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THE BASE POINTS OF INDEFINITE QUADRATIC FORMS IN THE CYCLE AND PROPER CYCLE OF AN INDEFINITE QUADRATIC FORM

Ahmet Tekcan*

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Abstract

Let $F = (a, b, c)$ be an indefinite quadratic form of discriminant $\Delta > 0$. In the first section, we give some preliminaries from binary quadratic forms. In the second section, we derive some results concerning the base points of indefinite quadratic forms in the cycle and proper cycle of F using the transformations $\tau(F) = (-a, b, -c)$, $\xi(F) = (c, b, a)$, $\chi(F) = (-c, b, -a)$, $\psi(F) = (-a, -b, -c)$, and the right neighbor $R^i(F)$ of F for $i \geq 0$.

Keywords: Quadratic form, Indefinite form, Cycle, Proper cycle, Right neighbor, Base point.

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1. Introduction.

A real binary quadratic form (or just a form) F is a polynomial in two variables x and y of the type

$$F = F(x, y) = ax^2 + bxy + cy^2$$

with real coefficients a, b, c. We denote F briefly by $F = (a, b, c)$. The discriminant of F is defined by the formula $b^2 - 4ac$ and is denoted by $\Delta = \Delta(F)$. The form F is an *integral form* if and only if $a, b, c \in \mathbb{Z}$, and is *indefinite* if and only if $\Delta(F) > 0$. An indefinite quadratic form $F = (a, b, c)$ of discriminant Δ is said to be *reduced* if

$$\left|\sqrt{\Delta} - 2|a|\right| < b < \sqrt{\Delta}. \tag{1.1}$$

*Uludag University, Faculty of Science, Department of Mathematics, Görükle 16059, Bursa, Turkey. E-mail: tekcan@uludag.edu.tr

Let $\mathrm{GL}(2,\mathbb{Z})$ be the modular multiplicative group of 2×2 matrices $g=\begin{pmatrix} r & s\\ t & u\end{pmatrix}$ such that $r,s,t,u\in\mathbb{Z}$ and $\det g=\pm 1$. Gauss (1777-1855) defined the group action of $\mathrm{GL}(2,\mathbb{Z})$ on the set of forms as follows: Let $F=(a,b,c)$ be a form and $g=\begin{pmatrix} r & s\\ t & u\end{pmatrix}\in \mathrm{GL}(2,\mathbb{Z})$. Then the form gF is defined by

$$\begin{aligned} gF(x,y) &= a(rx+ty)^2+b(rx+ty)(sx+uy)+c(sx+uy)^2\\ &= (ar^2+brs+cs^2)x^2+(2art+bru+bts+2csu)xy\\ &\qquad +(at^2+btu+cu^2)y^2, \end{aligned} \tag{1.2}$$

that is, gF is gotten from F by making the substitution $x\to rx+tu,\ y\to sx+uy$. Moreover, $\Delta(F)=\Delta(gF)$ for all $g\in\mathrm{GL}(2,\mathbb{Z})$, that is, the action of $\mathrm{GL}(2,\mathbb{Z})$ on forms leaves the discriminant invariant. If F is indefinite or integral, then so is gF for all $g\in\mathrm{GL}(2,\mathbb{Z})$.

Let F and G be two forms. If there exists a $g\in\mathrm{GL}(2,\mathbb{Z})$ such that $gF=G$, then F and G are called *equivalent.* If $\det g=1$, then F and G are called *properly equivalent.* If $\det g=-1$, then F and G are called *improperly equivalent.*

Let $\rho(F)$ denotes the *normalization* of $(c,-b,a)$, see [1, p. 88] for further details. To be more explicit, we set

$$\rho(F)=(c,-b+2cs,cs^2-bs+a),$$

where

$$s=s(F)=\begin{cases} \operatorname{sign}(c)\left\lfloor \frac{b}{2|c|}\right\rfloor & \text{for } |c|\geq\sqrt{\Delta}\\ \operatorname{sign}(c)\left\lfloor \frac{b+\sqrt{\Delta}}{2|c|}\right\rfloor & \text{for } |c|<\sqrt{\Delta}. \end{cases}$$

Note that, if F is reduced, then $\rho(F)$ is also reduced by (1.1). In fact, ρ is a permutation of the set of all reduced indefinite forms.

Let $F=(a,b,c)$ be a quadratic form. Define

$$\tau(F)=\tau(a,b,c)=(-a,b,-c).$$

Then $\tau(\rho(F))=\rho(\tau(F))=(-c,-b+2cs,-a+bs-cs^2)$. Hence F and $\tau(F)$ are equivalent, but not necessarily properly equivalent. If F is reduced, then $\tau(F)$ is also reduced. The *cycle of* F is the sequence $((\tau\rho)^i(G))$ for $i\in\mathbb{Z}$, where $G=(k,l,m)$ is a reduced form with $k>0$ which is equivalent to F. Similarly, the *proper cycle* of F is the sequence $(\rho^i(G))$ for $i\in\mathbb{Z}$, where G is a reduced form which is properly equivalent to F. The cycle and the proper cycle of F are invariants of the equivalence class of F. We represent the cycle (or proper cycle) of F by its period

$$F_0\sim F_1\sim\cdots\sim F_{l-1}$$

of length l. We explain how to compute the cycle of F : Let $F_0=F=(a_0,b_0,c_0)$ and let

$$s_i=|s(F_i)|=\left\lfloor \frac{b_i+\left\lfloor\sqrt{\Delta}\right\rfloor}{2|c_i|}\right\rfloor. \tag{1.3}$$

Then

$$\begin{aligned} F_{i+1} &= (a_{i+1},b_{i+1},c_{i+1})\\ &= \left(|c_i|,-b_i+2s_i|c_i|,-a_i-b_is_i-c_is_i^2\right) \end{aligned} \tag{1.4}$$

for $0\leq i\leq l-2$. Hence $F_0\sim F_1\sim F_2\sim\cdots\sim F_{l-1}$ is the cycle of F of length l. The proper cycle of F is given by the following lemma.

1.1. Lemma. [1, p. 106] *Let $F_0 \sim F_1 \sim F_2 \sim \cdots \sim F_{l-1}$ be the cycle of F of length l.*

(1) *If l is odd, then the proper cycle of F is the cycle*

$$\begin{aligned}&F_0 \sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \cdots \sim \tau(F_{l-2}) \sim F_{l-1}\\ &\quad \sim \tau(F_0) \sim F_1 \sim \tau(F_2) \sim \cdots \sim F_{l-2} \sim \tau(F_{l-1})\end{aligned}$$

of length $2l$. In this case the equivalence class of F is equal to the proper equivalence class of F.

(2) *If l is even, then the proper cycle of F is the cycle*

$$F_0 \sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \cdots \sim F_{l-2} \sim \tau(F_{l-1})$$

of length l. In this case the equivalence class of F is the disjoint union of the proper equivalence class of F and the proper equivalence class of $\tau(F)$.

The *right neighbor* of $F = (a, b, c)$, denoted by $R(F)$, is the form (A, B, C) determined by the three conditions:

i. $A = c$,
ii. $b + B \equiv 0 \pmod{2A}$ and $\sqrt{\Delta} - 2|A| < B < \sqrt{\Delta}$,
iii. $B^2 - 4AC = \Delta$.

It is clear from definition that

$$\begin{aligned}R(F) &= (A, B, C)\\ &= \begin{pmatrix}1 & 0\\ \delta & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}(a, b, c)\\ &= \begin{pmatrix}0 & -1\\ 1 & -\delta\end{pmatrix}(a, b, c),\end{aligned} \tag{1.5}$$

where

$$\delta = \frac{b + B}{2c}. \tag{1.6}$$

Therefore, F is properly equivalent to its right neighbor $R(F)$ (see [2, p.129]).

2. Base Points of Indefinite Quadratic Forms.

We assume that $F = (a, b, c)$ is integral and indefinite throughout this paper. The base point of F is

$$z = z(F) = \frac{-b + \sqrt{\Delta}}{2a}, \tag{2.1}$$

which is one of the zeros of $F(x, 1) = ax^2+bx+1$. The negative of F is $-F = (-a, -b, -c)$, and its base point is

$$\overline{z} = \overline{z}(-F) = \frac{b + \sqrt{\Delta}}{-2a} = \frac{-b - \sqrt{\Delta}}{2a}. \tag{2.2}$$

We recall that an element $g = \begin{pmatrix}r & s\\ t & u\end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$ is called *hyperbolic* if $|r + u| > 2$, *parabolic* if $|r + u| = 2$, and *elliptic* if $|r + u| < 2$.

A complex number z is a *fixed point* of g if

$$gz = z \iff \frac{rz + s}{tz + u} = z \iff tz^2 + (u - r)z - s = 0. \tag{2.3}$$

Note that $g^{-1} = \begin{pmatrix} u & -s \\ -t & r \end{pmatrix}$ and hence $(g^{-1})^t = \begin{pmatrix} u & -t \\ -s & r \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$. Similarly, a complex number z is a fixed point of $(g^{-1})^t$ if

$$(g^{-1})^t z = z \iff \frac{uz-t}{-sz+r} = z \iff sz^2+(u-r)z-t=0. \tag{2.4}$$

So, g fixes

$$z = \frac{r-u\pm\sqrt{(u-r)^2+4ts}}{2t} = \frac{r-u\pm\sqrt{(u+r)^2\pm 4}}{2t},$$

and $(g^{-1})^t$ fixes

$$z = \frac{r-u\pm\sqrt{(u-r)^2+4ts}}{2s} = \frac{r-u\pm\sqrt{(u+r)^2\pm 4}}{2s}.$$

Since $\mathrm{GL}(2,\mathbb{Z})$ is discrete, the *stabilizer*

$$\{g\in \mathrm{GL}(2,\mathbb{Z}) : gz=z\}$$

of any complex number z in $\mathrm{GL}(2,\mathbb{Z})$ is a cyclic subgroup of $\mathrm{GL}(2,\mathbb{Z})$. Hence we can call fixed points *hyperbolic*, *parabolic* or *elliptic* according to whether the matrices fixing them are hyperbolic, parabolic or elliptic, respectively.

2.1. Theorem. *Given any hyperbolic fixed z of g in $GL(2,\mathbb{Z})$, there exists an indefinite quadratic form F whose base point is z.*

Proof. Let $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$. Then $(g^{-1})^t = \begin{pmatrix} u & -t \\ -s & r \end{pmatrix}$. Further, g and $(g^{-1})^t$ generate the stabilizer of hyperbolic fixed point z in $\mathrm{GL}(2,\mathbb{Z})$. Then by (2.3), $gz=z$ gives rise to the equation

$$tz^2+(u-r)z-s=0,$$

and by (2.4), $(g^{-1})^t z = z$ gives rise the equation

$$sz^2+(u-r)z-t=0.$$

If we choose

$$g_z = \begin{cases} g & \text{if } z = \frac{r-u+\sqrt{(u-r)^2+4ts}}{2t} \\ \\ (g^{-1})^t & \text{if } z = \frac{r-u-\sqrt{(u-r)^2+4ts}}{2s} \end{cases}$$

then

$$F_z = \begin{cases} (t, u-r, -s) & \text{if } g_z = g \\ \\ (s, u-r, -t) & \text{if } g_z = (g^{-1})^t \end{cases}$$

is an indefinite quadratic form of discriminant

$$\Delta = (u-r)^2+4ts = (u+r)^2\pm 4$$

whose base point is z. □

2.2. Theorem. *Let z_1 and z_2 be two hyperbolic numbers and let $F_{z_1} = (a_1,b_1,c_1)$ and $F_{z_2} = (a_2,b_2,c_2)$ be two indefinite quadratic forms which correspond to z_1 and z_2, respectively. Let $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in GL(2,\mathbb{Z})$. Then F_{z_1} and F_{z_2} are equivalent if and only if $(g^{-1})^t z_1 = z_2$.*

Proof. Suppose that F_{z_1} and F_2 are (properly) equivalent. Then by (1.2), we have

$$\begin{aligned} gF_{z_1} &= (a_1r^2 + b_1rs + c_1s^2, 2a_1rt + b_1ru + b_1ts + 2c_1su, a_1t^2 + b_1tu + c_1u^2) \\ &= F_{z_2} = (a_2, b_2, c_2) \end{aligned}$$

for $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$ with $\det g = 1$. So by (2.1), the base point of F_{z_2} is

$$z_2 = \frac{-(2a_1rt + b_1ru + b_1ts + 2c_1su) + \sqrt{\Delta}}{2(a_1r^2 + b_1rs + c_1s^2)}.$$

Note that the base point of F_{z_1} is $z_1 = \frac{-b_1+\sqrt{\Delta}}{2a_1}$ and $(g^{-1})^t = \begin{pmatrix} u & -t \\ -s & r \end{pmatrix}$. So

$$\begin{aligned} (g^{-1})^t z_1 &= \frac{u\left(\frac{-b_1+\sqrt{\Delta}}{2a_1}\right) - t}{-s\left(\frac{-b_1+\sqrt{\Delta}}{2a_1}\right) + r} \\ &= \frac{-b_1u - 2a_1t + u\sqrt{\Delta}}{b_1s + 2a_1r - s\sqrt{\Delta}} \\ &= \frac{(-b_1u - 2a_1t + u\sqrt{\Delta})(b_1s + 2a_1r + s\sqrt{\Delta})}{(b_1s + 2a_1r - s\sqrt{\Delta})(b_1s + 2a_1r + s\sqrt{\Delta})} \\ &= \frac{\begin{pmatrix} [(-b_1u - 2a_1t)(b_1s + 2a_1r) + us\Delta] + \\ [s(-b_1u - 2a_1t) + u(b_1s + 2a_1r)]\sqrt{\Delta} \end{pmatrix}}{(b_1s + 2a_1r)^2 - s^2\Delta}. \end{aligned} \tag{2.5}$$

After some calculations, we see that

$$\begin{aligned} &(-b_1u - 2a_1t)(b_1s + 2a_1r) + us\Delta \\ &\qquad = -b_1^2us - 2a_1b_1ur - 2a_1b_1st - 4a_1^2rt + us(b_1^2 - 4a_1c_1) \\ &\qquad = -b_1^2us - 2a_1b_1ur - 2a_1b_1st - 4a_1^2rt + b_1^2us - 4a_1c_1us \\ &\qquad = -2a_1b_1ur - 2a_1b_1st - 4a_1^2rt - 4a_1c_1us \\ &\qquad = 2a_1\left[-(2a_1rt + b_1ru + b_1ts + 2c_1su)\right], \end{aligned}$$

$$\begin{aligned} s(-b_1u - 2a_1t) + u(b_1s + 2a_1r) &= -b_1us - 2a_1st + b_1us + 2a_1ru \\ &= 2a_1ru - 2a_1st \\ &= 2a_1(ru - st) \\ &= 2a_1 \end{aligned}$$

and

$$\begin{aligned} (b_1s + 2a_1r)^2 - s^2\Delta &= b_1^2s^2 + 4a_1b_1rs + 4a_1^2r^2 - s^2(b_1^2 - 4a_1c_1) \\ &= b_1^2s^2 + 4a_1b_1rs + 4a_1^2r^2 - b_1^2s^2 + 4a_1c_1s^2 \\ &= 4a_1b_1rs + 4a_1^2r^2 + 4a_1c_1s^2 \\ &= 2a_1\left[2(a_1r^2 + b_1rs + c_1s^2)\right]. \end{aligned}$$

So, (2.5) becomes

$$
\begin{aligned}
(g^{-1})^t z_1 &= \frac{u\left(\frac{-b_1+\sqrt{\Delta}}{2a_1}\right)-t}{-s\left(\frac{-b_1+\sqrt{\Delta}}{2a_1}\right)+r} \\
&= \frac{-b_1u-2a_1t+u\sqrt{\Delta}}{b_1s+2a_1r-s\sqrt{\Delta}} \\
&= \frac{(-b_1u-2a_1t+u\sqrt{\Delta})(b_1s+2a_1r+s\sqrt{\Delta})}{(b_1s+2a_1r-s\sqrt{\Delta})(b_1s+2a_1r+s\sqrt{\Delta})} \\
&= \frac{\left(\begin{array}{c}[(-b_1u-2a_1t)(b_1s+2a_1r)+us\Delta]+\\ [s(-b_1u-2a_1t)+u(b_1s+2a_1r)]\sqrt{\Delta}\end{array}\right)}{(b_1s+2a_1r)^2-s^2\Delta} \\
&= \frac{2a_1\left[-(2a_1rt+b_1ru+b_1ts+2c_1su)\right]+2a_1\sqrt{\Delta}}{2a_1\left[2(a_1r^2+b_1rs+c_1s^2)\right]} \\
&= \frac{-(2a_1rt+b_1ru+b_1ts+2c_1su)+\sqrt{\Delta}}{2(a_1r^2+b_1rs+c_1s^2)} \\
&= z_2.
\end{aligned}
$$

Conversely, let $(g^{-1})^t z_1 = z_2$. Let h be one of the two generators of the stabilizer of z_1 in $\mathrm{GL}(2,\mathbb{Z})$. Then $k=(g^{-1})^t hg$ is one of the two generators of the stabilizer of z_2 in $\mathrm{GL}(2,\mathbb{Z})$. Further h and k have the same trace (since the trace of a matrix is preserved by conjugation). Therefore, by Theorem 2.1, $hz_1=z_1$ gives rise to F_{z_1} and $kz_2=z_2$ gives rise to F_{z_2}. So F_{z_1} and F_{z_2} have the same discriminant Δ. We see as above that gF_{z_1} has the hyperbolic number

$$\frac{-(2a_1rt+b_1ru+b_1ts+2c_1su)+\sqrt{\Delta}}{2(a_1r^2+b_1rs+c_1s^2)} = (g^{-1})^t z_1 = z_2,$$

and $-gF_{z_1}$ has the hyperbolic number

$$\frac{-(2a_1rt+b_1ru+b_1ts+2c_1su)-\sqrt{\Delta}}{2(a_1r^2+b_1rs+c_1s^2)} = (g^{-1})^t\overline{z_1} = \overline{z_2}$$

by (2.2). Consequently, $gF_{z_1}(x,1)$ and $F_{z_2}(x,1)$ have the same discriminant Δ and the same zeros, so they are equal. Hence F_{z_1} and F_{z_2} are (properly) equivalent. □

2.3. Remark. Note that in the proof of Theorem 2.2 we assume that F_{z_1} and F_{z_2} are properly equivalent. If we assume that F_{z_1} and F_{z_2} are improperly equivalent, then the proof is same.

2.4. Example. The forms $F_{z_1}=(1,7,-6)$ and $F_{z_2}=(4,3,-4)$ with discriminant $\Delta=73$ are properly equivalent under $g=\begin{pmatrix}7 & 9\\ 10 & 13\end{pmatrix} \in\mathrm{GL}(2,\mathbb{Z})$, that is, $gF_{z_1}=F_{z_2}$. The base point of F_{z_1} is $z_1=\frac{-7+\sqrt{73}}{2}$ and the base point of F_{z_2} is $z_2=\frac{-3+\sqrt{73}}{8}$. Note that $(g^{-1})^t=\begin{pmatrix}13 & -10\\ -9 & 7\end{pmatrix}$, and hence

$$
\begin{aligned}
(g^{-1})^t z_1 &= \frac{13\left(\frac{-7+\sqrt{73}}{2}\right)-10}{-9\left(\frac{-7+\sqrt{73}}{2}\right)+7} \\
&= \frac{-111+13\sqrt{73}}{77-9\sqrt{73}} \\
&= \frac{(-111+13\sqrt{73})(77+9\sqrt{73})}{(77-9\sqrt{73})(77+9\sqrt{73})} \\
&= \frac{-6+2\sqrt{73}}{16} \\
&= \frac{-3+\sqrt{73}}{8} \\
&= z_2.
\end{aligned}
$$

Now we define the following transformations for an indefinite quadratic form $F = (a, b, c)$.

$$
\begin{aligned}
&\tau(F) = (-a, b, -c) \\
&\psi(F) = (-a, -b, -c) = -F \\
&\xi(F) = (c, b, a) \\
&\chi(F) = (-c, b, -a).
\end{aligned}
$$

First, we consider the connection between them.

2.5. Theorem. *Let $F = (a, b, c)$ be an indefinite quadratic form. Then*

$$\chi\tau = \xi, \quad \xi\tau = \chi, \quad \chi\xi = \tau, \quad \xi\chi = \tau \quad \text{and} \quad \chi\xi\tau\psi = \psi.$$

Proof. Let $F = (a, b, c)$ be an indefinite quadratic form. Then

$$\tau(F) = (-a, b - c) \iff \chi(\tau(F)) = (c, b, a) = \xi(F).$$

Similarly, it can be shown that

$$
\begin{aligned}
&\tau(F) = (-a, b, -c) \iff \xi(\tau(F)) = (-c, b, -a) = \chi(F). \\
&\xi(F) = (c, b, a) \iff \chi(\xi(F)) = (-a, b, -c) = \tau(F). \\
&\chi(F) = (-c, b, -a) \iff \xi(\chi(F)) = (-a, b, -c) = \tau(F). \\
&\psi(F) = (-a, -b, -c) \iff \tau(\psi(F)) = (a, -b, c) \iff \xi(\tau(\psi(F)) = (c, -b, a) \\
&\qquad\qquad\qquad\qquad\iff \chi(\xi(\tau(\psi(F))) = (-a, -b, -c) = \psi(F).
\end{aligned}
$$

This completes the proof. □

2.6. Remark. Note that $\tau^{-1} = \tau, \psi^{-1} = \psi, \xi^{-1} = \xi$ and $\chi^{-1} = \chi$. So all the results obtained in Theorem 2.5 can be easily verified.

Now we can give the following results concerning the base points of indefinite quadratic forms in the cycle and proper cycle of F.

2.7. Theorem. *In the cycle $F_0 \sim F_1 \sim \cdots \sim F_{l-1}$ of F,*

$$z(F_i) = z(\chi(F_{l-1-i}))$$

for $0 \leq i \leq l - 1$.

Proof. Let $F = F_0 = (a_0, b_0, c_0)$. Then by (1.3) and (1.4), we get

$$\begin{aligned}
F_0 &= (a_0, b_0, c_0)\\
F_1 &= (a_1, b_1, c_1)\\
F_2 &= (a_2, b_2, c_2)\\
&\cdots\cdots\cdots\cdots\cdots\\
F_{\frac{l-3}{2}} &= (a_{\frac{l-3}{2}}, b_{\frac{l-3}{2}}, c_{\frac{l-3}{2}})\\
F_{\frac{l-1}{2}} &= (a_{\frac{l-1}{2}}, b_{\frac{l-1}{2}}, -a_{\frac{l-1}{2}})\\
F_{\frac{l+1}{2}} &= (-c_{\frac{l-3}{2}}, b_{\frac{l-3}{2}}, -a_{\frac{l-3}{2}})\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
F_{l-3} &= (-c_2, b_2, -a_2)\\
F_{l-2} &= (-c_1, b_1, -a_1)\\
F_{l-1} &= (-c_0, b_0, -a_0).
\end{aligned}$$

Hence it is easily seen that

$$\begin{aligned}
&F_0 = \chi(F_{l-1}),\ F_1 = \chi(F_{l-2}),\ F_2 = \chi(F_{l-3}), \cdots, F_{\frac{l-3}{2}} = \chi(F_{\frac{l+1}{2}}),\\
&F_{\frac{l-1}{2}} = \chi(F_{\frac{l-1}{2}}),\ F_{\frac{l+1}{2}} = \chi(F_{\frac{l-3}{2}}), \cdots,\ F_{l-3} = \chi(F_2),\\
&F_{l-2} = \chi(F_1) \text{ and } F_{l-1} = \chi(F_0).
\end{aligned}$$

Therefore $F_i = \chi(F_{l-1-i})$ for $0 \leq i \leq l-1$. Consequently $z(F_i) = z(\chi(F_{l-1-i}))$ for $0 \leq i \leq l-1$. □

The proper cycle of F can be derived by using its consecutive right neighbors. Let $R^i(F_0)$ be the $i-$th right neighbor of $F = F_0$. We proved in [3] that if l is odd, then the proper cycle of F is

$$F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \cdots \sim R^{2l-2}(F_0) \sim R^{2l-1}(F_0)$$

of length $2l$, and if l is even, then the proper cycle of F is

$$F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \cdots \sim R^{l-2}(F_0) \sim R^{l-1}(F_0)$$

of length l. From now on, we assume that the length of the proper cycle of F is odd throughout this paper.

2.8. Theorem. *In the proper cycle of F,*

$$\begin{aligned}
z(R^i(F_0)) &= z\left(\tau(R^{i+l}(F_0))\right) \textit{ for } 0 \leq i \leq l-1,\\
z(R^l(F_0)) &= z(\tau(F_0))\\
z\left(R^i(F_0)\right) &= z\left(\tau(R^{i-l}(F_0))\right) \textit{ for } l+1 \leq i \leq 2l-1.
\end{aligned}$$

Proof. Let $F = F_0 = (a_0, b_0, c_0)$. Then by (1.5) and (1.6), we get

$$\begin{aligned} F_0 &= (a_0, b_0, c_0) \\ R^1(F_0) &= (a_1, b_1, c_1) \\ R^2(F_0) &= (a_2, b_2, c_2) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ R^{l-1}(F_0) &= (a_{l-1}, b_{l-1}, c_{l-1}) \\ R^{l}(F_0) &= (-a_0, b_0, -c_0) \\ R^{l+1}(F_0) &= (-a_1, b_1, -c_1) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ R^{2l-3}(F_0) &= (-a_{l-3}, b_{l-3}, -c_{l-3}) \\ R^{2l-2}(F_0) &= (-a_{l-2}, b_{l-2}, -c_{l-2}) \\ R^{2l-1}(F_0) &= (-a_{l-1}, b_{l-1}, -c_{l-1}). \end{aligned}$$

Hence, it is easily seen that $R^i(F_0) = \tau(R^{i+l}(F_0))$ for $0 \le i \le l-1$, $R^l(F_0) = \tau(F_0)$ and $R^i(F_0) = \tau(R^{i-l}(F_0))$ for $l+1 \le i \le 2l-1$.

Therefore $z(R^i(F_0)) = z\left(\tau(R^{i+l}(F_0))\right)$ for $0 \le i \le l-1$, $z(R^l(F_0)) = z(\tau(F_0))$ and $z\left(R^i(F_0)\right) = z\left(\tau(R^{i-l}(F_0))\right)$ for $l+1 \le i \le 2l-1$, as we claimed. □

2.9. Theorem. *In the proper cycle of F,*

$$z(R^i(F_0)) = z(\xi(R^{2l-1-i}(F_0)))$$

for $0 \le i \le 2l-1$.

Proof. Let $F = F_0 = (a_0, b_0, c_0)$. Then by (1.5) and (1.6), we get

$$\begin{aligned} F_0 &= (a_0, b_0, c_0) \\ R^1(F_0) &= (a_1, b_1, c_1) \\ R^2(F_0) &= (a_2, b_2, c_2) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ R^{l-1}(F_0) &= (a_{l-1}, b_{l-1}, c_{l-1}) \\ R^{l}(F_0) &= (c_{l-1}, b_{l-1}, a_{l-1}) \\ R^{l+1}(F_0) &= (c_{l-2}, b_{l-2}, a_{l-2}) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ R^{2l-3}(F_0) &= (c_2, b_2, a_2) \\ R^{2l-2}(F_0) &= (c_1, b_1, a_1) \\ R^{2l-1}(F_0) &= (c_0, b_0, a_0). \end{aligned}$$

It is easily seen that $R^i(F_0) = \xi(R^{2l-1-i}(F_0))$ for $0 \le i \le 2l-1$. Therefore $z(R^i(F_0)) = z(\xi(R^{2l-1-i}(F_0)))$ for $0 \le i \le 2l-1$. □

2.10. Theorem. *In the proper cycle of F,*

$$z(R^i(F_0)) = z(\chi(R^{l-1-i}(F_0))) \text{ for } 0 \le i \le l-1$$
$$z(R^i(F_0)) = z(\chi(R^{3l-1-i}(F_0))) \text{ for } l \le i \le 2l-1.$$

Proof. Let $F = F_0 = (a_0, b_0, c_0)$. Then by (1.5) and (1.6), we get

$$
\begin{aligned}
F_0 &= (a_0, b_0, c_0)\\
R^1(F_0) &= (a_1, b_1, c_1)\\
R^2(F_0) &= (a_2, b_2, c_2)\\
&\cdots\cdots\cdots\cdots\\
R^{\frac{l-3}{2}}(F_0) &= (a_{\frac{l-3}{2}}, b_{\frac{l-3}{2}}, c_{\frac{l-3}{2}})\\
R^{\frac{l-1}{2}}(F_0) &= (a_{\frac{l-1}{2}}, b_{\frac{l-1}{2}}, c_{\frac{l-1}{2}})\\
R^{\frac{l+1}{2}}(F_0) &= (-c_{\frac{l-3}{2}}, b_{\frac{l-3}{2}}, -a_{\frac{l-3}{2}})\\
&\cdots\cdots\cdots\cdots\\
R^{l-3}(F_0) &= (-c_2, b_2, -a_2)\\
R^{l-2}(F_0) &= (-c_1, b_1, -a_1)\\
R^{l-1}(F_0) &= (-c_0, b_0, -a_0)\\
R^{l}(F_0) &= (-a_0, b_0, -c_0)\\
R^{l+1}(F_0) &= (-a_1, b_1, -c_1)\\
R^{l+2}(F_0) &= (-a_2, b_2, -c_2)\\
&\cdots\cdots\cdots\cdots\\
R^{\frac{3l-3}{2}}(F_0) &= (-a_{\frac{l-3}{2}}, b_{\frac{l-3}{2}}, -c_{\frac{l-3}{2}})\\
R^{\frac{3l-1}{2}}(F_0) &= (-a_{\frac{l-1}{2}}, b_{\frac{l-1}{2}}, -c_{\frac{l-1}{2}})\\
R^{\frac{3l-1}{2}}(F_0) &= (-a_{\frac{l+1}{2}}, b_{\frac{l+1}{2}}, -c_{\frac{l+1}{2}})\\
&\cdots\cdots\cdots\cdots\\
R^{2l-3}(F_0) &= (-a_{l-3}, b_{l-3}, -c_{l-3})\\
R^{2l-2}(F_0) &= (-a_{l-2}, b_{l-2}, -c_{l-2})\\
R^{2l-1}(F_0) &= (-a_{l-1}, b_{l-1}, -c_{l-1}).
\end{aligned}
$$

Hence, $R^i(F_0) = \chi(R^{l-1-i}(F_0))$ for $0 \le i \le l-1$, and $R^i(F_0) = \chi(R^{3l-1-i}(F_0))$ for $l \le i \le 2l-1$. Therefore $z(R^i(F_0)) = z(\chi(R^{l-1-i}(F_0)))$ for $0 \le i \le l-1$ and $z(R^i(F_0)) = z(\chi(R^{3l-1-i}(F_0)))$ for $l \le i \le 2l-1$. □

2.11. Theorem. *In the proper cycle of* F,

(1)
$$
\begin{aligned}
z(R^i(F_0)) &= z(\chi(\tau(R^{2l-1-i}(F_0)))) \textit{ for } 1 \le i \le 2l-2\\
z(R^{2l-1}(F_0)) &= z(\chi(\tau(F_0)))
\end{aligned}
$$

(2)
$$
\begin{aligned}
z(R^i(F_0)) &= z(\tau(R^{i+l}(F_0))) \quad \textit{for} \quad l \le i \le l-1\\
z(R^l(F_0)) &= z(\tau(F_0))\\
z(R^i(F_0)) &= z(\tau(R^{i-l}(F_0))) \textit{ for } l+1 \le i \le 2l-1.
\end{aligned}
$$

Proof. (1) Let $F = F_0 = (a_0, b_0, c_0)$. Then by (1.5) and (1.6),

$$\begin{aligned}
F_0 &= (a_0,\, b_0,\, c_0)\\
R^1(F_0) &= (a_1, b_1, c_1)\\
R^2(F_0) &= (a_2,\, b_2,\, c_2)\\
R^3(F_0) &= (a_3, b_3, c_3)\\
&\cdots\cdots\cdots\cdots\\
R^{l-3}(F_0) &= (a_{l-3}, b_{l-3}, c_{l-3})\\
R^{l-2}(F_0) &= (a_{l-2}, b_{l-2}, c_{l-2})\\
R^{l-1}(F_0) &= (a_{l-1}, b_{l-1}, c_{l-1})\\
R^{l}(F_0) &= (c_{l-1}, b_{l-1}, a_{l-1})\\
R^{l+1}(F_0) &= (c_{l-2}, b_{l-2}, a_{l-2})\\
R^{l+2}(F_0) &= (c_{l-3}, b_{l-3}, a_{l-3})\\
&\cdots\cdots\cdots\cdots\\
R^{2l-4}(F_0) &= (c_3, b_3, a_3)\\
R^{2l-3}(F_0) &= (c_2, b_2, a_2)\\
R^{2l-2}(F_0) &= (c_1, b_1, a_1)\\
R^{2l-1}(F_0) &= (c_0, b_0, a_0).
\end{aligned}$$

It is clear that $R^i(F_0) = \chi(\tau(R^{2l-1-i}(F_0)))$ for $1 \le i \le 2l-2$, and $R^{2l-1}(F_0) = \chi(\tau(F_0))$. Consequently $z(R^i(F_0)) = z(\chi(\tau(R^{2l-1-i}(F_0))))$ for $1 \le i \le 2l-2$ and $z(R^{2l-1}(F_0)) = z(\chi(\tau(F_0)))$.

(2) This can be proved as for (1). □

Now we split the proper cycle

$$F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \cdots \sim R^{2l-2}(F_0) \sim R^{2l-1}(F_0)$$

of F into two equal part as follows:

$$F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \cdots \sim R^{l-2}(F_0) \sim R^{l-1}(F_0) \tag{2.6}$$

and

$$R^l(F_0) \sim R^{l+1}(F_0) \sim \cdots \sim R^{2l-2}(F_0) \sim R^{2l-1}(F_0), \tag{2.7}$$

each of length l. We call (2.6) the *first part* and (2.7) the *second part* of the proper cycle of F.

2.12. Theorem. *Let $F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \cdots \sim R^{2l-2}(F_0) \sim R^{2l-1}(F_0)$ be the proper cycle of F.*

(1) *In the first part,*

$$z(\chi(R^i(F_0))) = z(R^{l-1-i}(F_0)) \text{ for } 1 \le i \le l-2$$
$$z(\chi(R^{l-1}(F_0))) = z(F_0).$$

(2) *In the second part,*

$$z(\chi(R^i(F_0))) = z(R^{3l-1-i}(F_0)) \text{ for } l \le i \le 2l-1.$$

Proof. (1) Let $F_0 = (a_0,\, b_0,\, c_0)$. Then by (1.5) and (1.6),

$$\begin{aligned}
F_0 &= (a_0,\, b_0,\, c_0)\\
R^1(F_0) &= (a_1,\, b_1,\, c_1)\\
R^2(F_0) &= (a_2,\, b_2,\, c_2)\\
R^3(F_0) &= (a_3, b_3, c_3)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
R^{\frac{l-3}{2}}(F_0) &= \left(a_{\frac{l-3}{2}},\, b_{\frac{l-3}{2}},\, c_{\frac{l-3}{2}}\right)\\
R^{\frac{l-1}{2}}(F_0) &= \left(a_{\frac{l-1}{2}},\, b_{\frac{l-1}{2}},\, -a_{\frac{l-1}{2}}\right)\\
R^{\frac{l+1}{2}}(F_0) &= \left(-c_{\frac{l-3}{2}},\, b_{\frac{l-3}{2}}, -a_{\frac{l-3}{2}}\right)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
R^{l-4}(F_0) &= (-c_3, b_3, -a_3)\\
R^{l-3}(F_0) &= (-c_2, b_2, -a_2)\\
R^{l-2}(F_0) &= (-c_1,\, b_1,\, -a_1)\\
R^{l-1}(F_0) &= (-c_0,\, b_0, -a_0)\,.
\end{aligned}$$

It is easily seen that $\chi(R^i(F_0)) = R^{l-1-i}(F_0)$ for $1 \leq i \leq l-2$, and $\chi(R^{l-1}(F_0)) = F_0$. So $z(\chi(R^i(F_0))) = z(R^{l-1-i}(F_0))$ for $1 \leq i \leq l-2$ and $z(\chi(R^{l-1}(F_0))) = z(F_0)$.

(2) Similarly it can be shown that

$$\begin{aligned}
R^l(F_0) &= (a_l,\, b_l, c_l)\\
R^{l+1}(F_0) &= (a_{l+1},\, b_{l+1},\, c_{l+1})\\
R^{l+2}(F_0) &= (a_{l+2}, b_{l+2},\, c_{l+2})\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
R^{\frac{3l-3}{2}}(F_0) &= \left(a_{\frac{3l-3}{2}},\, b_{\frac{3l-3}{2}},\, c_{\frac{3l-3}{2}}\right)\\
R^{\frac{3l-1}{2}}(F_0) &= \left(a_{\frac{3l-1}{2}},\, b_{\frac{3l-1}{2}}, -a_{\frac{3l-1}{2}}\right)\\
R^{\frac{3l+1}{2}}(F_0) &= \left(-c_{\frac{3l-3}{2}},\, b_{\frac{3l-3}{2}},\, -a_{\frac{3l-3}{2}}\right)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
R^{2l-3}(F_0) &= (-c_{l+2}, b_{l+2}, -a_{l+2})\\
R^{2l-2}(F_0) &= (-c_{l+1},\, b_{l+1},\, -a_{l+1})\\
R^{2l-1}(F_0) &= (-c_l,\, b_l, -a_l)\,.
\end{aligned}$$

Hence, $\chi\left(R^i(F_0)\right) = R^{3l-1-i}(F_0)$ for $l \leq i \leq 2l-1$. Consequently $z(\chi(R^i(F_0))) = z(R^{3l-1-i}(F_0))$ for $l \leq i \leq 2l-1$. □

2.13. Example. The proper cycle of $F = (1, 7, -6)$ is

$$\begin{aligned} F_0 = (1,7,-6) \sim R^1(F_0) = (-6,5,2) \sim\ R^2(F_0) = (2,7,-3)\\ \sim R^3(F_0) = (-3,5,4) \sim\ R^4(F_0) = (4,3,-4) \sim R^5(F_0) = (-4,5,3)\\ \sim R^6(F_0) = (3,7,-2) \sim R^7(F_0) = (-2,5,6) \sim\ R^8(F_0) = (6,7,-1)\\ \sim R^9(F_0) = (-1,7,6) \sim R^{10}(F_0) = (6,5,-2) \sim R^{11}(F_0) = (-2,7,3)\\ \sim R^{12}(F_0) = (3,5,-4) \sim R^{13}(F_0) = (-4,3,4) \sim R^{14}(F_0) = (4,5,-3)\\ \sim R^{15}(F_0) = (-3,7,2) \sim R^{16}(F_0) = (2,5,-6) \sim R^{17}(F_0) = (-6,7,1) \end{aligned}$$

of length 18. In the first part of the cycle,

$$\begin{aligned} F_0 = (1,7,-6) \sim R^1(F_0) = (-6,5,2) \sim\ R^2(F_0) = (2,7,-3)\\ \sim R^3(F_0) = (-3,5,4) \sim\ R^4(F_0) = (4,3,-4) \sim R^5(F_0) = (-4,5,3)\\ \sim R^6(F_0) = (3,7,-2) \sim R^7(F_0) = (-2,5,6) \sim\ R^8(F_0) = (6,7,-1), \end{aligned}$$

so we have,

$$\begin{aligned} &\chi(R^1(F_0)) = R^7(F_0),\ \chi(R^2(F_0)) = R^6(F_0),\ \chi(R^3(F_0)) = R^5(F_0),\\ &\chi(R^4(F_0)) = R^4(F_0),\ \chi(R^5(F_0)) = R^3(F_0),\ \chi(R^6(F_0)) = R^2(F_0),\\ &\chi(R^7(F_0)) = R^1(F_0),\ \chi(R^8(F_0)) = F_0. \end{aligned}$$

Therefore,

$$\begin{aligned} &z(\chi(R^1(F_0))) = z(R^7(F_0)),\ z(\chi(R^2(F_0))) = z(R^6(F_0)),\\ &z(\chi(R^3(F_0))) = z(R^5(F_0)),\ z(\chi(R^4(F_0))) = z(R^4(F_0)),\\ &z(\chi(R^5(F_0))) = z(R^3(F_0)),\ z(\chi(R^6(F_0))) = z(R^2(F_0)),\\ &z(\chi(R^7(F_0))) = z(R^1(F_0)),\ \text{and},\ z(\chi(R^8(F_0))) = z(F_0). \end{aligned}$$

In the second part of the cycle,

$$\begin{aligned} R^9(F_0) = (-1,7,6) \sim R^{10}(F_0) = (6,5,-2) \sim R^{11}(F_0) = (-2,7,3)\\ \sim R^{12}(F_0) = (3,5,-4) \sim R^{13}(F_0) = (-4,3,4) \sim R^{14}(F_0) = (4,5,-3)\\ \sim R^{15}(F_0) = (-3,7,2) \sim R^{16}(F_0) = (2,5,-6) \sim R^{17}(F_0) = (-6,7,1), \end{aligned}$$

so we have,

$$\begin{aligned} &\chi(R^9(F_0)) = R^{17}(F_0),\ \chi(R^{10}(F_0)) = R^{16}(F_0),\ \chi(R^{11}(F_0)) = R^{15}(F_0),\\ &\chi(R^{12}(F_0)) = R^{14}(F_0),\ \chi(R^{13}(F_0)) = R^{13}(F_0),\ \chi(R^{14}(F_0)) = R^{12}(F_0),\\ &\chi(R^{15}(F_0)) = R^{11}(F_0),\ \chi(R^{16}(F_0)) = R^{10}(F_0),\ \chi(R^{17}(F_0)) = R^9(F_0). \end{aligned}$$

Therefore,

$$\begin{aligned} &z(\chi(R^9(F_0))) = z(R^{17}(F_0)),\ z(\chi(R^{10}(F_0))) = z(R^{16}(F_0)),\\ &z(\chi(R^{11}(F_0))) = z(R^{15}(F_0)),\ z(\chi(R^{12}(F_0))) = z(R^{14}(F_0)),\\ &z(\chi(R^{13}(F_0))) = z(R^{13}(F_0)),\ z(\chi(R^{14}(F_0))) = z(R^{12}(F_0)),\\ &z(\chi(R^{15}(F_0))) = z(R^{11}(F_0)),\ z(\chi(R^{16}(F_0))) = z(R^{10}(F_0)),\ \text{and},\\ &z(\chi(R^{17}(F_0))) = z(R^9(F_0)). \end{aligned}$$

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