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A RELATED FIXED POINT THEOREM ON THREE METRIC SPACES

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Abstract

A related fixed point theorem for three mappings, not all of which need be continuous, on three metric spaces of which one is a compact metric space is obtained. An example is obtained to illustrate the theorem.

Keywords: Compact space, Related fixed point.

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1. Introduction and Preliminaries

Related fixed point theorems on three complete metric spaces have been studied by Nung [5], Jain *et al.* [1,2,3,4] and Rao *et al.* [7]. In this paper, we prove a related fixed point theorem for three mappings, not all of which are necessarily continuous, on three metric spaces, of which one is a compact metric space. We also give an example to illustrate our theorem. Our theorem also improves Theorem 3 of Rao *et al.* [6].

2. Main Results

2.1. Theorem. *Let* (X, d), (Y, ρ) *and* (Z, σ) *be three metric spaces and* $T : X \to Y$, $S : Y \to Z$, *and* $R : Z \to X$ *mappings satisfying the inequalities:*

$$d(RSy, RSTx) < \max\{d(x, RSy), d(x, RSTx),\ \rho(y, Tx),\ \rho(y, TRSy),\ \rho(Tx, TRSy)\} \tag{2.1}$$

for all x *in* X *and* y *in* Y *with* $y \neq Tx$,

$$\rho(TRz, TRSy) < \max\{\rho(y, TRz),\ \rho(y, TRSy),\ \sigma(z, Sy),\ \sigma(z, STRz),\ \sigma(Sy, STRz)\} \tag{2.2}$$

for all y *in* Y *and* z *in* Z *with* $z \neq Sy$, *and*

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$$\sigma(STx, STRz) < max\{\sigma(z, STx),\ \sigma(z, STRz),\ d(x, Rz),\ d(x, RSTx),\ d(Rz, RSTx)\} \tag{2.3}$$

for all x in X and z in Z with $x \neq Rz$. Further, assume one of the following conditions:

(a) *(X, d) is compact and RST is continuous,*
(b) *(Y, ρ) is compact and TRS is continuous,*
(c) *(Z, σ) is compact and STR is continuous.*

Then RST has a unique fixed point w in X, TRS has a unique fixed point u in Y and STR has a unique fixed point v in Z. Further $Su = v$, $Rv = w$ and $Tw = u$.

Proof. Suppose (a) holds. Define $\phi(x) = d(x, RSTx)$ for $x \in X$. Then there exists p in X such that

$$\phi(p) = d(p, RSTp) = \inf\{\phi(x) : x \in X\}.$$

Suppose that $RSTRSTRSTp \neq RSTRSTp$. Then

$$STRSTRSTp \neq STRSTp, \quad TRSTRSTp \neq TRSTp, \quad RSTRSTp \neq RSTp,$$
$$STRSTp \neq STp, \quad TRSTp \neq Tp, \quad RSTp \neq p.$$

From (2.1), with $y = TRSTp, x = RSTRSTp$, we have

$$d(RSTRSTp, RSTRSTRSTp) < \max\{d(RSTRSTp, RSTRSTp), d(RSTRSTp, RSTRSTRSTp), \rho(TRSTp, TRSTRSTp), \rho(TRSTp, TRSTRSTp), \rho(TRSTRSTp, TRSTRSTp)\},$$

so that

$$\phi(RSTRSTp) < \rho(TRSTp, TRSTRSTp). \tag{2.4}$$

From (2.2), with $z = STp$, $y = TRSTp$, we have

$$\rho(TRSTp, TRSTRSTp) < \max\{\rho(TRSTp, TRSTp),\ \rho(TRSTp, TRSTRSTp), \sigma(STp, STRSTp),\ \sigma(STp, STRSTp), \sigma(STRSTp, STRSTp)\},$$

so that

$$\rho(TRSTp, TRSTRSTp) < \sigma(STp, STRSTp). \tag{2.5}$$

From (2.3) with $x = p$, $z = STp$ we have

$$\sigma(STp, STRSTp) < \max\{\sigma(STp, STp),\ \sigma(STp, STRSTp), d(p, RSTp),\ d(p, RSTp),\ d(RSTp, RSTp)\},$$

so that

$$\sigma(STp, STRSTp) < \phi(p). \tag{2.6}$$

From (2.4), (2.5) and (2.6), we have $\phi(RSTRSTp) < \phi(p)$, contradicting the existence of p. Hence, $RSTRSTRSTp = RSTRSTp$.

Putting $RSTRSTp = w$ in X we have,

$$RSTw = w.$$

Now let $Tw = u$ in Y and $Su = v$ in Z. Then $Rv = RSu = RSTw = w$, and it follows that

$$STRv = STw = Su = v$$

and

$$TRSu = TRv = Tw = u.$$

To prove uniqueness, suppose that RST has a second distinct fixed point w_0 in X. Then

$$RSTw \neq RSTw_0,\ STw \neq STw_0,\ Tw \neq Tw_0.$$

Using (2.1), with $y = Tw,\ x = w_0$, we get

$$d(RSTw, RSTw_0) < \max\{d(w_0, RSTw), d(w_0, RSTw_0), \rho(Tw, Tw_0),\ \rho(Tw, TRSTw),\ \rho(Tw_0, TRSTw)\},$$

so that

(2.7) $\quad d(w, w_0) < \rho(Tw, Tw_0)$.

Using (2.2) with $z = STw,\ y = Tw_0$, we get

$$\rho(TRSTw, TRSTw_0) < \max\{\rho(Tw_0, TRSTw),\ \rho(Tw_0, TRSTw_0), \sigma(STw, STw_0), \sigma(STw, STRSTw), \sigma(STw_0, STRSTw)\},$$

so that

(2.8) $\quad \rho(Tw, Tw_0) < \sigma(STw, STw_0)$.

Using (3) with $x = w,\ z = STw_0$, we get

$$\sigma(STw, STRSTw_0) < \max\{\sigma(STw_0, STw),\ \sigma(STw_0, STRSTw_0), d(w, RSTw_0),\ d(w, RSTw), d(RSTw_0, RSTw)\},$$

so that

(2.9) $\quad \sigma(STw, STw_0) < d(w, w_0)$.

From (2.7), (2.8) and (2.9), we have $d(w, w_0) < d(w, w_0)$ so that $w = w_0$, proving the uniqueness of w.

Similarly, we can show that v is the unique fixed point of STR and u is the unique fixed point of TRS.

It follows similarly that the theorem holds if (b) or (c) holds instead of (a). □

Now, we give the following example to illustrate our theorem.

2.2. Example. Let $X = [0, 1],\ Y = [1, 2),\ Z = (2, 3]$, and let $d = \rho = \sigma$ be the usual metric for the real numbers. Define $T : X \to Y$, $S : Y \to Z$ and $R : Z \to X$ by:

$$Tx = \begin{cases} 1 & \text{if } x \in [0, 3/4), \\ 3/2 & \text{if } x \in [3/4, 1], \end{cases}$$

$$Sy = 3\ \forall\, y \in Y,$$

$$Rz = \begin{cases} 3/4 & \text{if } z \in (2, 5/2], \\ 1 & \text{if } z \in (5/2, 3]. \end{cases}$$

Here Y and Z are not compact spaces and T and R are not continuous. However, all the conditions of Theorem 2.1 are satisfied. Clearly,

$$RST(1) = 1,\ TRS(3/2) = 3/2,\ STR(3) = 3,\ S(3/2) = 3,\ R3 = 1 \text{ and } T1 = 3/2.$$

2.3. Remark. Theorem 2.1 holds if (2.1), (2.2) and(2.3) are replaced by

$$(2.1)^1 \quad d(RSy, RSTx) < \frac{\max\{d(x, RSTx)\rho(y, TRSy),\ d(x, RSy)\rho(y, Tx)\}}{\max\{d(x, RSTx),\ d(x, RSy),\ \rho(Tx, TRSy)\}}$$

$\forall\, x \in X, y \in Y$ with denominator $\neq 0$,

$$(2.2)^1 \quad \rho(TRz, TRSy) < \frac{\max\{\rho(y, TRSy)\sigma(z, STRz),\ \rho(y, TRz)\sigma(z, Sy)\}}{\max\{\rho(y, TRSy),\ \rho(y, TRz),\ \sigma(Sy, STRz)\}}$$

$\forall\, z \in Z, y \in Y$ with denominator $\neq 0$,

$$(2.3)^1 \quad \sigma(STx, STRz) < \frac{\max\{\sigma(z, STRz)d(x, RSTx),\ \sigma(z, STx)d(x, Rz)\}}{\max\{\sigma(z, STRz),\ \sigma(z, STx),\ d(Rz, RSTx)\}}$$

$\forall\, z \in Z, x \in X$ with denominator $\neq 0$.

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