 5.1. Hence, $\omega(\mathfrak{T})$ is productive. □

5.5. Corollary. *Let $\mathfrak{T}$ be a topology on X and ω the Lowen mapping. Then $\omega(\mathfrak{T})$ satisfies $\tau_{\omega(\mathfrak{T})} = \mathfrak{T} \otimes \mathfrak{I}$, $\kappa_{\omega(\mathfrak{T})} = \mathfrak{T}^c \otimes \mathfrak{I}$. In particular, the set*

$$B^* = \{(\mathbf{s}, \emptyset) \mid s \in \mathbb{I}\} \cup \{(\mathbf{s}, X) \mid s \in \mathbb{I}\} \cup \{(\chi_Y, Y) \mid Y \in \mathfrak{T}\}$$

is a subbase for $\omega(\mathfrak{T})$.

Proof. By Proposition 5.4 (3) we have $\tau_{\omega(\mathfrak{T})} = \tau^1_{\omega(\mathfrak{T})} \times \tau^2_{\omega(\mathfrak{T})}$. By Lemma 3.10 (1) we obtain $\mathfrak{I} = \{[0, s) \mid s \in \mathbb{I}\} \cup \{[0, s] \mid s \in \mathbb{I}\} \subseteq \tau^2_{\omega(\mathfrak{T})} \subseteq \mathfrak{I}$ from Proposition 5.4 (1), and $\mathfrak{T} \subseteq \tau^1_{\omega(\mathfrak{T})}$ from Proposition 5.4 (2). To obtain the first equality it remains to prove that $\tau^1_{\omega(\mathfrak{T})} \subseteq \mathfrak{T}$. Take $G \in \tau^1_{\omega(\mathfrak{T})}$. If $G = \emptyset$ then $G \in \mathfrak{T}$ so assume $G \neq \emptyset$ and take $x \in G$.

Now $(\chi_G, G) \in \omega(\mathcal{T})$ and $(x_1, \{x\}) \in (\chi_G, G)$, so by the definition of subbase there exists $(\mu_i, M_i) \in B$, $1 \le i \le n$ with $(x_1, \{x\}) \in \bigsqcap_{i=1}^n (\mu_i, M_i) \sqsubseteq (\chi_G, G)$. Set

$$Y_i = \{y \mid (y_1, \{y\}) \in (\mu_i, M_i),\ i = 1, 2, \dots, n\}.$$

Then $x \in Y_i \in \mathcal{T}$ and so $x \in \bigcap_{i_1}^n Y_i \in \mathcal{T}$. Finally it is easy to verify that $\bigcap_{i=1}^n Y_i \subseteq G$, so $G \in \mathcal{T}$ as required.

This completes the proof of $\tau_{\omega(\mathcal{T})} = \mathcal{T} \otimes \mathcal{I}$, and $\kappa_{\omega(\mathcal{T})} = \mathcal{T}^c \otimes \mathcal{I}$ follows by taking the complement.

Since the sets $G \times \mathbb{I}$, $G \in \mathcal{T}$, $X \times [0, s)$, $X \times [0, s]$, $s \in \mathbb{I}$ form a subbase for the topology $\mathcal{T} \otimes \mathcal{I}$, the set B^* is a subbase for $\omega(\mathcal{T})$. □

This result shows that the generalized Lowen functor ω involves the discrete, codiscrete ditopology $(\mathcal{I}, \mathcal{I})$ on $(\mathbb{I}, \mathcal{I}, \omega)$, a result which is quite analogous to that for the classical Lowen functor [6, Theorem 5.14]. We wish to show that we may replace ω by a functor that involves the dicompact natural ditopology on $(\mathbb{I}, \mathcal{I}, \omega)$. As a first step we define a special class of soft fuzzy sets.

5.6. Definition. The soft fuzzy set $(\mu, M) \in SF(X)$ is called *rotund* if

$$(x_r, \{x\}) \in (\mu, M),\ r < 1 \implies \exists\, s > r \text{ with } (x_s, \{x\}) \in (\mu, M).$$

We denote by $SF_r(X)$ the set of rotund soft fuzzy subsets of X.

5.7. Example. For $s \in \mathbb{I}$ and $A \subseteq X$ we have

(1) $(\mathbf{s}, \emptyset)$ and (χ_A, A) are rotund.
(2) $(\mathbf{s}, X)$ is not rotund.

Clearly (1) follows from $(x_r, \{x\}) \in (\mathbf{s}, \emptyset) \iff r < s$ and $(x_r, \{x\}) \in (\chi_A, A) \iff x \in A$, while (2) follows from $(x_r, \{x\}) \in (\mathbf{s}, X) \iff r \le s$.

5.8. Lemma. *The family $SF_r(X)$ is closed under arbitrary joins and finite meets.*

Proof. First take $(\mu_j, M_j) \in SF_r(X)$, $j \in J$ and $(x_r, \{x\}) \in \bigsqcup_{j \in J} (\mu_j, M_j)$ with $r < 1$. Then $(x_r, \{x\}) \in (\mu_j, M_j)$ for some $j \in J$, whence we have $r < s$ with $(x_s, \{x\}) \in (\mu_j, M_j) \sqsubseteq \bigsqcup_{j \in J} (\mu_j, M_j)$.

It will be sufficient to prove that the meet of two rotund soft fuzzy sets is rotund, so take $(\mu_1, M_1), (\mu_2, M_2) \in SF_r(X)$ and $r < 1$ with $(x_r, \{x\}) \in (\mu_1, M_1) \sqcap (\mu_2, M_2)$. From $(x_r, \{x\}) \in (\mu_1, M_1)$ we have $r < s_1$ with $(x_{s_1}, \{x\}) \in (\mu_1, M_1)$, and likewise we have $r < s_2$ with $(x_{s_2}, \{x\}) \in (\mu_2, M_2)$. Setting $s = \min\{s_1, s_2\}$ now gives

$$(x_s, \{x\}) \in (x_{s_1}, \{x\}) \sqcap (x_{s_2}, \{x\}) \sqsubseteq (\mu_1, M_1) \sqcap (\mu_2, M_2),$$

whence $(x_s, \{x\}) \in (\mu_1, M_1) \sqcap (\mu_2, M_2)$, as required. □

5.9. Definition. Let $\mathcal{T}$ be a topology on X. We denote by $\omega_r(\mathcal{T})$ the SF–topology on X with subbase $B_r^{\mathcal{T}} = \{(\mu, M) \in SF_r(X) \mid \{x \mid (x_r, \{x\}) \in (\mu, M)\} \in \mathcal{T}\ \forall\, r \in \mathbb{I}\}$, and refer to ω_r as the *Lowen rotund mapping.*

We note that with the notation of Definition 5.1 we have $B_r^{\mathcal{T}} = B_{\mathcal{T}} \cap SF_r(X)$. Since $SF_r(X) \subset SF(X)$ we may again regard ω_r as a functor from **Top** to **SF-Top**.

5.10. Proposition. *Let $\mathcal{T}$ be a topology on X and ω_r the Lowen rotund functor. Then:*

(1) *For $s \in \mathbb{I}$ we have $(\mathbf{s}, \emptyset) \in \omega_r(\mathcal{T})$.*
(2) *For $G \in \mathcal{T}$ we have $(\chi_G, G) \in \omega_r(\mathcal{T})$.*
(3) *The SF-topology $\omega_r(\mathcal{T})$ is productive.*

Proof. (1) and (2) are clear since $(\mathbf{s},\emptyset)$, (χ_G, G), $G \in \mathcal{T}$, belong to $B_{\mathcal{T}}$ by the proof of Proposition 5.4, and they belong to $SF_r(X)$ by Example 5.7.

(3) Take $(x,r) \in X \times \mathbb{I}$ and $(\mu, M) \in B_r^{\mathcal{T}}$ with $(x_r, \{x\}) \in (\mu, M)$. Define $Y = \{y \in X \mid (y_r, \{y\}) \in (\mu, M)\}$. Then $x \in Y$, and $Y \in \mathcal{T}$ by the definition of $B_r^{\mathcal{T}}$. Since $\{u \in X \mid (u_s, \{u\}) \in (\chi_Y, Y)\} = Y \in \mathcal{T}$ for all $s \in \mathbb{I}$ we deduce that $(\chi_Y, Y) \in B_r^{\mathcal{T}} \subseteq \omega_r(\mathcal{T})$. Finally, for $r < s$ we have $(\mathbf{s}, X) \in \omega_r(\mathcal{T})$ by (1), and it is easy to verify that

$$(x_r, \{x\}) \in (\chi_Y \wedge \mathbf{s}, X) \sqsubseteq (\mu, M),$$

so the conditions of Lemma 3.10 (2 iii) hold for the subbase $B_r^{\mathcal{T}}$ of Definition 6.9. Hence, $\omega_r(\mathcal{T})$ is productive. □

5.11. Corollary. *Let $\mathcal{T}$ be a topology on X and ω_r the Lowen rotund functor. Then $\omega_r(\mathcal{T})$ satisfies $\tau_{\omega_r(\mathcal{T})} = \mathcal{T} \otimes \tau_{\mathbb{I}}$, $\kappa_{\omega_r(\mathcal{T})} = \mathcal{T}^c \otimes \kappa_{\mathbb{I}}$. In particular, the set*

$$B_r^* = \{(\mathbf{s},\emptyset) \mid s \in \mathbb{I}\} \cup \{(\chi_Y, Y) \mid Y \in \mathcal{T}\}$$

is a subbase for $\omega_r(\mathcal{T})$.

Proof. By Proposition 5.10 (3) we have $\tau_{\omega_r(\mathcal{T})} = \tau^1_{\omega_r(\mathcal{T})} \times \tau^2_{\omega_r(\mathcal{T})}$. By Lemma 3.10 (1) we obtain $\tau_{\mathbb{I}} = \{[0,s) \mid s \in \mathbb{I}\} \cup \{\mathbb{I}\} \subseteq \tau^2_{\omega_r(\mathcal{T})} \subseteq \tau_{\mathbb{I}}$ from Proposition 5.10 (1), and $\mathcal{T} \subseteq \tau^1_{\omega_r(\mathcal{T})}$ from Proposition 5.10 (2). To obtain the first equality it remains to prove that $\tau^1_{\omega_r(\mathcal{T})} \subseteq \mathcal{T}$. Take $G \in \tau^1_{\omega_r(\mathcal{T})}$. If $G = \emptyset$ then $G \in \mathcal{T}$ so assume $G \neq \emptyset$ and take $x \in G$. Now $(\chi_G, G) \in \omega_r(\mathcal{T})$ and $(x_r, \{x\}) \in (\chi_G, G)$, so by the definition of subbase there exists $(\mu_i, M_i) \in B$, $1 \leq i \leq n$ with $(x_r, \{x\}) \in \bigsqcap_{i=1}^n (\mu_i, M_i) \sqsubseteq (\chi_G, G)$. Set

$$Y_i = \{y \mid (y_1, \{y\}) \in (\mu_i, M_i),\ i = 1, 2, \ldots, n\}.$$

Then $x \in Y_i \in \mathcal{T}$ and so $x \in \bigcap_{i_1}^n Y_i \in \mathcal{T}$. Finally it is easy to verify that $\bigcap_{i=1}^n Y_i \subseteq G$, so $G \in \mathcal{T}$ as required.

This completes the proof of $\tau_{\omega_r(\mathcal{T})} = \mathcal{T} \otimes \tau_{\mathbb{I}}$, and $\kappa_{\omega_r(\mathcal{T})} = \mathcal{T}^c \otimes \kappa_{\mathbb{I}}$ follows by taking the complement.

Since the sets $G \times \mathbb{I}$, $G \in \mathcal{T}$, $X \times [0,s)$, $s \in \mathbb{I}$ form a subbase for the topology $\mathcal{T} \otimes \mathcal{I}$, the set B_r^* is a subbase for $\omega_r(\mathcal{T})$. □

6. The preservation of topological properties by the Lowen functors ω and ω_r

It is well known that the classical Lowen functor ω preserves the separation axioms but not compactness. We will see that the generalized Lowen functor ω from **Top** to **SF-Top** has just the same property, but that the Lowen rotund functor ω_r preserves both the separation axioms and compactness. The existence of a functor from **Top** to **SF-Top** that preserves, in particular, the compact Hausdorff property is one of the most important gains that is achieved by replacing classical fuzzy sets by soft fuzzy sets.

Let us consider the functor $\mathfrak{F}_{(\tau_{\mathbb{I}}, \kappa_{\mathbb{I}})}$: **Top** → **SF-Ditop**, where $(\tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ is the natural ditopology $\tau_{\mathbb{I}} = \{[0,s) \mid s \in \mathbb{I}\} \cup \{\mathbb{I}\}$, $\kappa_{\mathbb{I}} = \{[0,s] \mid s \in \mathbb{I}\} \cup \{\emptyset\}$ on $(\mathbb{I}, \mathcal{I}, \iota)$. Hence, for $(X, \mathcal{T}) \in$ **Top** we have

$$\mathfrak{F}_{(\tau_{\mathbb{I}}, \kappa_{\mathbb{I}})}(X, \mathcal{T}) = (X \times \mathbb{I},\ \mathcal{P}(X) \otimes \mathcal{I},\ \sigma_X,\ \mathcal{T} \otimes \tau_{\mathbb{I}},\ \mathcal{T}^c \otimes \kappa_{\mathbb{I}}).$$

Now by Corollary 5.11 we have $\mathcal{T} \otimes \tau_{\mathbb{I}} = \tau_{\omega_r(\mathcal{T})}$ and $\mathcal{T}^c \otimes \kappa_{\mathbb{I}} = \kappa_{\omega_r(\mathcal{T})}$. On the other hand, for the isomorphism $\mathfrak{E}$: **SF-Top** → **SF-Ditop** we have

$$\mathfrak{E}^{-1}(X \times \mathbb{I},\ \mathcal{P}(X) \otimes \mathcal{I},\ \sigma_X,\ \tau_{\omega_r(\mathcal{T})},\ \kappa_{\omega_r(\mathcal{T})}) = (X, \omega_r(\mathcal{T})).$$

We deduce at once that

$$\omega_r = \mathfrak{E}^{-1} \circ \mathfrak{F}_{(\tau_{\mathbb{I}}, \kappa_{\mathbb{I}})},$$

and likewise, using Corollary 5.5,

$$\omega = \mathfrak{E}^{-1} \circ \mathfrak{F}_{(\mathcal{I}, \mathcal{I})}.$$

Since $\mathfrak{E}_f^{-1}$ is an isomorphism it follows that we may consider $\mathfrak{F}_{(\tau_{\mathbb{I}}, \kappa_{\mathbb{I}})}$, $\mathfrak{F}_{(\mathcal{I}, \mathcal{I})}$ in place of ω_r, ω respectively. Hence the question we must ask is whether the fact that $\mathcal{T}$ has a certain topological property implies that $(\mathcal{T} \otimes \tau_{\mathbb{I}}, \mathcal{T}^c \otimes \kappa_{\mathbb{I}})$ or $(\mathcal{T} \otimes \mathcal{I}, \mathcal{T}^c \otimes \mathcal{I})$ has the corresponding ditopological property.

To be definite we will restrict our attention to the topological properties T_0, R_0, R_1, regular, and compact.

The first thing to notice is that the natural complemented ditopology $(\tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ has all the corresponding separation properties T_0, R_0, R_1 and regular [7], and also the dicompactness property [2]. On the other hand the finer ditopology $(\mathcal{I}, \mathcal{I})$ satisfies the separation axioms, but is not dicompact. Indeed, the set $[0, 1)$ is closed, but the open cover $\{[0, 1 - \epsilon) \mid 0 < s < 1\}$ has no finite subcover. Hence $(\mathcal{I}, \mathcal{I})$ is not stable and hence not dicompact.

Secondly, if $\mathcal{T}$ has one of the properties mentioned above, we must determine if $(\mathcal{T}, \mathcal{T}^c)$ has the corresponding ditopological property.

6.1. Proposition. *Let $(X, \mathcal{T})$ be a topological space and $(X, \mathcal{P}(X), \pi_X, \mathcal{T}, \mathcal{T}^c)$ the corresponding ditopological texture space. Then*

(1) *If $(X, \mathcal{T})$ is T_0, R_0, R_1 or regular then $(X, \mathcal{P}(X), \pi_X, \mathcal{T}, \mathcal{T}^c)$ is respectively T_0, R_0, R_1, or regular.*
(2) *If $(X, \mathcal{T})$ is compact then $(X, \mathcal{P}(X), \pi_X, \mathcal{T}, \mathcal{T}^c)$ is dicompact.*

Proof. (1) Let $(X, \mathcal{T})$ be T_0 and take x, y in X with $Q_x \not\subseteq Q_y$. Then $X \setminus \{x\} \not\subseteq X \setminus \{y\}$ and so $x \neq y$. Since $(X, \mathcal{T})$ is T_0 we have $G \in \mathcal{T}$ with

$$(x \notin G \text{ and } y \in G) \text{ or } (x \in G \text{ and } y \notin G).$$

In the first case take $B = G \in \mathcal{T} \subseteq \tau$, and in the second case take $B = X \setminus G \in \mathcal{T}^c$. Then in either case $P_x \not\subseteq B \not\subseteq Q_y$ and $B \in \mathcal{T} \cup \mathcal{T}^c$, so the ditopology $(\mathcal{T}, \mathcal{T}^c)$ is T_0.

Now let $(X, \mathcal{T})$ be R_0 and take $G \in \mathcal{T}$ and $x \in X$ with $G \not\subseteq Q_x$. Then $x \in G$ and so $K = \overline{\{x\}} \subseteq G$. Then in $(X, \mathcal{P}(X), \pi_X, \mathcal{T}, \mathcal{T}^c)$ we have $P_x \subseteq K \subseteq G$, $K \in \mathcal{T}^c$, whence $[P_x] \subseteq G$ and so $(\mathcal{T}, \mathcal{T}^c)$ is R_0.

Suppose that $(X, \mathcal{T})$ is R_1 and take $G \in \mathcal{T}$, $x, y \in X$ with $G \not\subseteq Q_x$, $P_y \not\subseteq G$. Then $x \notin \overline{\{y\}}$ so since $(X, \mathcal{T})$ is R_1 the points x and y are contained in disjoint open sets. Hence there exists $H \in \mathcal{T}$ with $x \in H$ and $y \notin \overline{H}$. We deduce that $H \not\subseteq Q_x$ and $P_y \not\subseteq [H]$, whence $(\mathcal{T}, \mathcal{T}^c)$ is R_1.

Finally, let $(X, \mathcal{T})$ be regular and take $G \in \mathcal{T}$, $x \in X$ with $G \not\subseteq Q_x$. Then $x \in G$ and since for a regular space each point has a base of closed neighborhoods there exists $H \in \mathcal{T}$ with $x \in H \subseteq \overline{H} \subseteq G$. This gives $H \not\subseteq Q_x$, $[H] \subseteq G$, so $(\mathcal{T}, \mathcal{T}^c)$ is regular.

(2) Let $(X, \mathcal{T})$ be compact. Then every closed subset of X is also compact, so $(\mathcal{T}, \mathcal{T}^c)$ is compact and stable. Since this ditopology is complemented it is also cocompact and costable, so $(\mathcal{T}, \mathcal{T}^c)$ is dicompact. □

In view of the fact that the ditopological properties T_0, R_0, R_1, R_2, regular and dicompact are all productive (see respectively, [7] and [11]) we obtain the following.

6.2. Theorem.

(1) *The Lowen rotund functor* ω_r *preserves the properties* R_0, R_1, *regular,* T_0, T_1, T_2, T_3 *and compactness.*

(2) *The generalized Lowen functor* ω *preserves the properties* R_0, R_1, *regular,* T_0, T_1, T_2 *and* T_3, *but not compactness.*

It is not known if the Lowen functors preserve normality.

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