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SOME GOOD EXTENSIONS OF COMPACTNESS IN ŠOSTAK'S L-FUZZY TOPOLOGY

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Abstract

In this paper, we introduce good definitions of some weaker and stronger forms of L-fuzzy compactness in L-fuzzy topological spaces in Šostak's sense, where L is a fuzzy lattice. We define these concepts on arbitrary L-fuzzy sets, obtain various characterizations and study some of their properties.

Keywords: L-fuzzy topology, Fuzzy lattice, α-compactness, γ-compactness, β-compactness, Good extension

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1. Introduction

Kubiak [11] and Šostak [19-21] introduced the notion of (L-) fuzzy topological spaces as a generalization of L-topological spaces (originally called (L-) fuzzy topological spaces by Chang [4] and Goguen [7]). It is the grade of openness of an L-fuzzy set. A general approach to the study of topological-type structures on fuzzy powersets was developed in [8-12].

The notion of compactness is one of the most important concepts in general topology. Therefore, the problem of generalizing classical compactness to fuzzy topological spaces has been intensively discussed over the past 30 years. Many papers on fuzzy compactness have been published and various kinds of fuzzy compactness have been presented and studied. Among these compactness, the fuzzy compactness in L-fuzzy topological spaces introduced by Warner and McLean [22] and extended to arbitrary L-fuzzy sets by Kudri [13] possesses several nice properties, such as: this compactness is defined for arbitrary L-fuzzy sets, is inherited by closed L-fuzzy sets, is preserved under fuzzy continuous functions and arbitrary products; is a good extension and every compact Hausdorff space

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is regular and normal. Good extensions of some weaker and stronger fuzzy covering properties were introduced and studied by Kudri and Warner.

Aygün *et al.* [2], introduced the notion of L-fuzzy compactness in L-fuzzy topological spaces in the sense of Šostak as a generalization of the L-fuzzy compactness introduced by Warner and McLean [22]. Based on this definition various kinds of compactness in L-fuzzy topological spaces in Šostak's sense have been introduced and studied in [2,3,15].

In this paper, good definitions of some weaker and stronger forms of L-fuzzy compactness on arbitrary fuzzy sets are introduced in L-fuzzy topological spaces in Šostak's sense along the same lines as the L-fuzzy compactness defined by Aygün *et al.* [2]. We prove the goodness of the proposed definitions, obtain various characterizations and study some of their properties.

2. Preliminaries

Throughout this paper X will be a non-empty ordinary set and $L = L(\leq, \vee, \wedge, ')$ will denote a fuzzy lattice, i.e., a completely distributive lattice with a smallest element 0 and a largest element 1 $(0 \neq 1)$ and with an order reversing involution $a \to a'$ $(a \in L)$ [14]. We shall denote by L^X the lattice of all L-fuzzy sets on X. For an ordinary subset A of X, we denote by χ_A the characteristic function of A.

2.1. Definition. [6]. An element p of L is called *prime* iff $p \neq 1$ and whenever $a, b \in L$ with $a \wedge b \leq p$ then $a \leq p$ or $b \leq p$. The set of all prime elements of L will be denoted by $P_r(L)$.

2.2. Definition. [6] An element q of L is called *union-irreducible* iff whenever $a, b \in L$ with $q \leq a \vee b$ then $q \leq a$ or $q \leq b$. The set of all nonzero union-irreducible elements of L will be denoted by $M(L)$.

Thus, $p \in P_r(L)$ iff $p' \in M(L)$.

2.3. Definition. [2] Let (X, T) be an ordinary topological space. A function $f : (X, T) \to L$, where L has its Scott topology (topology generated by the sets of the form $\{t \in L : t \not\leq p\}$ where $p \in P_r(L)$[22]), is said to be *Scott continuous* iff for every $p \in P_r(L)$, $f^{-1}\{t \in L : t \not\leq p\} \in T$.

2.4. Definition. [2] Let (X, T) be an ordinary topological space, and $q \in L$. A function $f : (X, T) \to L$, where L has its Scott topology, is said to be *q-Scott continuous* iff for every $p \in P_r(L)$ with $q \not\leq p$, $f^{-1}\{t \in L : t \not\leq p\} \in T$.

It is clear that if f is Scott continuous then f is q-Scott continuous for every $q \in L$. Moreover, f is 1-Scott continuous iff f is Scott continuous. Naturally, every function from (X, T) to L is 0-Scott continuous.

2.5. Definition. [20] An L-fuzzy topology on X is a map $\mathcal{T} : L^X \to L$ satisfying the following three axioms:

(O1) $\mathcal{T}(\chi_\phi) = \mathcal{T}(\chi_X) = 1$,
(O2) $\mathcal{T}(f \wedge g) \geq \mathcal{T}(f) \wedge \mathcal{T}(g)$, for every $f, g \in L^X$,
(O3) $\mathcal{T}(\bigvee_{i \in I} f_i) \geq \bigwedge_{i \in I} \mathcal{T}(f_i)$, for every family $(f_i)_{i \in I}$ in L^X.

The pair $(X, \mathcal{T})$ is called an *L-fuzzy topological space* (L-fts, for short). For every $f \in L^X$, $\mathcal{T}(f)$ is called the *degree of openness* of the L-fuzzy subset f.

2.6. Definition. [20] Let $(X, \mathcal{T})$ be an L-fts. The map $\mathcal{F}_\mathcal{T} : L^X \to L$ defined by $\mathcal{F}_\mathcal{T}(g) = \mathcal{T}(g')$ for every $g \in L^X$ is called the *degree of closedness* on X.

2.7. Definition. [5] Let $(X, \mathcal{T})$ be an L-fts and $f \in L^X$.

(1) The *closure of* f, denoted by $\mathrm{cl}\,(f)$, is defined by

$$\mathrm{cl}\,(f) = \bigwedge \Big\{ g \in L^X : \mathcal{T}(g') > 0,\ f \le g \Big\}.$$

(2) The *interior of* f, denoted by $\mathrm{int}\,(f)$, is defined by

$$\mathrm{int}\,(f) = \bigvee \Big\{ g \in L^X : \mathcal{T}(g) > 0,\ g \le f \Big\}.$$

2.8. Definition. [16-18] Let $(X, \mathcal{T})$ be an L-fts and $f \in L^X$.

(1) f is called *fuzzy semi-open* iff for every $p \in P_r(L)$ there exists $g \in L^X$ with $\mathcal{T}(g) \not\le p$ such that $g \le f \le \mathrm{cl}\,(g)$.
(2) f is called *fuzzy α-open* iff for every $p \in P_r(L)$ there exists $g \in L^X$ with $\mathcal{T}(g) \not\le p$ such that $g \le f \le \mathrm{int}\,(\mathrm{cl}\,(g))$.
(3) f is called *fuzzy regular open* iff $f = \mathrm{int}\,(\mathrm{cl}\,(f))$.
(4) f is called *fuzzy pre-open* iff $f \le \mathrm{int}\,(\mathrm{cl}\,(f))$.
(5) f is called *fuzzy β-open* iff $f \le \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(f)))$.
(6) f is called *fuzzy γ-open* iff $f \le \mathrm{int}\,(\mathrm{cl}\,(f)) \vee \mathrm{cl}\,(\mathrm{int}\,(f))$.
(7) f is called *fuzzy regular semi-open* iff there exists fuzzy regular open L-fuzzy set $g \in L^X$ with $g \le f \le \mathrm{cl}\,(g)$

2.9. Definition. Let $F : (X, \mathcal{T}) \to (Y, \mathcal{T}^*)$ be a function. Then,

(1) F is called *fuzzy continuous* iff $\forall g \in L^Y$, $\mathcal{T}(F^{-1}(g)) \ge \mathcal{T}^*(g)$ [20].
(2) F is called *fuzzy irresolute* iff for each fuzzy semi-open set $g \in L^Y$, $F^{-1}(g)$ is a fuzzy semi-open set of X [16].
(3) F is called *fuzzy α-irresolute* iff for each fuzzy α-open set $g \in L^Y$, $F^{-1}(g)$ is a fuzzy α-open set of X [18].

2.10. Theorem. [2] *Let (X, T) be an ordinary topological space. Then, the function $W(T) : L^X \to L$ defined by*

$$W(T)(f) = \bigvee \big\{ q \in L : f \text{ is } q\text{–Scott continuous} \big\},$$

for every $f \in L^X$, is an L-fuzzy topology on X.

2.11. Theorem. [2] *If $F : (X, T) \to (Y, T^*)$ is continuous, then $F : (X, W(T)) \to (Y, W(T^*))$ is fuzzy continuous.*

Thus, by Theorems 2.10, 2.11, we obtain an L-fts from a given ordinary topological space, and the functor W from the category **TOP** of ordinary topological space into the category **FTS** of L-fts. This provides a “goodness of extension” criterion for L-fuzzy topological properties. An L-fuzzy extension of a topological property of (X, T) is said to be *good* when it is possessed by the L-fts $(X, W(T))$ iff the original property is possessed by (X, T).

2.12. Lemma. *[15] Let (X, T) be a topological space and $f \in L^X$. Considering the L-fts $(X, W(T))$ we have:*

(i) $(\mathrm{cl}\,(f))^{-1}\big\{t \in L : t \not\le p\big\} \subseteq \mathrm{cl}\,\big(f^{-1}\big\{t \in L : t \not\le p\big\}\big)$.
(ii) $(\mathrm{int}\,(f))^{-1}\big\{t \in L : t \not\le p\big\} \subseteq \mathrm{int}\,\big(f^{-1}\big\{t \in L : t \not\le p\big\}\big)$.

2.13. Corollary. [15] *Let (X, T) be a topological space and $A \subseteq X$. Considering the L-fts $(X, W(T))$, we have*

$$\chi_{\mathrm{cl}\,(A)} = \mathrm{cl}\,(\chi_A) \text{ and } \chi_{\mathrm{int}\,(A)} = \mathrm{int}\,(\chi_A).$$

2.14. Definition. [2] Let $(X, \mathfrak{T})$ be an L-fts and $g \in L^X$. The L-fuzzy subset g is said to be *compact* iff for every $p \in P_r(L)$ and every collection $(f_i)_{i\in I}$ of L-fuzzy sets with $\mathfrak{T}(f_i) \not\leq p \,\forall\, i \in I$ and $(\bigvee_{i\in I} f_i)(x) \not\leq p \,\forall\, x \in X$ with $g(x) \geq p'$, there is a finite subset I_0 of I such that $(\bigvee_{i\in I_0} f_i)(x) \not\leq p \,\forall\, x \in X$ with $g(x) \geq p'$. If $g = 1_X$, we say that the L-fts $(X, \mathfrak{T})$ is *compact*.

In the crisp case of $\mathfrak{T}$, fuzzy compactness coincides with the compactness introduced by Warner and McLean [22], and extended to arbitrary L-fuzzy sets by Kudri [13].

2.15. Definition. [15] Let $(X, \mathfrak{T})$ be an L-fts and $g \in L^X$. The L-fuzzy subset g is said to be *almost compact* iff for every $p \in P_r(L)$ and every collection $(f_i)_{i\in I}$ of L-fuzzy sets with $\mathfrak{T}(f_i) \not\leq p\ \forall,\ i \in I$ and $\big(\bigvee_{i\in I} f_i\big)(x) \not\leq p,\ \forall,\ x \in X$ with $g(x) \geq p'$, there is a finite subset I_0 of I such that $\big(\bigvee_{i\in I_0} \mathrm{cl}\,(f_i)\big)(x) \not\leq p\ \forall,\ x \in X$ with $g(x) \geq p'$. If $g = 1_X$, we say that the L-fts $(X, \mathfrak{T})$ is *almost compact*.

2.16. Definition. [3] Let $(X, \mathfrak{T})$ be an L-fts and $g \in L^X$. The L-fuzzy subset g is said to be *RS-compact* (resp. *S-closed*) iff for every $p \in P_r(L)$ and every collection $(f_i)_{i\in I}$ of semi-open L-fuzzy sets with $\big(\bigvee_{i\in I} f_i\big)(x) \not\leq p,\ \forall\, x \in X$ with $g(x) \geq p'$, there is a finite subset I_0 of I such that $\big(\bigvee_{i\in I_\circ} \mathrm{int}\,(cl(f_i))\big)(x) \not\leq p,\ \forall\, x \in X$ (resp. $\big(\bigvee_{i\in I_\circ} \mathrm{cl}\,(f_i)\big)(x) \not\leq p,\ \forall\, x \in X$) with $g(x) \geq p'$. If $g = 1_X$, we say that the L-fts $(X, \mathfrak{T})$ is *RS-compact* (resp. *S-closed*).

3. Proposed Definitions and their Goodness Theorems

3.1. Definition. Let $(X, \mathfrak{T})$ be an L-fts and $g \in L^X$. The L-fuzzy subset g is said to be *α-compact* (resp. *α-closed*) iff for every $p \in P_r(L)$ and every collection $(f_i)_{i\in I}$ of α-open L-fuzzy sets with $\big(\bigvee_{i\in I} f_i\big)(x) \not\leq p,\ \forall\, x \in X$ with $g(x) \geq p'$, there is a finite subset I_0 of I such that $(\bigvee_{i\in I_\circ} (f_i))(x) \not\leq p,\ \forall\, x \in X$ (resp. $\big(\bigvee_{i\in I_\circ} \mathrm{cl}\,(f_i)\big)(x) \not\leq p,\ \forall\, x \in X$) with $g(x) \geq p'$. If $g = 1_X$, we say that the L-fts $(X, \mathfrak{T})$ is *α-compact* (resp. *α-closed*).

In the following theorem we shall prove the goodness of the notions of α-compactness and α-closedness in L-fts's.

3.2. Theorem. *Let (X, T) be an ordinary topological space. Then $(X, W(T))$ is α-compact (resp. α-closed) iff (X, T) is α-compact (resp. α-closed).*

Proof. (*Necessity*). Let $(A_i)_{i\in I}$ be an α-open cover of (X, T). Then for each $i \in I$, there exists $B_i \in T$ such that $B_i \subseteq A_i \subseteq \mathrm{int}\,(\mathrm{cl}\,(B_i))$. But, the characteristic function of every open set is 1-Scott continuous, so $W(T)(\chi_{B_i}) = 1 \not\leq p,\ \forall\, i \in I$ and $p \in P_r(L)$. Also, $\chi_{B_i} \leq \chi_{A_i} \leq \chi_{\mathrm{int}\,(cl(B_i))} = \mathrm{int}\,(cl(\chi_{B_i}))$ (Corollary 2.13). Hence $(\chi_{A_i})_{i\in I}$ is a family of α-open L-fuzzy sets in X with $\big(\bigvee_{i\in I} \chi_{A_i}\big)(x) = 1 \not\leq p,\ \forall x \in X$. From the α-closedness of $(X, W(T))$, there exists a finite subset $I_\circ$ of I such that $\big(\bigvee_{i\in I_\circ} \mathrm{cl}\,(\chi_{A_i}\big)(x) \not\leq p,\ \forall\, x \in X$. Then by Corollary 2.13, we have $\big(\bigvee_{i\in I_\circ} \chi_{\mathrm{cl}\,(A_i)}\big)(x) \not\leq p,\ \forall\, x \in X$, i.e., $\forall\, x \in X$, $x \in \big(\bigvee_{i\in I_\circ} \chi_{\mathrm{cl}\,(A_i)}\big)^{-1}\big\{t \in L : t \not\leq p\big\}$, then by [2, Lemma 3.3], $x \in \bigcup_{i\in I_\circ} (\chi_{\mathrm{cl}\,(A_i)})^{-1}\big\{t \in L : t \not\leq p\big\} = \bigcup_{i\in I_\circ} \mathrm{cl}\,(A_i)$. Hence, $X \subseteq \bigcup_{i\in I_\circ} \mathrm{cl}\,(A_i))$. Thus, (X, T) is α-closed.

(*Sufficiency*). Let $p \in P_r(L)$ and let $(f_i)_{i\in I}$ be a family of α-open L-fuzzy sets in $(X, W(T))$ with $\big(\bigvee_{i\in I} f_i\big)(x) \not\leq p,\ \forall\, x \in X$. For each $x \in X$, there is $i \in I$ with $f_i(x) \not\leq p$, i.e., there exists $i \in I$ with $x \in f_i^{-1}\big\{t \in L : t \not\leq p\big\}$. Then, $X \subseteq \bigcup_{i\in I} f_i^{-1}\big\{t \in L : t \not\leq p\big\}$. We also have that, for each $i \in I$, $f_i^{-1}\{t \in L : t \not\leq p\}$ is α-open in (X, T) because, for each $i \in I$, there is $g_i \in L^X$, $W(T)(g_i) \not\leq p$ with $g_i \leq f_i \leq \operatorname{int}(\operatorname{cl}(g_i))$. Then

$$\begin{aligned} g_i^{-1}\{t \in L : t \not\leq p\} &\subseteq f_i^{-1}\{t \in L : t \not\leq p\} \\ &\subseteq (\operatorname{int}(\operatorname{cl}(g_i)))^{-1}\{t \in L : t \not\leq p\} \\ &\subseteq \operatorname{int}(\operatorname{cl}(g_i^{-1}\{t \in L : t \not\leq p\}), \quad \text{(Lemma 2.12)} \end{aligned}$$

$\forall p \in P_r(L)$. Since $W(T)(g_i) \not\leq p$ then $g_i^{-1}\{t \in L : t \not\leq p\} \in T$. Hence $f_i^{-1}\{t \in L : t \not\leq p\}$ is α-open for every $i \in I$. Therefore, the family $(f_i^{-1}\{t \in L : t \not\leq p\})_{i\in I}$ is a family of α-open sets in (X, T) covering X. From the α-closedness, there is a finite subset $I_\circ$ of I such that $X \subseteq \bigcup_{i\in I_\circ} \operatorname{cl}(f_i^{-1}\{t \in L : t \not\leq p\}))$. Then for every $x \in X$, there is $i \in I_\circ$ such that $\operatorname{cl}(f_i))(x) \not\leq p$. So, $\left(\bigvee_{i\in I_\circ} \operatorname{cl}(x)\right) \not\leq p$. Hence $(X, W(T))$ is α-closed.

The proof for α-compactness is similar. □

3.3. Definition. Let $(X, \mathfrak{T})$ be an L-fts and $g \in L^X$. The L-fuzzy subset g is said to be *γ-compact* (resp. *γ-closed*) iff for every $p \in P_r(L)$ and every collection $(f_i)_{i\in I}$ of γ-open L-fuzzy sets with $\left(\bigvee_{i\in I} f_i\right)(x) \not\leq p$, $\forall x \in X$ with $g(x) \geq p'$, there is a finite subset I_0 of I such that $\left(\bigvee_{i\in I_\circ}(f_i)\right)(x) \not\leq p$, $\forall x \in X$ (resp. $\left(\bigvee_{i\in I_\circ} \operatorname{cl}(f_i)\right)(x) \not\leq p$, $\forall x \in X$) with $g(x) \geq p'$. If $g = 1_X$, we say that the L-fts $(X, \mathfrak{T})$ is *γ-compact* (resp. *γ-closed*).

3.4. Lemma. *Let (X, T) be a topological space and $A \subseteq X$. Then A is γ-open in (X, T) iff χ_A is γ-open in the L-fts $(X, W(T))$.*

Proof. A is γ-open in (X, T) iff $A \subseteq \operatorname{int}(\operatorname{cl}(A)) \cup \operatorname{cl}(\operatorname{int}(A))$ [1] iff

$$\begin{aligned} \chi_A &\leq \chi_{[\operatorname{int}(\operatorname{cl}(A))\cup\operatorname{cl}(\operatorname{int}(A))]} \\ &= \chi_{\operatorname{int}(\operatorname{cl}(A))} \vee \chi_{\operatorname{cl}(\operatorname{int}(A))} \\ &= \operatorname{int}(\operatorname{cl}(\chi_A) \vee \operatorname{cl}(\operatorname{int}(\chi_A)). \text{ (this equality is due to Corollary 2.13)} \end{aligned}$$

Then, χ_A is γ-open in $(X, W(T))$. □

3.5. Lemma. *Let (X, T) be a topological space, $p \in P_r(L)$ and $f \in L^X$. Then f is γ-open in the L-fts $(X, W(T))$ if $f^{-1}\{t \in L : t \not\leq p\}$ is γ-open in (X, T).*

Proof. Suppose that f is a γ-open L-fuzzy set in $(X, W(T))$. Then $f \leq \operatorname{int}(\operatorname{cl}(f)) \vee \operatorname{cl}(\operatorname{int}(f))$, which implies that for $p \in P_r(L)$,

$$\begin{aligned} f^{-1}\{t \in L : t \notin p\} &\subseteq \left(\operatorname{int}(\operatorname{cl}(f)) \vee \operatorname{cl}(\operatorname{int}(f))\right)^{-1}\{t \in L : t \not\leq p\} \\ &\subseteq \left(\operatorname{int}(\operatorname{cl}(f))\right)^{-1}\{t \in L : t \not\leq p\} \cup \left(\operatorname{cl}(\operatorname{int}(f))\right)^{-1}\{t \in L : t \not\leq p\} \\ &\qquad \text{(Lemma 3.3[2])} \\ &\subseteq \operatorname{int}\left(\operatorname{cl}\left(f^{-1}\{t \in L : t \not\leq p\}\right)\right) \cup \operatorname{cl}\left(\operatorname{int}\left(f^{-1}\{t \in L : t \not\leq p\}\right)\right). \\ &\qquad \text{(Lemma 2.12)} \end{aligned}$$

So, $f^{-1}\{t \in L : t \not\leq p\}$ is γ-open in (X, T). □

With the following theorem we prove the goodness of the notions of γ-compactness and γ-closedness in L-fts's.

3.6. Theorem. *Let (X, T) be an ordinary topological space. Then $(X, W(T))$ is γ-compact (resp. γ-closed) iff (X, T) is γ-compact (resp. γ-closed).*

Proof. By Lemmas 3.4, 3.5, the proof is similar to that of Theorem 3.2. □

3.7. Definition. Let $(X, \mathcal{T})$ be an L-fts and $g \in L^X$. The L-fuzzy subset g is said to be *β-compact* (resp. *strongly-compact*) iff for every $p \in P_r(L)$ and every collection $(f_i)_{i\in I}$ of β-open (resp. preopen) L-fuzzy sets with $(\bigvee_{i\in I} f_i)(x) \not\leq p$, $\forall\, x \in X$ with $g(x) \geq p'$, there is a finite subset I_0 of I such that $(\bigvee_{i\in I_\circ} f_i)(x) \not\leq p$, $\forall\, x \in X$ with $g(x) \geq p'$. If $g = 1_X$, we say that the L-fts $(X, \mathcal{T})$ is *β-compact* (resp. *strongly-compact*).

With the following theorem we prove the goodness of the notions of β-compactness and strong-compactness in L-fts's.

3.8. Theorem. *Let (X, T) be an ordinary topological space. Then $(X, W(T))$ is β-compact (resp. strongly compact) iff (X, T) is β-compact (resp. strongly-compact).*

Proof. The proof is similar to that of Theorem 3.2. □

3.9. Definition. Let $(X, \mathcal{T})$ be an L-fts and $g \in L^X$. The L-fuzzy subset g is said to be *$\mathcal{S}^*$-closed* iff for every $p \in P_r(L)$ and every collection $(f_i)_{i\in I}$ of regular semi-open L-fuzzy sets with $\big(\bigvee_{i\in I} f_i\big)(x) \not\leq p$, $\forall\, x \in X$ with $g(x) \geq p'$, there is a finite subset I_0 of I such that $\big(\bigvee_{i\in I_\circ} f_i\big)(x) \not\leq p$, $\forall x \in X$ with $g(x) \geq p'$. If $g = 1_X$, we say that the L-fts $(X, \mathcal{T})$ is *$\mathcal{S}^*$-closed.*

To prove the goodness of $\mathcal{S}^*$-closedness, we need the following lemma.

3.10. Lemma. *Let (X, T) be an ordinary topological space, f a regular open L-fuzzy set in the L-fts $(X, W(T))$ and $p \in P_r(L)$. Then we have*

(i) $\operatorname{int}\big(\operatorname{cl}(f^{-1}\{t \in L : t \not\leq p\})\big) \subseteq f^{-1}\big\{t \in L : t \not\leq p\big\}$.

(ii) $f^{-1}\big\{t \in L : t \not\leq p\big\}$ *is regular open in* (X, T).

Proof. (i) We are going to prove that any regular open set C in (X, T) with $f^{-1}\big\{t \in L : t \not\leq p\big\} \subseteq C$ satisfies $\operatorname{int}\big(cl\big(f^{-1}\big\{t \in L : t \not\leq p\big\}\big)\big) \subseteq C$. Let $f^{-1}\big\{t \in L : t \not\leq p\big\} \subseteq C$, and let $g : X \to L$ be a function defined by

$$g(x) = \begin{cases} 1 & \text{if } x \in C, \\ p & \text{otherwise.} \end{cases}$$

Since for every $e \in L$,

$$g^{-1}\big\{t \in L : t \geq e\big\} = \begin{cases} X & \text{if } e \leq p, \\ C & \text{if } e \not\leq p, \end{cases}$$

we have $g^{-1}\big\{t \in L : t \geq e\big\}$ is regular open in (X, T) for all $e \in L$. Also, we have $f \leq g$. Then $f^{-1}\big\{t \in L : t \not\leq p\big\} \subseteq g^{-1}\big\{t \in L : t \not\leq p\big\} = C$. So, $\operatorname{int}\big(\operatorname{cl}\big(f^{-1}\{t \in L : t \not\leq p\}\big)\big) \subseteq \operatorname{int}(\operatorname{cl}(C)) = C$. Hence,

$$\operatorname{int}\big(\operatorname{cl}(f^{-1}\big\{t \in L : t \not\leq p\big\})\big) \subseteq f^{-1}\big\{t \in L : t \not\leq p\big\}.$$

(ii) Since f is a regular open L-fuzzy set in $(X, W(T))$, we have $f = \operatorname{int}(\operatorname{cl}(f)$. Then $f^{-1}\big\{t \in L : t \not\leq p\big\} = \operatorname{int}\big(\operatorname{cl}(f)\big)^{-1}\big\{t \in L : t \not\leq p\big\} \subseteq \operatorname{int}\big(\operatorname{cl}(f^{-1}\big\{t \in L : t \not\leq p\big\})\big)$ (Lemma 2.12). Hence by using (i), we have $f^{-1}\big\{t \in L : t \not\leq p\big\} = \operatorname{int}(\operatorname{cl}(f^{-1}\big\{t \in L : t \not\leq p\big\})\big)$. □

3.11. Theorem. *Let (X, T) be an ordinary topological space. Then $(X, W(T))$ is $\mathcal{S}^*$-closed iff (X, T) is $\mathcal{S}^*$-closed.*

Proof. By Lemma 3.10, the proof is similar to that of Theorem 3.2. □

4. Characterizations and Comparisons

The next theorem provides a different description of α-compactness in L-fuzzy topological spaces.

4.1. Theorem. *Let $(X, \mathfrak{T})$ be an L-fts and $g \in L^X$. The L-fuzzy set g is α-compact iff for every $p \in P_r(L)$ and every collection $(f_i)_{i\in I}$ of α-open L-fuzzy sets with $\big(\bigvee_{i\in I} f_i \vee g'\big)(x) \not\leq p$ for all $x \in X$, there is a finite subset $I_\circ$ of I such that $\big(\bigvee_{i\in I_\circ} f_i \vee g'\big)(x) \not\leq p$ for all $x \in X$.*

Proof. (*Necessity*). Let $p \in P_r(L)$ and $(f_i)_{i\in I}$ a collection of α-open L-fuzzy sets with $\big(\bigvee_{i\in I} f_i \vee g'\big)(x) \not\leq p$ for all $x \in X$. Then, $\big(\bigvee_{i\in I} f_i\big)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$. Since, g is α-compact, there is a finite subset $I_\circ$ of I such that $\big(\bigvee_{i\in I_\circ} f_i\big)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$.

Take an arbitrary $x \in X$. If $g'(x) \leq p$ then $g'(x) \vee \big(\bigvee_{i\in I_\circ} f_i\big)(x) = \big(\bigvee_{i\in I_\circ} f_i \vee g'\big)(x) \not\leq p$ because $\big(\bigvee_{i\in I_\circ} f_i\big)(x) \not\leq p$. If $g'(x) \not\leq p$ then we have

$$g'(x) \vee \big(\bigvee_{i\in I_\circ} f_i\big)(x) = \big(\bigvee_{i\in I_\circ} f_i \vee g'\big)(x) \not\leq p.$$

Thus, we have $\big(\bigvee_{i\in I_\circ} f_i \vee g'\big)(x) \not\leq p$ for all $x \in X$.

(*Sufficiency*). Let $p \in P_r(L)$, and let $(f_i)_{i\in I}$ be a collection of α-open L-fuzzy sets of X with $\big(\bigvee_{i\in I} f_i\big)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$. Hence, $\big(\bigvee_{i\in I} f_i \vee g'\big)(x) \not\leq p$ for all $x \in X$. From the hypothesis, there is a finite subset $I_\circ$ of I such that $\big(\bigvee_{i\in I_\circ} f_i \vee g'\big)(x) \not\leq p$ for all $x \in X$. Then, $\big(\bigvee_{i\in I_\circ} f_i\big)(x) \not\leq p$ for all $x \in X$ with $g'(x) \leq p$. Hence, g is α-compact. □

As stated in the next theorems similar descriptions are valid for the properties of α-closed, γ-compact, γ-closed, β-compact, strongly compact, S-closed, RS-compact and $\mathcal{S}^*$-closed in L-fts.

4.2. Theorem. *Let $(X, \mathfrak{T})$ be an L-fts and $g \in L^X$. The L-fuzzy set g is α-closed iff for every $p \in P_r(L)$ and every collection $(f_i)_{i\in I}$ of α-open L-fuzzy sets with $\bigvee_{i\in I} \big[(f_i) \vee g'\big](x) \not\leq p$ for all $x \in X$, there is a finite subset $I_\circ$ of I such that $\big(\bigvee_{i\in I_\circ} \operatorname{cl}(f_i) \vee g'\big)(x) \not\leq p$ for all $x \in X$.*

Proof. Very similar to the proof of Theorem 4.1. □

4.3. Theorem. *Let $(X, \mathfrak{T})$ be an L-fts and $g \in L^X$. The L-fuzzy set g is γ-compact iff for every $p \in P_r(L)$ and every collection $(f_i)_{i\in I}$ of γ-open L-fuzzy sets with $\big(\bigvee_{i\in I} f_i \vee g'\big)(x) \not\leq p$ for all $x \in X$, there is a finite subset $I_\circ$ of I such that $\big(\bigvee_{i\in I_\circ} f_i \vee g'\big)(x) \not\leq p$ for all $x \in X$.*

Proof. Very similar to the proof of Theorem 4.1. □

4.4. Theorem. *Let $(X, \mathfrak{T})$ be an L-fts and $g \in L^X$. The L-fuzzy set g is γ-closed iff for every $p \in P_r(L)$ and every collection $(f_i)_{i\in I}$ of γ-open L-fuzzy sets with $\bigvee_{i\in I} \big[(f_i) \vee g'\big](x) \not\leq p$ for all $x \in X$, there is a finite subset $I_\circ$ of I such that $\big(\bigvee_{i\in I_\circ} \operatorname{cl}(f_i) \vee g'\big)(x) \not\leq p$ for all $x \in X$.*

Proof. Very similar to the proof of Theorem 4.1. □

4.5. Theorem. *Let $(X, \mathfrak{T})$ be an L-fts and $g \in L^X$. The L-fuzzy set g is β-compact (resp. strong compact) iff for every $p \in P_r(L)$ and every collection $(f_i)_{i\in I}$ of β-open (resp. preopen) L-fuzzy sets with $\big(\bigvee_{i\in I} f_i \vee g'\big)(x) \not\leq p$ for all $x \in X$, there is a finite subset $I_\circ$ of I such that $\big(\bigvee_{i\in I_\circ} f_i \vee g'\big)(x) \not\leq p$ for all $x \in X$.*

Proof. Very similar to the proof of Theorem 4.1. □

4.6. Theorem. *Let $(X, \mathcal{T})$ be an L-fts and $g \in L^X$. The L-fuzzy set g is S-closed iff for every $p \in P_r(L)$ and every collection $(f_i)_{i\in I}$ of semiopen L-fuzzy sets with $\bigvee_{i\in I}\big[(f_i) \vee g'\big](x) \not\leq p$ for all $x \in X$, there is a finite subset $I_\circ$ of I such that $\big(\bigvee_{i\in I_\circ} \mathrm{cl}\,(f_i) \vee g'\big)(x) \not\leq p$ for all $x \in X$.*

Proof. Very similar to the proof of Theorem 4.1. □

4.7. Theorem. *Let $(X, \mathcal{T})$ be an L-fts and $g \in L^X$. The L-fuzzy set g is RS-compact iff for every $p \in P_r(L)$ and every collection $(f_i)_{i\in I}$ of semiopen L-fuzzy sets with $\bigvee_{i\in I}\big[(f_i) \vee g'\big](x) \not\leq p$ for all $x \in X$, there is a finite subset $I_\circ$ of I such that $\big(\bigvee_{i\in I_\circ} int(cl(f_i)) \vee g'\big)(x) \not\leq p$ for all $x \in X$.*

Proof. Very similar to the proof of Theorem 4.1. □

4.8. Theorem. *Let $(X, \mathcal{T})$ be an L-fts and $g \in L^X$. The L-fuzzy set g is $\mathcal{S}^*$-closed iff for every $p \in P_r(L)$ and every collection $(f_i)_{i\in I}$ of regular semiopen L-fuzzy sets with $\bigvee_{i\in I}\big[(f_i) \vee g'\big](x) \not\leq p$ for all $x \in X$, there is a finite subset $I_\circ$ of I such that $\big(\bigvee_{i\in I_\circ}(f_i)\big) \vee g')(x) \not\leq p$ for all $x \in X$.*

Proof. Very similar to the proof of Theorem 4.1. □

4.9. Theorem. *Let $(X, \mathcal{T})$ be an L-fts and $g \in L^X$. Then we have the following implications:*

g is strongly compact $\Longrightarrow^{(\mathrm{i})}$ g is α-compact $\Longrightarrow^{(\mathrm{ii})}$ g is compact.

That is, α-compactness is stronger than compactness and is weaker than strong compactness.

Proof. (i) Since every α-open L-fuzzy set is preopen, this follows directly from the definitions.

(ii) Since every $f \in L^X$ with $\mathcal{T}(f) \not\leq p$, $p \in P_r(L)$, is α-open, this follows directly from the definitions. □

4.10. Theorem. *Let $(X, \mathcal{T})$ be an L-fts and $g \in L^X$. Then we have the following implications:*

g is S-closed $\Longrightarrow^{(\mathrm{i})}$ g is α-closed $\Longrightarrow^{(ii)}$ g is almost compact.

That is, α-closedness is stronger than almost compactness and is weaker than S-closedness.

Proof. (i) Since every α-open L-fuzzy set is semi-open, this follows directly from the definitions.

(ii) Since every $f \in L^X$ with $\mathcal{T}(f) \not\leq p$, $p \in P_r(L)$, is α-open, this follows directly from the definitions. □

4.11. Theorem. *Let $(X, \mathcal{T})$ be an L-fts and $g \in L^X$. Then we have the following implications:*

g is γ-compact $\Longrightarrow^{(\mathrm{i})}$ g is γ-closed $\Longrightarrow^{(ii)}$ g is S-closed.

That is, γ-closedness is stronger than S-closedness and weaker than γ-compactness.

Proof. (i) Since for every $f \in L^X$, $f \leq \mathrm{cl}\,(f)$, this follows directly from the definitions.

(ii) Since every semi-open L-fuzzy set is γ-open, this follows directly from the definitions. □

4.12. Theorem. *Let $(X, \mathfrak{T})$ be an L-fts and $g \in L^X$. Then we have the following implication:*

g *is* β*-compact* $\Longrightarrow$ g *is strongly compact.*

Proof. Since every preopen L-fuzzy set is β-open, this follows directly from the definitions. □

4.13. Definition. [3] An L-fts $(X, \mathfrak{T})$ is said to be a *fuzzy extremely disconnected space* iff for each $p \in P_r(L)$, $\mathfrak{T}(\mathrm{cl}\,(g)) \not\leq p$ for every $g \in L^X$ with $\mathfrak{T}(g) \not\leq p$.

4.14. Theorem. *For an fuzzy extremely disconnected L-fts $(X, \mathfrak{T})$, the following statements are equivalent:*

(i) X *is* β*-compact.*
(ii) X *is strongly compact.*

Proof. (i) implies (ii). This follows from Theorem 4.12.

(ii) implies (i). Let $p \in P_r(L)$ and let $(f_i)_{i\in I}$ be a family of β-open L-fuzzy sets of X with $\big(\bigvee_{i\in I} f_i\big)(x) \not\leq p$, $\forall\, x \in X$. Then for each $i \in I$, $f_i \leq \mathrm{cl}\,(int(\mathrm{cl}\,(f_i))) = \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,)(f_i)))) = \mathrm{int}\,(\mathrm{cl}\,(f_i))$ from the fuzzy extremely disconnectedness of X. Hence $f_i \leq \mathrm{int}\,\mathrm{cl}\,(f_i))$ for each $i \in I$ and so $(f_i)_{i\in I}$ is a family of preopen L-fuzzy subsets of X with $\big(\bigvee_{i\in I} f_i\big)(x) \not\leq p$. So, there exists a finite subset $I_\circ$ of I such that $\big(\bigvee_{i\in I_\circ} f_i\big)(x) \not\leq p$. □

5. Applications of S-closedness in L-fts's

5.1. Theorem. *Let $F : (X, \mathfrak{T}) \to (Y, \mathfrak{T}^*)$ be a fuzzy continuous, α-irresolute function and let $g \in L^X$ be S-closed relative to X. Then $F(g)$ is α-closed relative to Y.*

Proof. Let $p \in P_r(L)$ and $(h_i)_{i\in I}$ be a collection of α-open L-fuzzy sets of Y such that

$$\big(\bigvee_{i\in I} h_i\big)(y) \not\leq p,\ \forall\, y \in Y \text{ with } F(g)(y) \geq p'.$$

Then, from the α-irresoluteness of the function F, $(F^{-1}(h_i))_{i\in I}$ is a family of α-open L-fuzzy sets of X which is also a family of semiopen L-fuzzy sets of X, with

$$\big(\bigvee_{i\in I} F^{-1}(h_i)\big)(x) \not\leq p,\ \forall\, x \in X \text{ with } g(x) \geq p',$$

because, if $g(x) \geq p'$, then $F(g)(F(x)) \geq p'$. So,

$$\big(\bigvee_{i\in I} F^{-1}(h_i)\big)(x) = \big(\bigvee_{i\in I} h_i\big)(F(x)) \not\leq p.$$

From the S-closedness of g in $(X, \mathfrak{T})$, there exists a finite subset $I_\circ$ of I such that

$$\big(\bigvee_{i\in I_\circ} \mathrm{cl}\,(F^{-1}(h_i))\big)(x) \not\leq p,\ \forall\, x \in X \text{ with } g(x) \geq p'.$$

We also have that

$$\big(\bigvee_{i\in I_\circ} \mathrm{cl}\,(h_i)\big)(y) \not\leq p,\ \forall\, y \in Y \text{ with } F(g)(y)) \geq p'.$$

In fact, if $F(g)(y) \geq p'$, then $\bigvee_{x \in F^{-1}(y)} g(x) \geq p'$ which implies that there is $x \in X$ with $g(x) \geq p'$ and $F(x) = y$. So,

$$\begin{aligned} F\big(\bigvee_{i \in I_\circ} \mathrm{cl}\,(F^{-1}(h_i))\big)(F(x)) &= \bigvee_{i \in I_\circ} F\big(\mathrm{cl}\,(F^{-1}(h_i))\big)(F(x)) \\ &\leq \big(\bigvee_{i \in I_\circ} FF^{-1}(\mathrm{cl}\,(h_i))\big)(F(x)) \quad [5, \text{Proposition } 2.4] \\ &= \bigvee_{i \in I_\circ} \big(\mathrm{cl}\,(h_i)\big)(y) \not\leq p, \ \forall\, y \in Y, \end{aligned}$$

with $F(g)(y) \geq p'$. So, $F(g)$ is α-closed in $(Y, \mathfrak{T}^*)$. □

5.2. Corollary. *If a function $F : (X, \mathfrak{T}) \to (Y, \mathfrak{T}^*)$ is fuzzy continuous and α-irresolute, and X is a S-closed L-fts, then $F(X)$ is α-closed.* □

5.3. Theorem. *Let $F : (X, \mathfrak{T}) \to (Y, \mathfrak{T}^*)$ be a fuzzy continuous, irresolute function and let $g \in L^X$ be γ-closed relative to X. Then $F(g)$ is S-closed relative to Y.*

Proof. This is very similar to the proof of Theorem 5.1, and is omitted. □

5.4. Corollary. *If a function $F : (X, \mathfrak{T}) \to (Y, \mathfrak{T}^*)$ is fuzzy continuous and irresolute, and X is a γ-closed L-fts, then $F(X)$ is S-closed.* □

References

[1] Andrijevic, D. *On b-open sets*, Mathematicki Vesnik **48**, 59–64, 1996.

[2] Aygün, H., Warner, M. W. and Kudri, S. R. T. *On smooth L-fuzzy topological spaces* J. Fuzzy Math. **5** (2), 321–338, 1997.

[3] Aygün, H. and Abbas, S. E. *On characterization of some covering properties in L-fuzzy topological spaces in Šostak sense* Information Sciences **165**, 221–233, 2004.

[4] Chang, C. L. *Fuzzy topological spaces*, J. Math. Anal. Appl. **24**, 182–190, 1968.

[5] Demirci, M. *On several types of compactness in smooth topological spaces* Fuzzy Sets and Systems **90** (1), 83–88, 1997.

[6] Gierz, G., Hofman, K. H., Keimel, K., Lawson, J. D., Misolve, M. and Scott, D. S. *A compendium of continuous lattices* (Springer Verlag, 1980).

[7] Goguen, J. A. *The fuzzy Tychonoff theorem*, J. Math. Anal. Appl. **43**, 734–742, 1973.

[8] Höhle, U. *Upper semicontinuous fuzzy sets and applications*, J. Math. Anal. Appl. **78**, 659–673, 1980.

[9] Höhle, U. and Šostak, A. P. *A general theory of fuzzy topological spaces* Fuzzy Sets and Systems **73**, 131–149, 1995.

[10] Höhle, U. and Šostak, A. P. *Axiomatic Foundations of Fixed-Basis fuzzy topology* in The Handbooks of Fuzzy sets series, Volume 3 (Kluwer Academic Publishers, Dordrecht (Chapter 3), 1999).

[11] Kubiak, T.*On fuzzy topologies* (Ph. D. Thesis, A. Mickiewicz, Poznan, 1985).

[12] Kubiak, T. and Šostak, A. P. *Lower set-valued fuzzy topologies*, Quaestiones Math. **20** (3), 423–429, 1997.

[13] Kudri, S. R. T. *Compactness in L-fuzzy topological spaces*, Fuzzy Sets and Systems **67**, 229–236, 1994.

[14] Ying-Ming, Liu and Mao-Kang, Luo *Fuzzy Topology* (World Scientific Publishing Co., Singapore, 1997).

[15] Ramadan, A. A. and Abbas, S.E. *On L-smooth compactness* J. Fuzzy Math. **9** (1), 59–73, 2001.

[16] Ramadan, A. A., Abbas, S. E. and Chan Kim, Yong *Fuzzy irresolute mappings in smooth fuzzy topological spaces* J. Fuzzy Math. **9** (4), 865–877, 2001.

[17] Ramadan, A. A., Abbas, S. E. and Çoker, D. *Fuzzy γ-continuity in Šostak fuzzy topology*, J. Fuzzy Math. **10** (1), 2002.

[18] Ramadan, A. A., Y. C. Kim, Y. C. and Abbas, S. E. *Weaker forms of continuity is Šostak's fuzzy topology*, Indian J. Pure and Appl. Math. **34** (2), 311–333, 2003.
[19] Šostak, A. P. *On a fuzzy topological structure*, Suppl. Rend. Circ. Matem. Palerms ser II **11**, 89–103, 1985.
[20] Šostak, A. P. *Two decades of fuzzy topology: basic ideas, notions and results*, Russian Math. Surveys **44** (6), 125–186, 1989.
[21] Šostak, A. P. *Basic structures of fuzzy topology*, J. Math. Sci. **78** (6), 662–701, 1996.
[22] Warner, M. W. and R. C. McLean, R. C. *On compact Hausdorff L-fuzzy spaces*, Fuzzy Sets and Systems **55**, 103–110, 1993.