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SOME RESULTS RELATED TO A CERTAIN VECTOR FIELD SATISFYING THE LOCAL MÖBIUS EQUATION

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Abstract

In this paper we prove some results related to a certain vector field satisfying the local Möbius equation on vector fields.

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1. Introduction

A vector field Z on a Riemannian manifold (M, g) is said to satisfy the local Möbius equation if

$$(\nabla^2 Z)(X, Y) = g(\Delta Z, X)Y$$

for all vector fields X, Y.

It is known that the existence of solutions Z to the local Möbius equation is related to the conformal structure of the manifold, since the divergence div Z is a solution of the local Möbius equation, i.e.

$$\mathrm{Hess}_{\mathrm{div\,Z}} = \frac{\nabla \mathrm{div\,Z}}{n} Id$$

and moreover, in such cases $\nabla\mathrm{div\,Z}$ is a conformal vector field, since $\pounds_{\nabla \mathrm{div\,Z}} = 2\,\mathrm{Hess}_{\mathrm{div\,Z}}$. (See also the first four in references.)

The purpose of this paper is to point out such a connection by considering the vector field Z itself. We prove the following:

(*Theorem* 3.4). A nonzero solution Z of the local Möbius equation is conformal, provided that M is compact.

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(*Theorem* 3.5). A nonzero conformal vector field Z satisfying

$$R(X,Y)Z = -[g(\nabla Z, Y)X - g(\nabla Z, X)Y],$$

(which is a consequence of the local Möbius equation), is a solution of the local Möbius equation.

2. Preliminaries

Here, we briefly state the main concepts and definitions used throughout this paper.

Let (M, g) be a Riemannian manifold of dimension n, ∇ the Levi-Civita connection and

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

the curvature tensor. We write also $< X, Y >$ if this is convenient. The Ricci curvature (tensor) is the trace of R : trace$(X \to R(X,Y)Z)$ and is denoted by Ric(Y, Z). If $\{X_1, \ldots, X_n\}$ is a local orthonormal frame for TM, then

$$\mathrm{Ric}(Y,Z) = \sum_{i=1}^{n} g(R(X_i, Y)Z, X_i) = \sum_{i=1}^{n} g(R(Y, X_i)X_i, Z).$$

Thus Ric is a symmetric bilinear form. It could also be defined as a symmetric (1,1) tensor

$$\mathrm{Ric}(Z) = \sum_{i=1}^{n} R(Z, X_i)X_i.$$

The scalar curvature is defined by $Sc = \mathrm{tr}\,\mathrm{Ric}$. Let Z be a vector field on this n-dimensional Riemannian manifold (M, g) with Levi-Civita connection ∇. The second covariant differential $\nabla^2 Z$ of Z is defined by

$$(\nabla^2 Z)(X,Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z,$$

where $X, Y \in \Gamma(TM)$. We define the Laplacian ΔZ of Z on (M, g) to be the trace of $\nabla^2 Z$ with respect to g, that is,

$$\Delta Z = \mathrm{trace}\,\nabla^2 Z = \sum_{i=1}^{n} (\nabla^2 Z)(X_i, X_i),$$

where $\{X_1, \ldots, X_n\}$ is a local orthonormal frame for TM.

Also, the affinity tensor $L_Z \nabla$ of Z is defined by

$$(L_Z \nabla)(X,Y) = L_Z \nabla_X Y - \nabla_{L_Z X} Y - \nabla_X L_Z Y,$$

where L_Z is the Lie derivative with respect to Z and $X, Y \in \Gamma(TM)$. (See, for example page 109 of [5]). We define the tension field $\Box Z$ of Z on (M, g) to be the trace of $L_Z \nabla$ with respect to g, that is

$$\Box Z = \mathrm{trace}\, L_Z \nabla = \sum_{i=1}^{n} (L_Z \nabla)(X_i, X_i),$$

where $\{X_1, \ldots, X_n\}$ is a local orthonormal frame for TM.

By a straightforward computation, it can be shown by using the torsion-free property of ∇ that

$$(L_Z \nabla)(X,Y) = (\nabla^2 Z)(X,Y) + R(Z, X)Y,$$

(see page 110 of [5]), and hence

$$\Box Z = \Delta Z + \mathrm{Ric}(Z),$$

where $X, Y \in \Gamma(TM)$. (Also see page 40 of [6]).

The divergence of a vector field Z, $divZ$, on (M, g) is defined as

$$\mathrm{div\,Z = tr(\nabla Z)} = \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{g(\nabla_{X_i} Z, X_i)}$$

if $\{X_i\}$ is an orthonormal basis of TM.

3. Some Results Related to a certain Vector Field

The elementary results of this section could also be collected from [1]. First we state an elementary lemma to be used in the proof of the results of this paper which is, for example, Lemma 3.3 in [1].

3.1. Lemma. *Let (M, g) be an n-dimensional Riemannian manifold. If Z is a vector field on (M, g) satisfying the local Möbius equation*

$$(\nabla^2 Z)(X, Y) = g(\Delta Z, X)Y,$$

for all $X, Y \in \Gamma TM$, then

(i) $R(X, Y)Z = -[g(\Delta Z, Y)X - g(X, \Delta Z)Y]$ *for all $X, Y \in \Gamma(TM)$, and hence* $\mathrm{Ric}(Z) = -(n-1)\Delta Z$,

(ii) $\nabla \mathrm{div\,Z = n\Delta Z}$, *and hence* $\nabla^2 \mathrm{div\,Z = n\nabla\Delta Z}$, *where* $\nabla^2 \mathrm{div\,Z}$ *is the Hessian tensor of* $\mathrm{div\,Z}$.

For completeness, we give the proof of this and the other lemmas.

Proof. (i). Let $X, Y \in \Gamma(TM)$. Then

$$\begin{aligned} R(X, Y)Z &= \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z \\ &= g(\Delta Z, X)Y - g(\Delta Z, Y)X \\ &= -[g(\Delta Z, Y)X - g(\Delta Z, X)Y]. \end{aligned}$$

Hence

$$\begin{aligned} g(\mathrm{Ric}(Z), X) &= g(\sum_{i=1}^{n} R(Z, X_i)X_i, X) \\ &= \sum_{i=1}^{n} g(R(Z, X_i)X_i, X) \\ &= \sum_{i=1}^{n} R(Z, X_i, X_i, X) \\ &= \sum_{i=1}^{n} R(X_i, X, Z, X_i) \\ &= \sum_{i=1}^{n} g(R(X_i, X)Z, X_i) \\ &= \sum_{i=1}^{n} g(-g(\Delta Z, X)X_i + g(\Delta Z, X_i)X, X_i) \\ &= -g(\Delta Z, X)\sum_{i=1}^{n} g(X_i, X_i) + \sum_{i=1}^{n} g(\Delta Z, X_i)g(X, X_i), \end{aligned}$$

where $\{X_1, \cdots, X_n\}$ is an orthonormal frame for TM near $p \in M$. Hence,

$$\begin{aligned} g(\mathrm{Ric}(Z), X) &= -ng(\Delta Z, X) + g(\Delta Z, X) \\ &= (-n+1)g(\Delta Z, X) \\ &= -(n-1)g(\Delta Z, X) \\ &= g(-(n-1)\Delta Z, X). \end{aligned}$$

(ii). Let $\{X_1, \cdots, X_n\}$ be an adapted orthonormal frame near $p \in M$, that is, $\{X_1, \cdots, X_n\}$ is an orthonormal frame in TM with $(\nabla X_i)_p = 0$ for $i = 1, \dots, n$, and let $X \in \Gamma(TM)$. Then at $p \in M$,

$$\begin{aligned} g(\nabla div\, Z, X) &= X(div\, Z) \\ &= \sum_{i=1}^{n} Xg(\nabla_{X_i} Z, X_i) \\ &= \sum_{i=1}^{n} [g(\nabla_X \nabla_{X_i} Z, X_i) + g(\nabla_{X_i} Z, \nabla_X X_i)] \\ &= \sum_{i=1}^{n} [g((\nabla^2 Z)(X, X_i), X_i) - g(\nabla_{\nabla_X X_i} Z, X_i)] \\ &= \sum_{i=1}^{n} g(g(\Delta Z, X)X_i, X_i) \\ &= g(\Delta Z, X) \sum_{i=1}^{n} g(X_i, X_i) \\ &= ng(\Delta Z, X) \\ &= g(n\Delta Z, X). \end{aligned}$$

Hence, it follows that $\nabla div\, Z = n\Delta Z$ and hence $\nabla^2 div\, Z = n\nabla\Delta Z$. □

3.2. Lemma. *Let (M, g) be an n-dimensional Riemannian manifold. If Z is a non-zero vector field on (M, g) satisfying the local Möbius equation*

$$(\nabla^2 Z)(X, Y) = g(\Delta Z, X)Y,$$

for all $X, Y \in \Gamma(TM)$ and ΔZ is a non-zero conformal vector field on (M, g), then

$$\nabla^2 div\, Z = \frac{\Delta div\, Z}{n} id.$$

Proof. Since ΔZ is non-zero, it follows from *Lemma* 3.1 that $div\, Z$ is non-constant and $\nabla^2 div\, Z = n\nabla\Delta Z$. Hence,$\nabla\Delta Z$ is self-adjoint and can be written as $\nabla\Delta Z = \frac{div\,\Delta Z}{n} id + \sigma$, where σ is the traceless self-adjoint part of $\nabla\Delta Z$. But, since ΔZ is assumed to be a conformal vector field, $\sigma = 0$ (see page 173 0f [5]), and it follows that

$$\begin{aligned} \nabla^2 div\, Z &= n\nabla\Delta Z \\ &= n(\frac{div\,\Delta Z}{n} id) \\ &= div\,\Delta Z id \\ &= \frac{\Delta div\, Z}{n} id, \end{aligned}$$

since $\Delta div\, Z = ndiv\,\Delta Z$ by *Lemma* 3.1. □

3.3. Lemma. *Let (M, g) be an $n(\geq 2)$-dimensional Riemannian manifold. If Z is a non-zero vector field on (M, g) satisfying the local Möbius equation*

$$(\nabla^2 Z)(X, Y) = g(\Delta Z, X)Y,$$

for all $X, Y \in \Gamma(TM)$, then it also satisfies the equation

$$\Box Z + \frac{n-2}{n} \nabla div\, Z = 0$$

on (M, g).

Proof.

$$\begin{aligned}
\Box Z + \frac{n-2}{n}\nabla div\, Z &= \Delta Z + Ric(Z) + \frac{n-2}{n}\nabla div\, Z \\
&= \frac{1}{n}\nabla div\, Z - (n-1)\Delta Z + \frac{n-2}{n}\nabla div\, Z \\
&= \frac{1}{n}\nabla div\, Z - \frac{n-1}{n}\nabla div\, Z + \frac{n-2}{n}\nabla div\, Z \\
&= \frac{-n+2}{n}\nabla div\, Z + \frac{n-2}{n}\nabla div\, Z \\
&= 0.
\end{aligned}$$

□

3.4. Theorem. *Let (M, g) be an n-dimensional compact Riemannian manifold. If Z is a non-zero vector field on (M, g) satisfying the local Möbius equation*

$$(\nabla^2 Z)(X, Y) = g(\Delta Z, X)Y,$$

for all $X, Y \in \Gamma(TM)$, then Z is a conformal vector field on M.

Proof. Follows from *Lemma* 3.3 (see page 47 of [6]). □

3.5. Theorem. *Let (M, g) be an $n(\geq 2)$-dimensional Riemannian manifold. If Z is a non-zero conformal vector field on (M, g) satisfying the equation*

$$R(X, Y)Z = -[g(\Delta Z, Y)X - g(X, \Delta Z)Y],$$

for all $X, Y \in \Gamma(TM)$, then Z satisfies the local Möbius equation

$$(\nabla^2 Z)(X, Y) = g(\Delta Z, X)Y.$$

for all $X, Y \in \Gamma(TM)$.

Proof. This can be easily obtained from the equation

$$\Box Z = \Delta Z + Ric(Z),$$

which implies

$$\begin{aligned}
\Delta Z &= \frac{2-n}{n}\nabla div\, Z + \frac{n-1}{n}\nabla div\, Z \\
&= \frac{1}{n}\nabla div\, Z,
\end{aligned}$$

since Z is conformal (see page 47 of [6]) and by *Lemma* 3.1. □

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