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NEW GENERALIZED ESTIMATORS FOR THE POPULATION VARIANCE USING AUXILIARY INFORMATION

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Abstract

We suggest a generalized class of estimators for the population variance using an auxiliary variable. Triphati et al. (T.P. Tripathi, Singh, H.P. and Upadhyaya, L.N. *A general method of estimation and its application to the estimation of coefficient of variation*, Statistics in Transition **5**(6), 887–908, 2002) proposed some generalized estimators for a population parameter. By adapting these estimators to population variance, we develop some estimators. We obtain the mean square error (MSE) equation of the proposed estimators. We illustrate the results with an application using original data.

Keywords: Variance estimation, Ratio estimator, Auxiliary variable, Bias, Minimum mean square error.

2000 AMS Classification: 62D05.

1. Introduction

Using auxiliary information in estimation takes advantage of the correlation between the auxiliary variate (x) and the variate of interest (y). When information about the auxiliary variable is available, it increases the precision of the population variance.

The problem of estimating a parameter θ_0 using a single parameter θ_1 was dealt with by Das and Tripathi [1]. They defined the following general classes of estimators when a

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population parameter of the auxiliary variable is known:

$$
\begin{aligned}
&d_1 = \frac{\hat{\theta}_0 - t_1(\hat{\theta}_1 - \theta_1)}{\hat{\theta}_1 - t_2(\hat{\theta}_1 - \theta_1)}(\theta_1)^{\alpha}, \\
&d_2 = \hat{\theta}_0 \, h\left(\frac{\hat{\theta}_1}{\theta_1}\right), \\
&d_3 = h(\hat{\theta}_0, \hat{\theta}_1, \theta_1),
\end{aligned} \tag{1}
$$

where $h(\cdot)$ and $g(\cdot)$ are functions.

In case $\hat{\theta}_0$ and $\hat{\theta}_1$ are unbiased estimators, the minimum mean square error (MSE) of the estimators in these classes is found to be

$$
M_{\min}(d_j) = (1 - \rho_{01}^2) V(\hat{\theta}_0), \; j = 1, 2, 3. \tag{2}
$$

When there is *a priori* information about two parameters, another general class of estimators was suggested by Tripathi et al. [6]. When two parameters (θ_1, θ_2) of the auxiliary variable are known they defined the following class of estimators for θ_0:

$$
d_g^* = g\big(\hat{\theta}_0, \hat{\theta}_1, \hat{\theta}_2\big). \tag{3}
$$

Here, $\hat{\theta}_0$ is the estimate of the variable of interest, $\hat{\theta}_1$ and $\hat{\theta}_2$ are respectively, the estimates of the parameters θ_1 and θ_2 of the auxiliary variable.

To study the properties of d_g^*, Tripathi *et al.* [6] assumed that

(i) $t = (\hat{\theta}_0, \hat{\theta}_1, \hat{\theta}_2)$ assumes values in a closed subset, S, of three dimensional real space containing the point $T = (\theta_0, \theta_1, \theta_2)$.
(ii) $g(\cdot)$ is a function of t such that $g(t)\,|_{t=T} = \theta_0$.
(iii) The function $g(\cdot)$ is continuous and bounded in S.
(iii) The first and second order partial derivatives of $g(\cdot)$ exist and are continuous and bounded.

Expanding $g(\cdot)$ about the point T in a second order Taylor's series, they obtained

$$
\begin{aligned}
d_g^* = \theta_0 &+ (\hat{\theta}_0 - \theta_0) g_1(t) + (\hat{\theta}_1 - \theta_1) g_2(t) + (\hat{\theta}_2 - \theta_2) g_3(t) \\
&+ \frac{1}{2}\left\{(\hat{\theta}_0 - \theta_0)^2 \, g_{11}^2(t^*) + (\hat{\theta}_1 - \theta_1)^2 g_{22}^2(t^*) + (\hat{\theta}_2 - \theta_2)^2 g_{33}^2(t^*)\right\} \\
&+ (\hat{\theta}_0 - \theta_0)(\hat{\theta}_1 - \theta_1) g_{12}(t) + (\hat{\theta}_0 - \theta_0)(\hat{\theta}_2 - \theta_2) g_{13}(t) \\
&+ (\hat{\theta}_1 - \theta_1)(\hat{\theta}_2 - \theta_2) g_{23}(t)
\end{aligned} \tag{4}
$$

for some $t^* \in S$, where g_i, $(i = 1, 2, 3)$ denote the partial derivatives and g_{ij}, $(i, j = 1, 2, 3)$ the second order partial derivatives of the function $g(\cdot)$.

By considering the expectation of both sides and noting $g_1(T) = 1$, the bias of the estimator was obtained as:

$$
\mathrm{Bias}(d_g^*) \cong \mathrm{Bias}(\hat{\theta}_0) g_1(T) + \mathrm{Bias}(\hat{\theta}_1) g_2(T) + \mathrm{Bias}(\hat{\theta}_2) g_3(T).
$$

If we take the square and expected value of Eq. (4) then the MSE of the estimator is found as:

$$
\begin{aligned}
\mathrm{MSE}\left(d_g^*\right) \cong{}& \mathrm{MSE}(\hat{\theta}_0) + \mathrm{MSE}\left(\hat{\theta}_1\right) g_2^2(T) + \mathrm{MSE}\left(\hat{\theta}_2\right) g_3^2(T) \\
&+ 2\mathrm{Cov}(\hat{\theta}_0, \hat{\theta}_1) g_2(T) + 2\mathrm{Cov}(\hat{\theta}_0, \hat{\theta}_2) g_3(T) + 2\mathrm{Cov}(\hat{\theta}_1, \hat{\theta}_2) g_2(T) g_3(T)
\end{aligned} \tag{5}
$$

When the derivatives of Eq. (5) are taken with respect to $g_2(T)$, $g_3(T)$ and set equal to zero,

$$\frac{\partial \mathrm{MSE}(d_g^*)}{\partial g_2(T)} = 0 \implies \mathrm{MSE}(\hat{\theta}_1)g_2(T) + \mathrm{Cov}(\hat{\theta}_0,\hat{\theta}_1) + \mathrm{Cov}(\hat{\theta}_1,\hat{\theta}_2)g_3(T) = 0,$$
$$\frac{\partial \mathrm{MSE}(d_g^*)}{\partial g_3(T)} = 0 \implies \mathrm{Cov}(\hat{\theta}_1,\hat{\theta}_2)g_2(T) + \mathrm{Cov}(\hat{\theta}_0,\hat{\theta}_2) + \mathrm{MSE}(\hat{\theta}_2)g_3(T) = 0$$

Solving for $g_2(T)$ and $g_3(T)$ one obtains:

$$g_2(T) = \frac{\mathrm{Cov}(\hat{\theta}_0,\hat{\theta}_2)\mathrm{Cov}(\hat{\theta}_1,\hat{\theta}_2) - \mathrm{MSE}(\hat{\theta}_2)\mathrm{Cov}(\hat{\theta}_0,\hat{\theta}_1)}{\mathrm{MSE}(\hat{\theta}_1)\mathrm{MSE}(\hat{\theta}_2) - \left[\mathrm{Cov}(\hat{\theta}_1,\hat{\theta}_2)\right]^2} \tag{6}$$

$$g_3(T) = \frac{\mathrm{Cov}(\hat{\theta}_0,\hat{\theta}_1)\mathrm{Cov}(\hat{\theta}_1,\hat{\theta}_2) - \mathrm{MSE}(\hat{\theta}_1)\mathrm{Cov}(\hat{\theta}_0,\hat{\theta}_2)}{\mathrm{MSE}(\hat{\theta}_1)\mathrm{MSE}(\hat{\theta}_2) - \left[\mathrm{Cov}(\hat{\theta}_1,\hat{\theta}_2)\right]^2} \tag{7}$$

When the values of $g_2(T)$ and $g_3(T)$ are included in Eq. (5), the minimum MSE can be obtained as follows:

$$\mathrm{MSE}_{\min}(e_i) = \mathrm{MSE}_{\min}(\hat{\theta}_0) - M,\ i = 1,\ldots,9, \tag{8}$$

where

$$M = \frac{\mathrm{MSE}(\hat{\theta}_1)\left[\mathrm{Cov}(\hat{\theta}_0,\hat{\theta}_2)\right]^2 + \mathrm{MSE}(\hat{\theta}_2)\left[\mathrm{Cov}(\hat{\theta}_0,\hat{\theta}_1)\right]^2}{\mathrm{MSE}(\hat{\theta}_1)\mathrm{MSE}(\hat{\theta}_2) - \left[\mathrm{Cov}(\hat{\theta}_1,\hat{\theta}_2)\right]^2} - \frac{2\mathrm{Cov}(\hat{\theta}_0,\hat{\theta}_1)\mathrm{Cov}(\hat{\theta}_0,\hat{\theta}_2)\mathrm{Cov}(\hat{\theta}_1,\hat{\theta}_2)}{\mathrm{MSE}(\hat{\theta}_1)\mathrm{MSE}(\hat{\theta}_2) - \left[\mathrm{Cov}(\hat{\theta}_1,\hat{\theta}_2)\right]^2}$$

In case the population parameters (θ_1,θ_2) of the auxiliary variable are known, Tripathi et al. [6] proposed the following estimators which are sub classes of d_g^*:

$$e_1 = \sum_{i=1}^{2} w_i\left\{\hat{\theta}_0 - \lambda_i(\hat{\theta}_i - \theta_i)\right\},\ \sum_{i=1}^{2} w_i = 1, \tag{9}$$

$$e_2 = \hat{\theta}_0 + w_1(\hat{\theta}_1 - \theta_1) + w_2(\hat{\theta}_2 - \theta_2),\ \sum_{i=1}^{2} w_i = 1, \tag{10}$$

$$e_3 = \hat{\theta}_0 - \sum_{i=1}^{2} \lambda_i(\hat{\theta}_i - \theta_i),\ i = 1,2, \tag{11}$$

$$e_4 = \hat{\theta}_0 u_1^{\alpha_1} u_2^{\alpha_2} u_i = \hat{\theta}_i/\theta_i,\ i = 1,2, \tag{12}$$

$$e_5 = \left[\hat{\theta}_0 - \lambda_1(\hat{\theta}_1 - \theta_1)\right](\hat{\theta}_2/\theta_2)^{\alpha}, \tag{13}$$

$$e_6 = \frac{\hat{\theta}_0 - \lambda_1(\hat{\theta}_1 - \theta_1) - \lambda_2(\hat{\theta}_2 - \theta_2)}{\hat{\theta}_1 - \lambda_1^*(\hat{\theta}_1 - \theta_1) - \lambda_2^*(\hat{\theta}_2 - \theta_2)}(\theta_1), \tag{14}$$

$$e_7 = \theta_0 \frac{1 + \alpha_1(u_1 - 1)}{1 + \alpha_2(u_2 - 1)}, \tag{15}$$

$$e_8 = \frac{\hat{\theta}_0}{\left[1 + \alpha_1(u_1 - 1) - \alpha_2(u_2 - 1)\right]}, \tag{16}$$

$$e_9 = \hat{\theta}_0(w_1 u_1 + w_2 u_2^{\alpha_2}),\ \sum_{i=1}^{2} w_i = 1, \tag{17}$$

where $(\lambda_1,\lambda_2,\lambda_1^*,\lambda_2^*)$, $(\alpha,\alpha_1,\alpha_2)$, (w_0,w_1,w_2) are suitably chosen constants.

The MSE of these estimators is given by Eq. (8).

2. The proposed estimators

We develop some estimators for the population variance by adapting the estimators to the population variance.

First, when one parameter of the auxiliary variable is known, say S_x^2, then from Eq. (1) we propose the following estimators:

$$\hat{S}^2_{pr_{o1}} = \frac{\hat{S}_y^2 - t_1(\hat{S}_x^2 - S_x^2)}{\hat{S}_x^2 - t_2(\hat{S}_x^2 - S_x^2)}(S_x^2)^\alpha \tag{18}$$

$$\hat{S}^2_{pr_{o2}} = \hat{S}_y^2 h\Big(\frac{\hat{S}_x^2}{S_x^2}\Big) \tag{19}$$

$$\hat{S}^2_{pr_{o3}} = h(\hat{S}_y^2, \hat{S}_x^2, S_x^2) \tag{20}$$

The minimum MSE for all these estimators above is the same and given by:

$$\begin{aligned} \mathrm{MSE}_{\min}(pr_{oj}) &= (1 - \rho^2_{\hat{S}_y^2, \hat{S}_x^2}) V(\hat{S}_y^2) \\ &= \lambda S_y^4(\beta_y - 1)\big(1 - \lambda^2 S_y^2 S_x^2(\theta - 1)^2\big), \ j = 1, 2, 3, \end{aligned} \tag{21}$$

where $\lambda = \frac{1-f}{n}$, $f = \frac{n}{N}$.

The population mean $\bar{X}$ and the population variance S_x^2 of the auxiliary variable can be estimated.

Second, when two parameters θ_1, θ_2 of the auxiliary variable, say the population mean $\bar{X}$ and the population variance S_y^2, respectively, of the auxiliary variable are known, then in line with Triphati *et al.* [6], we propose the following estimators for the population variance θ_0:

$$\hat{S}^2_{pr1} = w_1\big\{\hat{S}_y^2 - \lambda_1\big(\bar{x} - \bar{X}\big)\big\} + w_2\big\{\hat{S}_y^2 - \lambda_2\big(\hat{S}_x^2 - S_x^2\big)\big\}, \ \sum_{i=1}^{2} w_i = 1, \tag{22}$$

$$\hat{S}^2_{pr2} = w_0\hat{S}_y^2 + w_1\big(\bar{x} - \bar{X}\big) + w_2\big(\hat{S}_x^2 - S_x^2\big), \ \sum_{i=0}^{2} w_i = 1, \tag{23}$$

$$\hat{S}^2_{pr3} = \hat{S}_y^2 - \lambda_1\big(\bar{x} - \bar{X}\big) - \lambda_2\big(\hat{S}_x^2 - S_x^2\big), \tag{24}$$

$$\hat{S}^2_{pr4} = \hat{S}_y^2\Big(\frac{\bar{x}}{\bar{X}}\Big)^{\alpha_1}\Big(\frac{\hat{S}_x^2}{S_x^2}\Big)^{\alpha_2} \tag{25}$$

$$\hat{S}^2_{pr5} = \big[\hat{S}_y^2 - \lambda_1\big(\bar{x} - \bar{X}\big)\big]\big(\hat{S}_x^2/S_x^2\big)^\alpha, \tag{26}$$

$$\hat{S}^2_{pr6} = \frac{\hat{S}_y^2 - \lambda_1\big(\bar{x} - \bar{X}\big) - \lambda_2\big(\hat{S}_x^2 - S_x^2\big)}{\bar{x} - \lambda_1^*\big(\bar{x} - \bar{X}\big) - \lambda_2^*\big(\hat{S}_x^2 - S_x^2\big)}(\bar{X}), \tag{27}$$

$$\hat{S}^2_{pr7} = \hat{S}_y^2 \frac{1 + \alpha_1\big(\big(\frac{\bar{x}}{\bar{X}}\big) - 1\big)}{1 + \alpha_2\big(\big(\frac{\hat{S}_x^2}{S_x^2}\big) - 1\big)}, \tag{28}$$

$$\hat{S}^2_{pr8} = \frac{\hat{S}_y^2}{\big[1 + \alpha_1\big(\big(\frac{\bar{x}}{\bar{X}}\big) - 1\big) - \alpha_2\big(\big(\frac{\hat{S}_x^2}{S_x^2}\big) - 1\big)\big]}, \tag{29}$$

$$\hat{S}^2_{pr9} = \hat{S}_y^2\Big(w_1\Big(\frac{\bar{x}}{\bar{X}}\Big) + w_2\Big(\frac{\hat{S}_x^2}{S_x^2}\Big)^{\alpha_2}\Big), \ \sum_{i=1}^{2} w_i = 1, \tag{30}$$

where $(\lambda_1, \lambda_2, \lambda_1^*, \lambda_2^*)$, $(\alpha, \alpha_1, \alpha_2)$, (w_0, w_1, w_2) are again suitably chosen constants.

The minimum MSE of these estimators for the selected constants are the same as in Eq. (8), and are given as follows:

$$MSE_{\min}(\hat{S}^2_{pri}) = MSE_{\min}(\hat{S}^2_y) - M, \tag{31}$$

where

$$M = \frac{\mathrm{MSE}(\bar{x})\left[\mathrm{Cov}(\hat{S}^2_y, \hat{S}^2_x)\right]^2 + \mathrm{MSE}(\hat{S}^2_x)\left[\mathrm{Cov}(\hat{S}^2_y, \bar{x})\right]^2}{\mathrm{MSE}(\bar{x}) MSE(\hat{S}^2_x) - \left[\mathrm{Cov}(\bar{x}, \hat{S}^2_x)\right]^2} - \frac{2\mathrm{Cov}(\hat{S}^2_y, \bar{x})\mathrm{Cov}(\hat{S}^2_y, \hat{S}^2_x)\mathrm{Cov}(\bar{x}, \hat{S}^2_x)}{\mathrm{MSE}(\bar{x})\mathrm{MSE}(\hat{S}^2_x) - \left[\mathrm{Cov}(\bar{x}, \hat{S}^2_x)\right]^2}$$

Here, (see, Kadilar and Çingi [3]),

$$\begin{aligned}
\mathrm{MSE}(\bar{x}) &= \lambda S^2_x, \\
\mathrm{MSE}(\hat{S}^2_x) &= \lambda S^4_x(\beta_x - 1), \\
\mathrm{Cov}(\hat{S}^2_y, \hat{S}^2_x) &= \lambda S^2_y S^2_x(\theta - 1), \\
\mathrm{Cov}(\hat{S}^2_y, \bar{x}) &= \lambda \mu_{21}, \\
\mathrm{Cov}(\bar{x}, \hat{S}^2_x) &= \lambda \mu_{03},
\end{aligned} \tag{32}$$

where β_x and β_y are the population kurtosis of the auxiliary variable and variable of interest, respectively,

$$\mu_{rs} = \frac{\sum_{i=1}^{N}(y_i - \bar{Y})^r(x_i - \bar{X})^s}{N}, \text{ and } \theta = \frac{\mu_{22}}{\mu_{20}\mu_{02}}$$

(see, Kendall and Stuart [4]).

Using these definitions, we can write (31) as

$$\begin{aligned}
\mathrm{MSE}_{\min}(\hat{S}^2_{pr_i}) &= \lambda S^4_y(\beta_y - 1) - \lambda\left\{\frac{S^2_x\left[S^2_y S^2_x(\theta - 1)\right]^2 + S^4_x(\beta_x - 1)\left[\mu_{21}\right]^2}{S^2_x S^4_x(\beta_x - 1) - \left[\mu_{03}\right]^2} - \frac{2\mu_{21} S^2_x S^2_y(\theta - 1)\mu_{03}}{S^2_x S^4_x(\beta_x - 1) - (\mu_{03})^2}\right\} \\
&= \lambda S^4_y(\beta_y - 1) \\
&\quad - \lambda\left\{\frac{S^6_x S^4_y(\theta - 1)^2 + S^4_x(\beta_x - 1)(\mu_{21})^2 - 2S^2_y S^2_x \mu_{21}\mu_{03}(\theta - 1)}{S^6_x(\beta_x - 1) - (\mu_{03})^2}\right\}.
\end{aligned}$$

Hence,

$$\mathrm{MSE}_{\min}(\hat{S}^2_{pr_i}) = \lambda S^4_y \beta^*_y - \frac{\lambda}{S^6_x \beta^*_x - \mu^2_{03}}\left\{S^6_x S^4_y \theta^{*2} + S^4_x \beta^*_x \mu^2_{21} - 2S^2_y S^2_x \mu_{21}\mu_{03}\theta^*\right\}, \tag{33}$$

where $\theta^* = \theta - 1$, $\beta^*_x = \beta_x - 1$, $\beta^*_y = \beta_y - 1$.

Independently of the sampling method, if the mean and the variance of the auxiliary variable are known, the minimum MSE of the population variance is as shown in Eq. (33). Obtaining the MSE of the fourth proposed estimator by the classical method is presented in the Appendix.

3. Numerical Study

We use data in Unyazici's PhD. Thesis [7] to find the minimum MSE of the proposed estimators. This data base (see, Savcı [5]) concerns the height of the flower *Asteraceae* (x) as the auxiliary variate and the pappus height (y) of the *Asteraceae* as the variate of interest. The pappus height of the flower is related to the flower's height ($\rho = 0,63$). Therefore, we can use ratio-type estimators.

Table 1. Data statistics of the population

$N = 450$	$n = 81$	$\rho = 0,63$	$\bar{Y} = 2,291$	$\bar{X} = 3,455$	$S_x^2 = 0,518$
$\lambda = 0,012$	$\theta = 2,530$	$\mu_{03} = -0,19255$	$\mu_{21} = 0,06208$	$\beta_y = 5,707$	$\beta_x = 3,697$

In Table 1, we observe the statistics about the population. We would like to remind the reader that the sampling method has no effect on the MSE of the estimators.

The MSE values are computed as follows:

$$\mathrm{MSE}_{\min}(\hat{S}^2_{pr_{0i}}) = 0,013671$$
$$\mathrm{MSE}_{\min}(\hat{S}^2_{pri}) = 0,010854$$

4. Discussion and Conclusion

From the theoretical discussion in Section 2 and the numerical example, we infer that if one parameter of the auxiliary variable is known, the minimum MSE for the estimators are the same. If the mean and variance of the auxiliary variable are known, the minimum MSE of the population variance is the same for all of the proposed estimators. The sampling method has no effect on the MSE of estimators. Using Eq. (33), the minimum MSE can be found easily.

Also, knowing two parameters of the auxiliary variable is can be more advantageous than knowing one parameter. This is shown in the numerical example:

$$\mathrm{MSE}_{\min}(\hat{S}^2_{pri}) < \mathrm{MSE}_{\min}(\hat{S}^2_{pr_{oi}})$$

Appendix

We give a classical deduction of the MSE of the proposed estimator

$$\hat{S}^2_{pr4} = \hat{S}^2_y \left(\frac{\bar{x}}{\bar{X}}\right)^{\alpha_1} \left(\frac{\hat{S}^2_x}{S^2_x}\right)^{\alpha_2}$$

by using the first degree approximation in Taylor Series method (see, Kadilar and Cingi [2]) defined by

$$d = \begin{bmatrix} \frac{\partial h(a,b,c)}{\partial a}\Big|_{S^2_y,\bar{X},S^2_x} & \frac{\partial h(a,b,c)}{\partial b}\Big|_{S^2_y,\bar{X},S^2_x} & \frac{\partial h(a,b,c)}{\partial c}\Big|_{S^2_y,\bar{X},S^2_x} \end{bmatrix}$$

where $h(a,b,c) = h(\hat{S}^2_y, \bar{X}, \hat{S}^2_x)$. The vector of first derivatives is therefore

$$\text{(A.1)} \qquad d = \begin{bmatrix} 1 & \alpha_1 \frac{S^2_y}{\bar{X}} & \alpha_2 \frac{S^2_y}{S^2_x} \end{bmatrix}$$

The variance- covariance matrix is:

$$\text{(A.2)} \qquad \Sigma = \begin{bmatrix} \mathrm{V}(\hat{S}^2_y) & \mathrm{C}(\hat{S}^2_y, \bar{x}) & \mathrm{C}(\hat{S}^2_y, \hat{S}^2_x) \\ & \mathrm{V}(\bar{x}) & \mathrm{C}(\bar{x}, \hat{S}^2_x) \\ & & \mathrm{V}(\hat{S}^2_x) \end{bmatrix}$$

The MSE of the proposed estimator is:

$$
\begin{aligned}
\mathrm{MSE}(\hat{S}^2_{pr4}) &= d\Sigma d' \\
&= \mathrm{V}(\hat{S}^2_y) + \alpha_1^2 \frac{S_y^4}{\bar{X}^2}\mathrm{V}(\bar{x}) + \alpha_2^2 \frac{S_y^4}{S_x^4}\mathrm{V}(\hat{S}^2_x) + 2\alpha_1 \frac{S_y^2}{\bar{X}}\mathrm{C}(\hat{S}^2_y, \bar{x}) \\
&\qquad + 2\alpha_2 \frac{S_y^2}{S_x^2}\mathrm{C}(\hat{S}^2_y, \hat{S}^2_x) + 2\alpha_1\alpha_2 \frac{S_y^4}{\bar{X}S_x^2}\mathrm{C}(\bar{x}, \hat{S}^2_x).
\end{aligned}
\tag{A.3}
$$

Here, $\mathrm{V}(\hat{S}^2_y) = \lambda S_y^4(\beta_y - 1)$, and the other values are as given in Eq. (28). When these values are substituted in Eq. (A.3) we have:

$$
\begin{aligned}
\mathrm{MSE}(\hat{S}^2_{pr4}) &= \lambda S_y^4 \beta_y^* + \alpha_1^2 \frac{S_y^4}{\bar{X}^2}\lambda S_x^2 + \alpha_2^2 \frac{S_y^4}{S_x^2}\lambda S_x^4 \beta_x^* + 2\alpha_1 \frac{S_y^2}{\bar{X}}\lambda \mu_{21} \\
&\qquad + 2\alpha_2 \frac{S_y^2}{S_x^2}\lambda S_y^2 S_x^2 \theta^* + 2\alpha_1\alpha_2 \frac{S_y^4}{\bar{X}S_x^2}\lambda \mu_{03} \\
&= \lambda S_y^4 \big\{\beta_y^* + \alpha_1^2 C_x^2 + \alpha_2^2 \beta_x^* + 2\alpha_1 \frac{\mu_{21}}{\bar{X}S_y^2} + 2\alpha_2\theta^* + 2\alpha_1\alpha_2 \frac{\mu_{03}}{\bar{X}S_x^2}\big\}
\end{aligned}
\tag{A.4}
$$

Taking the derivatives of the MSE with respect to α_1 and α_2 and equating to 0 gives $\frac{\partial MSE(\hat{S}^2_{pr4})}{\partial \alpha_1} = 0 = \frac{\partial MSE(\hat{S}^2_{pr4})}{\partial \alpha_2}$, and so

$$
\begin{aligned}
&\alpha_1 C_x^2 + \alpha_2 \frac{\mu_{03}}{\bar{X}S_x^2} = -\frac{\mu_{21}}{\bar{X}S_y^2}, \\
&\alpha_1 \frac{\mu_{03}}{\bar{X}S_x^2} + \alpha_2 \beta_x^* = -\theta^*.
\end{aligned}
$$

Solving for α_1, α_2 yields the optimal values as

$$
\alpha_1^* = \frac{\bar{X}}{S_y^2}\frac{\left(S_y^2 S_x^2 \theta^* \mu_{03} - S_x^4 \beta_x^* \mu_{21}\right)}{\left(\beta_x^* S_x^6 - \mu_{03}^2\right)} \tag{A.5}
$$

$$
\alpha_2^* = \frac{S_x^2}{S_y^2}\Big(\frac{\mu_{03}\mu_{21} - S_y^2 S_x^4(\theta - 1)}{\left(\beta_x^* S_x^6 - \mu_{03}^2\right)}\Big) \tag{A.6}
$$

When these optimal values are put in Eq. (A.4) the minimum MSE of the proposed estimator is obtained as follows:

$$
\mathrm{MSE}_{\min}(\hat{S}^2_{pr4}) = \lambda S_y^4 \beta_y^* - \frac{\lambda}{S_x^6 \beta_x^* - \mu_{03}^2}\big\{S_x^6 S_y^4 \theta^{*2} + S_x^4 \beta_x^* \mu_{21}^2 - 2 S_y^2 S_x^2 \mu_{21}\mu_{03}\theta^*\big\}
$$

This clearly takes some time and effort, whereas using Eq. (33) directly gives the minimum MSE in one step.

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