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RESULTS ON THE NEUTRIX COMPOSITION OF THE DELTA FUNCTION

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Abstract

Let F be a distribution in $\mathcal{D}'$ and f a locally summable function. The composition $F(f(x))$ of F and f is said to exist and be equal to the distribution $h(x)$ if the neutrix limit of the sequence $\{F_n(f(x))\}$ is equal to $h(x)$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \ldots$ and $\{\delta_n(x)\}$ is a certain regular sequence converging to the Dirac delta function. It is proved that the neutrix composition $\delta^{(s)}[\ln^r(1 + |x|)]$ exists and that

$$\delta^{(s)}[\ln^r(1+|x|)] = \sum_{k=0}^{sr+r-1} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^{s-i}[1 + (-1)^k]s!(i + 1)^{rs+r-1}}{2r(rs + r - 1)!k!} \delta^{(k)}(x)$$

for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$.

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1. Introduction

In the theory of distributions, many arguments may be used to show that generally no meaning can be given to expressions of the form $F(f(x))$, where F is a distribution and f is a locally summable function.

Using the concepts of a neutrix and neutrix limit due to van der Corput [1], the first author gave a general principle for the discarding of unwanted infinite quantities from asymptotic expansions, and this has been exploited in the context of distributions, particularly in connection with the composition of distributions, see [2, 3]. Using Fisher's definition, Koh and Li give a meaning to δ^r and $(\delta')^r$ for $r = 2, 3, \ldots$, see [13], and the

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more general form $(\delta^{(s)}(x))^r$ was considered by Kou and Fisher in [14]. Recently the rth powers of the Dirac function $\delta(x)$ and the Heaviside function $H(x)$ for negative integers have been defined in [15] and [16] respectively.

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support, $\mathcal{D}[a,b]$ the space of infinitely differentiable functions with support contained in the interval $[a,b]$ and $\mathcal{D}'$ the space of distributions defined on $\mathcal{D}$.

Now let $\rho(x)$ be a function in $\mathcal{D}$ having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$, and
(iv) $\displaystyle\int_{-1}^{1} \rho(x)\,dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, we have

$$\begin{aligned}\lim_{n\to\infty} \langle \delta_n(x), \varphi \rangle &= \lim_{n\to\infty} \int_{-1/n}^{1/n} \delta_n(x)\varphi(x)\,dx = \lim_{n\to\infty} \int_{-1}^{1} \rho(t)\varphi(t/n)\,dt \\ &= \varphi(0) = \langle \delta(t), \varphi(t) \rangle,\end{aligned}$$

for arbitrary φ in $\mathcal{D}$. It follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. Further, if F is an arbitrary distribution in $\mathcal{D}'$ and $F_n(x) = F(x) * \delta_n(x) = \langle F(x-t), \delta_n(t) \rangle$, then $\{F_n(x)\}$ is a regular sequence converging to $F(x)$.

If f is an infinitely differentiable function having a single simple zero at the point $x = x_0$, then the distribution $\delta^{(r)}(f(x))$ is defined by

$$\delta^{(r)}(f(x)) = \frac{1}{|f'(x_0)|}\left[\frac{1}{|f'(x)|}\frac{d}{dx}\right]^r \delta(x - x_0) \tag{1}$$

for $r = 0, 1, 2, \ldots$, see [12].

In [2] the first author generalized Eq.(1) as follows:

1.1. Definition. Let f be a infinitely differentiable function. We say that *the neutrix composition $\delta^{(r)}(f(x))$ exists and is equal to h on the open interval (a,b), with $-\infty < a < b < \infty$*, if

$$\operatorname*{N-lim}_{n\to\infty} \int_{-\infty}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)dx = \langle h(x), \varphi(x) \rangle$$

for all φ in $\mathcal{D}[a,b]$, where N is the neutrix, see [1], having domain N' the positive and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n:\ \lambda > 0,\ r = 1, 2, \ldots,$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Note that taking the neutrix limit of a function $f(n)$ is equivalent to taking the usual limit of Hadamard's finite part of $f(n)$.

Definition 1.1 was later generalized in [3] using the following definition, and was originally called the *neutrix composition of distributions*.

1.2. Definition. Let F be a distribution in $\mathcal{D}'$ and f a locally summable function. We say that *the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a,b), with $-\infty < a < b < \infty$*, if

$$\operatorname*{N-lim}_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x) \rangle$$

for all φ in $\mathcal{D}[a,b]$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \ldots$. In particular, we say that *the composition* $F(f(x))$ *exists and is equal to* h *on the open interval* (a,b) if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a,b]$.

The following theorem was proved in [4].

1.3. Theorem. *The neutrix composition* $\delta^{(s)}(\operatorname{sgn} x|x|^\lambda)$ *exists and*

$$\delta^{(s)}(\operatorname{sgn} x|x|^\lambda) = 0$$

for $s = 0, 1, 2, \ldots$ *and* $(s+1)\lambda = 1, 3, \ldots$ *and*

$$\delta^{(s)}(\operatorname{sgn} x|x|^\lambda) = \frac{(-1)^{(s+1)(\lambda+1)}s!}{\lambda[(s+1)\lambda - 1]!}\delta^{((s+1)\lambda-1)}(x)$$

for $s = 0, 1, 2, \ldots$ *and* $(s+1)\lambda = 2, 4, \ldots$.

The next two theorems were proved in [5].

1.4. Theorem. *The compositions* $\delta^{(2s-1)}(\operatorname{sgn} x|x|^{1/s})$ *and* $\delta^{(s-1)}(|x|^{1/s})$ *exist and*

$$\delta^{(2s-1)}(\operatorname{sgn} x|x|^{1/s}) = \tfrac{1}{2}(2s)!\delta'(x),$$

$$\delta^{(s-1)}(|x|^{1/s}) = (-1)^{s-1}\delta(x)$$

for $s = 1, 2, \ldots$.

1.5. Theorem. *The neutrix composition* $\delta^{(s)}[\ln(1+|x|)]$ *exists and*

$$\delta^{(s)}[\ln(1+|x|)] = \sum_{k=0}^{s}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{s+i}[1+(-1)^k](i+1)^s}{2k!}\delta^{(k)}(x),$$

for $s = 0, 1, 2, \ldots$.

2. Main Results

We now prove the following generalization of Theorem 1.3.

2.1. Theorem. *The neutrix composition* $\delta^{(s)}[\ln^r(1+|x|)]$ *exists and*

$$\delta^{(s)}[\ln^r(1+|x|)] = \sum_{k=0}^{sr+r-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{s-i}[1+(-1)^k]s!(i+1)^{rs+r-1}}{2r(rs+r-1)!k!}\delta^{(k)}(x) \tag{2}$$

for $s = 0, 1, 2, \ldots$ *and* $r = 1, 2, \ldots$.

In particular, the composition $\delta[\ln(1+|x|)]$ *exists and*

$$\delta[\ln(1+|x|)] = \delta(x).$$

Proof. To prove equation (2), we will first of all evaluate

$$\operatorname*{N-lim}_{n\to\infty}\langle\delta_n^{(s)}[\ln^r(1+|x|)], \varphi(x)\rangle,$$

for an arbitrary function $\varphi(x)$ in $\mathcal{D}[-1,1]$.

By Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{rs+r-1}\frac{\varphi^{(k)}(0)}{k!}x^k + \frac{x^{rs+r}}{(rs+r)!}\varphi^{(rs+r)}(\xi x),$$

where $0 < \xi < 1$. Then if $\varphi(x)$ in $\mathcal{D}[-1,1]$, we have

$$\begin{aligned} \operatorname*{N-lim}_{n\to\infty}\langle\delta_n^{(s)}[\ln^r(1+|x|)],\varphi(x)\rangle &= \operatorname*{N-lim}_{n\to\infty}\sum_{k=0}^{rs+r-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{1}\delta_n^{(s)}[\ln^r(1+|x|)]x^k\,dx \\ &+ \operatorname*{N-lim}_{n\to\infty}\frac{1}{(rs+r)!}\int_{-1}^{1}\delta_n^{(s)}[\ln^r(1+|x|)]x^{rs+r}\varphi^{(rs+r)}(\xi x)\,dx. \end{aligned} \tag{3}$$

For large enough n, we have

$$\begin{aligned} \int_{-1}^{1}\delta_n^{(s)}[\ln^r(1+|x|)]x^k\,dx &= n^{s+1}\int_{-1}^{1}\rho^{(s)}[n\ln^r(1+|x|)]x^k\,dx \\ &= n^{s+1}[1+(-1)^k]\int_0^1\rho^{(s)}[n\ln^r(1+x)]x^k\,dx. \end{aligned} \tag{4}$$

Making the substitution $t = n\ln^r(1+x)$, we have

$$\begin{aligned} n^{s+1}\int_0^1 & \rho^{(s)}[n\ln^r(1+x)]x^k\,dx = \\ &= \frac{n^{s+1-1/r}}{r}\int_0^1 t^{1/r-1}\{\exp[(t/n)^{1/r}]-1\}^k\exp[(t/n)^{1/r}]\rho^{(s)}(t)\,dt \\ &= \frac{n^{s+1-1/r}}{r}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\int_0^1 t^{1/r-1}\exp[(i+1)(t/n)^{1/r}]\rho^{(s)}(t)\,dt, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \frac{n^{s+1-1/r}}{r}\int_0^1 t^{1/r-1}\exp[(i+1)(t/n)^{1/r}]\rho^{(s)}(t)\,dt &= \\ &= \sum_{j=0}^{\infty}\int_0^1\frac{(i+1)^j t^{(j+1)/r-1}}{rj!n^{(j+1)/r-s-1}}\rho^{(s)}(t)\,dt. \end{aligned}$$

It follows that

$$\begin{aligned} \operatorname*{N-lim}_{n\to\infty}\frac{n^{s+1-1/r}}{r}\int_0^1 t^{1/r-1}\exp[(i+1)(t/n)^{1/r}]\rho^{(s)}(t)\,dt &= \\ &= \int_0^1\frac{(i+1)^{rs+r-1}t^s}{r(rs+r-1)!}\rho^{(s)}(t)\,dt \\ &= \frac{(-1)^s s!(i+1)^{rs+r-1}}{2r(rs+r-1)!} \end{aligned}$$

for $i = 0,1,2,\ldots k$, and so

$$\operatorname*{N-lim}_{n\to\infty} n^{s+1}\int_0^1\rho^{(s)}[n\ln^r(1+x)]x^k\,dx = \sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{s+k-i}s!(i+1)^{rs+r-1}}{2r(rs+r-1)!}. \tag{6}$$

It now follows from equations (4), (5) and (6) that

$$\operatorname*{N-lim}_{n\to\infty}\int_{-1}^{1}\delta_n^{(s)}[\ln^r(1+|x|)]x^k\,dx = [1+(-1)^k]\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{s+k-i}s!(i+1)^{rs+r-1}}{2r(rs+r-1)!}, \tag{7}$$

for $k = 0,1,2,\ldots,rs+r-1$.

When $k = rs + r$, we have

$$\begin{aligned}\int_0^1 \Big|\delta_n^{(s)}[\ln^r(1+|x|)]x^{rs+r}\Big|\,dx &\le \frac{n^{s+1-1/r}}{r}\int_0^1 t^{1/r-1}\{\exp[(t/n)^{1/r}]-1\}^{rs+r}\exp[(t/n)^{1/r}]\rho^{(s)}(t)\,dt \\ &= \frac{n^{s+1-1/r}}{r}\int_0^1 t^{1/r-1}[(t/n)^{1/r}+O(n^{-2/r})]^{rs+r}[1+O(n^{-1/r})]\rho^{(s)}(t)\,dt \\ &\le \frac{n^{s+1-1/r}}{r}\int_0^1 t^{1/r-1}[(t/n)^{s+1}+O(n^{r+1+1/r})]\rho^{(s)}(t)\,dt \\ &= O(n^{-1/r}).\end{aligned}$$

Thus, if ψ is a continuous function, then

$$\lim_{n\to\infty}\int_0^1 \Big|\delta_n^{(s)}[\ln^r(1+|x|)]x^{rs+r}\psi(x)\Big|\,dx = 0. \tag{8}$$

Now let φ be an arbitrary function in $\mathcal{D}[-1,1]$. Then it follows from equations (3), (7) and (8) that

$$\begin{aligned}\mathop{\mathrm{N-lim}}_{n\to\infty}\langle\delta_n^{(s)}[\ln^r(1+|x|)],\varphi(x)\rangle &= \\ &= \sum_{k=0}^{sr+r-1}[1+(-1)^k]\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{s+k-i}s!(i+1)^{rs+r-1}\varphi^{(k)}(0)}{2r(rs+r-1)!k!}+0 \\ &= \sum_{k=0}^{sr+r-1}[1+(-1)^k]\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{s-i}s!(i+1)^{rs+r-1}}{2r(rs+r-1)!k!}\langle\delta^{(k)}(x),\varphi(x)\rangle,\end{aligned}$$

proving equation (2) on the interval $[-1,1]$. However, it is clear that $\delta_n^{(s)}[\ln^r(1+|x|)] = 0$ outside this interval and so equation (2) is proved.

Note that when $r = 1$ and $s = 0$, all the neutrix limits exist as ordinary limits and so the composition $\delta_n^{(s)}[\ln^r(1+|x|)]$ exists. This completes the proof of the theorem. □

2.2. Theorem. *The neutrix composition $\delta^{(s)}(\ln^r|1+x|)$ exists and*

$$\delta^{(s)}(\ln^r|1+x|) = \sum_{k=0}^{sr+r-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{s-i}s!(i+1)^{rs+r-1}}{r(rs+r-1)!k!}\delta^{(k)}(x) \tag{9}$$

for $s = 0,1,2,\ldots$ and $r = 1,2,\ldots$.

In particular, the composition $\delta(\ln|1+x|)$ exists and

$$\delta(\ln|1+x|) = \delta(x).$$

Proof. To prove equation (9), we will now have to evaluate

$$\mathop{\mathrm{N-lim}}_{n\to\infty}\langle\delta_n^{(s)}(\ln^r|1+x|),\varphi(x)\rangle,$$

for an arbitrary function $\varphi(x)$ in $\mathcal{D}[-1,1]$. By Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{rs+r-1}\frac{\varphi^{(k)}(0)}{k!}x^k + \frac{x^{rs+r}}{(rs+r)!}\varphi^{(rs+r)}(\xi x),$$

where $0 < \xi < 1$. Then if φ is in $\mathcal{D}[-1,1]$, we have

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty}\langle\delta_n^{(s)}[\ln^r|1+x|)],\varphi(x)\rangle &= \operatorname*{N-lim}_{n\to\infty}\sum_{k=0}^{rs+r-1}\frac{\varphi^{(k)}(0)}{k!}\int_0^1\delta_n^{(s)}[\ln^r(1+x)]x^k\,dx \\ &+ \operatorname*{N-lim}_{n\to\infty}\sum_{k=0}^{rs+r-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^0\delta_n^{(s)}[\ln^r(1+x)]x^k\,dx \\ &+ \operatorname*{N-lim}_{n\to\infty}\frac{1}{(rs+r)!}\int_{-1}^1\delta_n^{(s)}[\ln^r|1+x|]x^{rs+r}\varphi^{(rs+r)}(\xi x)\,dx.\end{aligned} \tag{10}$$

It follows as in the proof of Theorem 2.1 that

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty}\int_0^1\delta_n^{(s)}[\ln^r(1+x)]x^k\,dx &= \operatorname*{N-lim}_{n\to\infty}n^{s+1}\int_0^1\rho^{(s)}[n\ln^r(1+x)]x^k\,dx \\ &= \sum_{i=0}^k\binom{k}{i}\frac{(-1)^{s+k-i}s!(i+1)^{rs+r-1}}{2r(rs+r-1)!}\end{aligned} \tag{11}$$

for $k = 0,1,2,\ldots,rs+r-1$.

Next, for large enough n, we have on making the substitution $t = n\ln^r(1-x)$,

$$\begin{aligned}\int_{-1}^0\delta_n^{(s)}[\ln^r(1+x)]x^k\,dx &= (-1)^k n^{s+1}\int_0^1\rho^{(s)}[n\ln^r(1-x)]x^k\,dx \\ &= (-1)^k\frac{n^{s+1-1/r}}{r}\int_0^1 t^{1/r-1}\{1-\exp[(t/n)^{1/r}]\}^k\exp[(t/n)]^{1/r}\rho^{(s)}(t)\,dt \\ &= \frac{n^{s+1-1/r}}{r}\sum_{i=0}^k\binom{k}{i}(-1)^{k-i}\int_0^1 t^{1/r-1}\exp[(i+1)(t/n)^{1/r}]\rho^{(s)}(t)\,dt,\end{aligned}$$

and it follows as above that

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty}\int_{-1}^0\delta_n^{(s)}[\ln^r(1+x)]x^k\,dx &= \operatorname*{N-lim}_{n\to\infty}(-1)^k n^{s+1}\int_0^1\rho^{(s)}[n\ln^r(1-x)]x^k\,dx \\ &= \sum_{i=0}^k\binom{k}{i}\frac{(-1)^{s+k-i}s!(i+1)^{rs+r-1}}{2r(rs+r-1)!}\end{aligned} \tag{12}$$

for $k = 0,1,2,\ldots,rs+r-1$.

It follows as above that when $k = rs+r$, we have

$$\int_{-1}^1\left|\delta_n^{(s)}[\ln^r|1+x|)]x^{rs+r}\right|dx = O(n^{-1/r})$$

and so if ψ is a continuous function, then

$$\lim_{n\to\infty}\int_{-1}^1\left|\delta_n^{(s)}[\ln^r|1+x|)]x^{rs+r}\psi(x)\right|dx = 0. \tag{13}$$

Now let φ be an arbitrary function in $\mathcal{D}[-1,1]$. Then it follows from equations (10), (11), (12) and (13) that

$$\begin{aligned}&\operatorname*{N-lim}_{n\to\infty}\langle\delta_n^{(s)}[\ln^r|1+x|)],\varphi(x)\rangle = \\ &\qquad = \sum_{k=0}^{sr+r-1}\sum_{i=0}^k\binom{k}{i}\frac{(-1)^{s+k-i}s!(i+1)^{rs+r-1}\varphi^{(k)}(0)}{r(rs+r-1)!k!} + 0 \\ &\qquad = \sum_{k=0}^{sr+r-1}\sum_{i=0}^k\binom{k}{i}\frac{(-1)^{s-i}s!(i+1)^{rs+r-1}}{r(rs+r-1)!k!}\langle\delta^{(k)}(x),\varphi(x)\rangle,\end{aligned}$$

so proving equation (9) on the interval $[-1,1]$. However, it is clear that $\delta_n^{(s)}[\ln^r|1+x|)] = 0$ outside this interval and so equation (9) is proved.

Note that when $r=1$ and $s=0$, all the neutrix limits again exist as ordinary limits and so the composition $\delta(\ln|1+x|)$ exists. This completes the proof of the theorem. □

2.3. Theorem. *The neutrix composition* $\delta^{(r^s-1)}[\ln^{1/r}(1+|x|)]$ *exists and*

$$\delta^{(r^s-1)}[\ln^{1/r}(1+|x|)] = \tag{14}$$
$$= \sum_{k=0}^{r^s-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{r^s-i-1}[1+(-1)^k]r(r^s-1)!(i+1)^{r^{s-1}-1}}{2(r^{s-1}-1)!k!}\delta^{(k)}(x)$$

for $s=1,2,\ldots$ *and* $r=2,3,\ldots$.

Proof. This time we must evaluate

$$\mathrm{N-}\lim_{n\to\infty}\langle\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)],\varphi(x)\rangle,$$

for an arbitrary function $\varphi(x)$ in $\mathcal{D}[-1,1]$.

By Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{r^s-1}\frac{\varphi^{(k)}(0)}{k!}x^k + \frac{x^{r^s}}{(r^s)!}\varphi^{(r^s)}(\xi x),$$

where $0<\xi<1$. Then, if $\varphi(x)$ in $\mathcal{D}[-1,1]$, we have

$$\mathrm{N-}\lim_{n\to\infty}\langle\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)],\varphi(x)\rangle = \tag{15}$$
$$= \mathrm{N-}\lim_{n\to\infty}\sum_{k=0}^{r^s-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{1}\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]x^k\,dx$$
$$+ \mathrm{N-}\lim_{n\to\infty}\frac{1}{(r^s)!}\int_{-1}^{1}\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]x^{r^s}\varphi^{(r^s)}(\xi x)\,dx.$$

For large enough n, we have

$$\int_{-1}^{1}\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]x^k\,dx = n^{r^s}\int_{-1}^{1}\rho^{(r^s-1)}[n\ln^{1/r}(1+|x|)]x^k\,dx \tag{16}$$
$$= n^{r^s}[1+(-1)^k]\int_0^1\rho^{(r^s-1)}[n\ln^{1/r}(1+x)]x^k\,dx.$$

Making the substitution $t=n\ln^{1/r}(1+x)$, we have

$$n^{r^s}\int_0^1\rho^{(r^s)}[n\ln^{1/r}(1+x)]x^k\,dx =$$
$$= rn^{r^s-r}\int_0^1 t^{r-1}\{\exp[(t/n)^r]-1\}^k\exp[(t/n)^r]\rho^{(r^s-1)}(t)\,dt$$
$$= rn^{r^s-r}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\int_0^1 t^{r-1}\exp[(i+1)(t/n)^r]\rho^{(r^s-1)}(t)\,dt,$$

where

$$rn^{r^s-r}\int_0^1 t^{r-1}\exp[(i+1)(t/n)^r]\rho^{(r^s-1)}(t)\,dt =$$
$$= \sum_{j=0}^{\infty}\int_0^1\frac{r(i+1)^j t^{r(j+1)-1}}{j!n^{r(j+1)-r^s}}\rho^{(r^s-1)}(t)\,dt.$$

It follows that

$$\mathrm{N-}\lim_{n\to\infty} rn^{r^s-r}\int_0^1 t^{r-1}\exp[(i+1)(t/n)^r]\rho^{(r^s-1)}(t)\,dt =$$

$$= \int_0^1 \frac{r(i+1)^{r^{s-1}-1}t^{r^s-1}}{(r^{s-1}-1)!}\rho^{(r^s-1)}(t)\,dt$$

$$= \frac{(-1)^{r^s-1}r(r^s-1)!(i+1)^{r^{s-1}-1}}{2(r^{s-1}-1)!}$$

for $i = 0, 1, 2, \ldots, k$, and so

$$\mathrm{N-}\lim_{n\to\infty} n^{r^s}\int_0^1 \rho^{(r^s-1)}[n\ln^{1/r}(1+x)]x^k\,dx = \tag{17}$$

$$= \sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{r^s+k-i-1}r(r^s-1)!(i+1)^{r^{s-1}-1}}{2(r^{s-1}-1)!}.$$

It now follows from equations (16) and (17) that

$$\mathrm{N-}\lim_{n\to\infty}\int_{-1}^1 \delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]x^k\,dx = \tag{18}$$

$$= [1+(-1)^k]\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{r^s+k-i-1}r(r^s-1)!(i+1)^{r^{s-1}-1}}{2(r^{s-1}-1)!},$$

for $k = 0, 1, 2, \ldots, r^s - 1$.

When $k = r^s$, it follows as above that

$$\int_{-1}^1 \left|\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]x^{r^s}\right| = O(n^{-r}),$$

and so if ψ is a continuous function, then

$$\lim_{n\to\infty}\int_{-1}^1 \left|\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]x^{r^s}\psi(x)\right|dx = 0. \tag{19}$$

Now let φ be an arbitrary function in $\mathcal{D}[-1,1]$. Then it follows from equations (15), (18) and (19) that

$$\mathrm{N-}\lim_{n\to\infty}\langle\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]x^k, \varphi(x)\rangle =$$

$$= \sum_{k=0}^{s^r-1}[1+(-1)^k]\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{r^s+k-i-1}r(r^s-1)!(i+1)^{r^{s-1}-1}\varphi^{(k)}(0)}{2(r^{s-1}-1)!k!} + 0$$

$$= \sum_{k=0}^{s^r-1}[1+(-1)^k]\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{r^s-i-1}r(r^s-1)!(i+1)^{r^{s-1}-1}}{2(r^{s-1}-1)!k!}\langle\delta^{(k)}(x), \varphi(x)\rangle,$$

so proving equation (14) on the interval $[-1, 1]$. However, it is clear that $\delta_n^{(s)}[\ln^r(1+|x|)] = 0$ outside this interval, and so equation (14) is proved. □

Finally we have:

2.4. Theorem. *The neutrix composition* $\delta^{(r^s-1)}(\ln^{1/r}|1+x|)$ *exists, and*

$$\delta^{(r^s-1)}(\ln^{1/r}|1+x|) = \sum_{k=0}^{r^s-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{r^s-i-1}r(r^s-1)!(i+1)^{r^{s-1}-1}}{(r^{s-1}-1)!k!}\delta^{(k)}(x) \tag{20}$$

for $s = 1, 2, \ldots$ *and* $r = 2, 3, \ldots$.

Proof. As in the proof of Theorem 2.3 we must evaluate

$$\operatorname*{N-lim}_{n\to\infty}\langle \delta_n^{(r^s-1)}(\ln^{1/r}|1+x|), \varphi(x)\rangle,$$

for an arbitrary function $\varphi(x)$ in $\mathcal{D}[-1,1]$.

By Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{r^s-1} \frac{\varphi^{(k)}(0)}{k!}x^k + \frac{x^{r^s}}{(r^s)!}\varphi^{(r^s)}(\xi x),$$

where $0<\xi<1$. Then if φ is in $\mathcal{D}[-1,1]$, we have

$$\begin{aligned}
&\operatorname*{N-lim}_{n\to\infty}\langle \delta_n^{(r^s-1)}(\ln^{1/r}|1+x|), \varphi(x)\rangle = \\
&\qquad = \operatorname*{N-lim}_{n\to\infty} \sum_{k=0}^{r^s-1} \frac{\varphi^{(k)}(0)}{k!} \int_0^1 \delta_n^{(r^s-1)}[\ln^{1/r}(1+x)]x^k\,dx \\
&\qquad\quad + \operatorname*{N-lim}_{n\to\infty} \sum_{k=0}^{r^s-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^0 \delta_n^{(r^s-1)}[\ln^{1/r}(1+x)]x^k\,dx \\
&\qquad\quad + \operatorname*{N-lim}_{n\to\infty} \frac{1}{(r^s)!}\int_{-1}^1 \delta_n^{(r^s-1)}(\ln^{1/r}(|1+x|)x^{r^s}\varphi^{(r^s)}(\xi x)\,dx.
\end{aligned} \tag{21}$$

It follows that

$$\begin{aligned}
&\operatorname*{N-lim}_{n\to\infty} \int_0^1 \delta_n^{(r^s-1)}[\ln^{1/r}(1+x)]x^k\,dx = \\
&\qquad = \operatorname*{N-lim}_{n\to\infty} n^{r^s} \int_0^1 \rho^{(r^s-1)}[n\ln^{1/r}(1+x)]x^k\,dx \\
&\qquad = \sum_{i=0}^k \binom{k}{i} \frac{(-1)^{r^s+k-i-1} r(r^s-1)!(i+1)^{r^{s-1}-1}}{2(r^{s-1}-1)!}
\end{aligned} \tag{22}$$

for $k=0,1,2,\dots,r^s-1$.

Next, for large enough n, we have on making the substitution $t=n\ln^r(1-x)$,

$$\begin{aligned}
&\operatorname*{N-lim}_{n\to\infty} \int_{-1}^0 \delta_n^{(r^s-1)}[\ln^{1/r}(1+x)]x^k\,dx = \\
&\qquad = \operatorname*{N-lim}_{n\to\infty} (-1)^k n^{r^s} \int_0^1 \rho^{(r^s-1)}[n\ln^{1/r}(1-x)]x^k\,dx \\
&\qquad = \sum_{i=0}^k \binom{k}{i} \frac{(-1)^{r^s+k-i-1} r(r^s-1)!(i+1)^{r^{s-1}-1}}{2(r^{s-1}-1)!}
\end{aligned} \tag{23}$$

for $k=0,1,2,\dots,r^s-1$.

When $k=r^s$, it follows as above that

$$\int_{-1}^1 \left|\delta_n^{(r^s-1)}[\ln^{1/r}(|1+x|)]x^{r^s}\right| = O(n^{-r})$$

and so if ψ is a continuous function, then

$$\lim_{n\to\infty}\int_{-1}^1 \left|\delta_n^{(r^s-1)}[\ln^{1/r}(|1+x|)]x^{r^s}\psi(x)\right|dx = 0. \tag{24}$$

Now let φ be an arbitrary function in $\mathcal{D}[-1,1]$. Then it follows from equations (21) to (24) that

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty}\langle\delta_n^{(r^s-1)}[\ln^{1/r}(|1+x|)]x^k,\varphi(x)\rangle &=\\ &= \sum_{k=0}^{r^s-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{r^s+k-i-1}r(r^s-1)!(i+1)^{r^{s-1}-1}\varphi^{(k)}(0)}{(r^{s-1}-1)!k!}+0\\ &= \sum_{k=0}^{r^s-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{r^s-i-1}r(r^s-1)!(i+1)^{r^{s-1}-1}}{(r^{s-1}-1)!k!}\langle\delta^{(k)}(x),\varphi(x)\rangle,\end{aligned}$$

so proving equation (20) on the interval $[-1,1]$. However, it is clear that $\delta_n^{(s)}[\ln^r(|1+x|)] = 0$ outside this interval and so equation (20) is proved. □

For further results on the neutrix composition of distributions, see [7], [8], [9], [10] and [11].

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