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A DIFFERENTIAL OPERATOR AND ITS APPLICATIONS TO CERTAIN MULTIVALENTLY ANALYTIC FUNCTIONS

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Abstract

In the present paper, the authors obtain some interesting properties of (ordinary) differential operators. Several useful consequences associated with the main results are also considered.

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1. Introduction and definitions

Let $\mathcal{A}(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \ (k, p \in \mathbb{N} = \{1, 2, \dots\}; \ a_k \in \mathbb{C}), \tag{1.1}$$

which are *multivalently analytic* in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, where $\mathbb{C}$ denotes the set of complex numbers.

We also denote by $\mathcal{S}(p)$, $\mathcal{C}(p)$ and $\mathcal{K}(p)$, respectively, the subclasses of all multivalently analytic functions which are *multivalently starlike*, *multivalently convex* and *multivalently close-to-convex* with respect to the origin in $\mathbb{U}$ (see [2,3,9] for further details).

Upon differentiating both sides of (1.1), q-times with respect to z, we easily obtain the following (ordinary) differential operator:

$$f^{(q)}(z) = \chi(p; q) z^{p-q} + \sum_{k=1}^{\infty} \chi(k + p; q) a_{k+p} z^{k+p-q}, \tag{1.2}$$

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where

$$\chi(m;q)=\frac{m!}{(m-q)!}\quad (m>q;\ m\in\mathbb{N};\ q\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}). \tag{1.3}$$

Now, using the operator in (1.2) together with (1.3), we introduce a subclass $\mathcal{T}_q(p)$ of the class $\mathcal{A}(p)$ of multivalently analytic functions which consists of functions $f(z)$ satisfying the inequality:

$$\left|p+\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}-q\right|<1\quad (z\in\mathbb{U};\ p>q;\ p\in\mathbb{N};\ q\in\mathbb{N}_0). \tag{1.4}$$

The purpose of the present investigation is to focus on the functions $f(z)$ in the class $\mathcal{T}_q(p)$ defined by (1.4), and to obtain some interesting results which will be useful in geometric function theory (cf., e.g., [9]). Certain geometric consequences of our main results are also pointed out. We remark in passing that the operator defined by (1.2) has been studied earlier by several researchers (see, for example, [3-6]).

The following assertion will be required in our present investigations.

1.1. Lemma. [7, 8] *Let the function $w(z)$ be non-constant and analytic in $\mathbb{U}$ with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point z_0, then $z_0w'(z_0)=cw(z_0)$, where c is a real number satisfying $c\geq 1$.*

2. Main results and their consequences

We begin by proving the following theorem.

2.1. Theorem. *Let $f(z)\in\mathcal{A}(p)$ be given by (1.1). Then,*

$$\Re e\left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\right)<p-q \implies \left|f^{(q)}(z)\right|<\chi(p;q)\,|z|^{p-q-1}, \tag{2.1}$$

where $\chi(p;q)$ is defined by (1.3).

Proof. Take $f(z)\in\mathcal{A}(p)$ and let the function $w(z)$ be defined by

$$\begin{aligned} f^{(q)}(z)&=\chi(p;q)z^{p-q-1}\left(z+\sum_{k=1}^{\infty}\frac{\chi(k+p;q)}{\chi(p;q)}a_{k+p}z^{k+1}\right)\\ &=\chi(p;q)z^{p-q-1}w(z). \end{aligned} \tag{2.2}$$

By differentiating (2.2), we obtain

$$zf^{(q+1)}(z)=\chi(p;q)w(z)\left(p-q-1+\frac{zw'(z)}{w(z)}\right)z^{p-q-1}. \tag{2.3}$$

In view of (2.2) and (2.3), we easily obtain

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}=p-q-1+\frac{zw'(z)}{w(z)},\ (w(z)\neq 0). \tag{2.4}$$

Supposing now that there exists a point $z_0\in\mathbb{U}$ such that $z_0w'(z_0)=cw(z_0)$, $(c\geq 1)$, setting z to be z_0 in (2.4) and using Lemma 1.1 and also (2.4), we then obtain the following assertion:

$$\begin{aligned} \Re e\left(\frac{z_0f^{(q+1)}(z_0)}{f^{(q)}(z_0)}\right)&=p-q-1+\Re e\left(\frac{z_0w'(z_0)}{w(z_0)}\right)\\ &\geq p-q, \end{aligned}$$

which is a contradiction to (2.1). Therefore, we conclude that $|w(z)| < 1$ for all $z \in \mathbb{U}$, and the definition (2.2) immediately yields the inequality:

$$\left| \frac{f^{(q)}(z)}{z^{p-q-1}} \right| = \chi(p;q)|w(z)| < \chi(p;q) \quad (z \in \mathbb{U};\ p > q;\ p \in \mathbb{N};\ q \in \mathbb{N}_0),$$

which completes the proof of Theorem 2.1. □

2.2. Theorem. *Let $f(z) \in \mathcal{A}(n)$, $g(z) \in \mathcal{A}(m)$ with $p := n - m$, $(p, n, m \in \mathbb{N})$, and suppose also that*

$$\Re e \left(\frac{g^{(q)}(z)}{zg^{(q+1)}(z)} \right) > \alpha \quad (z \in \mathbb{U};\ q \in \mathbb{N}_0;\ \alpha \geq 1). \tag{2.5}$$

If the inequality

$$\left| \frac{g^{(q)}(z)}{f^{(q)}(z)} \left(\frac{f^{(q+1)}(z)}{g^{(q+1)}(z)} - \frac{n!(m-q)!}{m!(n-q)!} z^p \right) \right| < \alpha \left(p + \frac{1}{2} \right) - \frac{1}{2} \tag{2.6}$$

is satisfied, then

$$\left| \frac{m!(n-q)!}{n!(m-q)!} \frac{f^{(q)}(z)}{g^{(q)}(z)} - z^p \right| < |z|^p, \quad (z \in \mathbb{U};\ m > q;\ m \in \mathbb{N};\ q \in \mathbb{N}_0). \tag{2.7}$$

Proof. Let the functions $f(z)$ and $g(z)$ satisfy the hypotheses of Theorem 2.2. In view of (1.1), (1.2) and (1.3), it is easily seen that

$$\frac{m!(n-q)!}{n!(m-q)!} \frac{f^{(q)}(z)}{g^{(q)}(z)} = z^p \left(1 + \sum_{k=1}^{\infty} c_{p+k} z^k \right) \in \mathcal{A}(p) \equiv \mathcal{A}(n-m), \quad (n - m \in \mathbb{N}), \tag{2.8}$$

and we define a new function $w(z)$ by

$$\frac{m!(n-q)!}{n!(m-q)!} \frac{f^{(q)}(z)}{g^{(q)}(z)} = z^p[1 + w(z)]. \tag{2.9}$$

Clearly, $w(z)$ satisfies the hypotheses of Lemma 1.1. Thus, on differentiating (2.8), we obtain

$$\frac{m!(n-q)!}{n!(m-q)!} \frac{f^{(q)}(z)}{z^p g^{(q+1)}(z)} - 1 = w(z) + [zw'(z) + p(1 + w(z)] \frac{g^{(q)}(z)}{zg^{(q+1)}(z)}. \tag{2.10}$$

It follows from (2.8) and (2.9) that

$$\begin{aligned} \mathcal{H}(z) &= \frac{m!(n-q)!}{n!(m-q)!} \frac{\dfrac{f^{(q+1)}(z)}{z^p g^{(q+1)}(z)} - 1}{\dfrac{m!(n-q)!}{n!(m-q)!} \dfrac{f^{(q)}(z)}{z^p g^{(q)}(z)}} \\ &= \frac{w(z)}{1 + w(z)} + \left(p + \frac{zw'(z)}{1 + w(z)} \right) \frac{g^{(q)}(z)}{zg^{(q+1)}(z)}. \end{aligned} \tag{2.11}$$

After making the same assumption as in in the proof of Theorem 2.2, (2.10) yields:

$$\begin{aligned}|\mathcal{H}(z_0)| &= \left|\frac{w(z_0)}{1+w(z_0)} + \left(p + \frac{z_0 w'(z_0)}{1+w(z_0)}\right)\frac{g^{(q)}(z_0)}{z_0 g^{(q+1)}(z_0)}\right| \\ &\geq \left|\left(p + \frac{z_0 w'(z_0)}{1+w(z_0)}\right)\frac{g^{(q)}(z_0)}{z_0 g^{(q+1)}(z_0)}\right| - \left|\frac{w(z_0)}{1+w(z_0)}\right| \\ &\geq \Re e\left\{p + \frac{z_0 w'(z_0)}{1+w(z_0)}\right\}\Re e\left\{\frac{g^{(q)}(z_0)}{z_0 g^{(q+1)}(z_0)}\right\} - \Re e\left\{\frac{w(z_0)}{1+w(z_0)}\right\} \\ &\geq \alpha\left(p+\frac{1}{2}\right) - \frac{1}{2},\end{aligned}$$

which contradicts (2.6). Hence, we conclude that $|w(z)| < 1$ for all $z \in \mathbb{U}$, and (2.8) evidently yields the inequality (2.7). □

2.3. Theorem. *Let $f(z) \in \mathcal{A}(p)$. Then,*

$$\left|1 - \frac{1 + \dfrac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - p + q}{\dfrac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q}\right| < \frac{1}{p-q+1} \implies f(z) \in \mathcal{T}_q(p). \tag{2.12}$$

Proof. Let the function $f(z)$ of the form (1.1) belong to the class $\mathcal{A}(p)$, and also let the function $w(z)$ be defined by

$$q + \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p = w(z), \ (z \in \mathbb{U};\ p > q;\ p \in \mathbb{N};\ q \in \mathbb{N}_0). \tag{2.13}$$

By logarithmically differentiating (2.13), we easily arrive at:

$$1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - p + q = w(z)\left(1 + \frac{zw'(z)}{w(z)}\frac{f^{(q)}(z)}{zf^{(q+1)}(z)}\right),$$

or, equivalently,

$$1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - p + q = w(z)\left(1 + \frac{zw'(z)}{w(z)}\frac{1}{p-q+w(z)}\right). \tag{2.14}$$

From the equations (2.13) and (2.14), we find that

$$\mathcal{G}(z) := \frac{1 + \dfrac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - p + q}{\dfrac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q} - 1 = \frac{zw'(z)}{w(z)} \cdot \frac{1}{p-q+w(z)}. \tag{2.15}$$

Therefore, from Lemma 1.1 and (2.15), we obtain

$$|\mathcal{G}(z_0)| = \left|\frac{z_0 w'(z_0)}{w(z_0)} \cdot \frac{1}{p-q+w(z_0)}\right| \geq \frac{c}{p-q+|w(z_0)|},$$

which is a contradiction to the first assumption of (2.11) when $c = 1$. Therefore, we conclude that $|w(z)| < 1$ for all $z \in \mathbb{U}$. Thus, the second property immediately follows from the inequality (2.12). This completes the proof. □

2.4. Theorem. *Let $f(z) \in \mathcal{A}(p)$. Then,*

$$\left|f^{(q)}(z)\left(q + \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p\right)\right| < |z|^{p-q} \implies \left|f^{(q)}(z) - \chi(p;q)z^{p-q}\right| < |z|^{p-q}. \tag{2.16}$$

Proof. Let the function $f(z)$ belong to the class $\mathcal{A}(p)$. If we again define $w(z)$ by

$$\frac{f^{(q)}(z)}{z^{p-q}} - \chi(p;q) = \sum_{k=1}^{\infty} \chi(k+p;q) a_{k+p} z^k = w(z), \ (w(z) \neq 0), \tag{2.17}$$

then clearly $w(z)$ is an analytic function in $\mathbb{U}$ and $w(0) = 0$. Furthermore, by differentiating (2.17), we have that

$$\frac{f^{(q+1)}(z)}{z^{p-q-1}} = (p-q)\left(\chi(q;p) + w(z)\right) + zw'(z), \tag{2.18}$$

which, in view of (2.17), readily yields

$$\mathcal{F}(z) = \frac{f^{(q)}(z)}{z^{p-q}} \left(q + \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p \right) = zw'(z). \tag{2.19}$$

If we first set z equal to z_0 in (2.19), and use Lemma 1.1 in the same equation once again, then we obtain:

$$|\mathcal{F}(z_0)| = |z_0 w'(z_0)| = c|w(z_0)| = c \geq 1,$$

which contradicts the first inequality of (2.16). So, we conclude that $|w(z)| < 1$ for all z in $\mathbb{U}$, which completes the proof. □

We conclude this paper by remarking that by choosing suitable values of the parameters q and p in the above theorems, one can infer several special results which will be useful in the study of geometric function theory and are therefore worthwhile consequences of our main results. For example, we give the following corollaries as special cases of the main results.

By taking $q = 0$ and $q = 1$, respectively, in Theorem 2.3, we obtain the following corollaries.

2.5. Corollary. *If $f(z) \in \mathcal{A}(p)$ satisfies*

$$\left| \frac{1 + \dfrac{zf''(z)}{f'(z)} - p}{\dfrac{zf'(z)}{f(z)} - p} - 1 \right| < \frac{1}{p+1}, \ (z \in \mathbb{U}; \ p \in \mathbb{N}),$$

then $f(z) \in \mathcal{S}(p)$, i.e. $f(z)$ is multivalently starlike in $\mathbb{U}$.

2.6. Corollary. *If $f(z) \in \mathcal{A}(p)$ satisfies*

$$\left| \frac{2 + \dfrac{zf'''(z)}{f''(z)} - p}{1 + \dfrac{zf''(z)}{f'(z)} - p} - 1 \right| < \frac{1}{p}, \ (z \in \mathbb{U}; \ p \in \mathbb{N}),$$

then $f(z) \in \mathcal{C}(p)$, i.e. $f(z)$ is multivalently convex in $\mathbb{U}$.

By taking $q = 1$ in Theorem 2.4 we get:

2.7. Corollary. *If $f(z) \in \mathcal{T}(p)$ satisfies*

$$\left| f'(z) \left(1 + \frac{zf''(z)}{f'(z)} - p \right) \right| < |z|^{p-1}, \ (z \in \mathbb{U}; \ p \in \mathbb{N}),$$

then $f(z) \in \mathcal{K}(p)$, i.e. $f(z)$ is multivalently close-to-convex in $\mathbb{U}$.

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