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A NEW DIFFERENTIAL INEQUALITY

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Abstract

We find conditions on the complex-valued function $A : U \to \mathbb{C}$ defined in the unit disc U such that the differential inequality

$$\operatorname{Re}\left[A(z)p^2(z) - \alpha(zp'(z) - 1)^2 + 2\beta(zp'(z))^2 + \gamma\right] > 0$$

implies $\operatorname{Re} p(z) > 0$, where $p \in \mathcal{H}[1, n]$, $\alpha, \beta \in \mathbb{C}$ and n is a positive integer.

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1. Introduction and preliminaries

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}[U] : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots ,\ z \in U\}.$$

In order to prove the new results we shall use the following lemma, which is a particular form of Theorem 2.3.i in [1, p.35].

1.1. Lemma. [1, p.35]. *Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ be a function which satisfies*

$$\operatorname{Re} \psi(\rho i, \sigma; z) \le 0,$$

where $\rho, \sigma \in \mathbb{R}$, $\sigma \le -\frac{n}{2}(1 + \rho^2)$, $z \in U$ and $n \ge 1$.

If $p \in \mathcal{H}[1, n]$ and

$$\operatorname{Re} \psi(p(z), zp'(z); z) > 0$$

then

$$\operatorname{Re} p(z) > 0.$$

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Oros and Cătaş [2] (see also [3]) obtained a condition on the complex-valued function $A : U \to \mathbb{C}$ defined in the unit disc U such that the differential inequality

$$\operatorname{Re}[A(z)p^2(z) - \alpha(zp'(z))^2 + \beta zp'(z) + \gamma] > 0$$

implies $\operatorname{Re} p(z) > 0$, where $p \in \mathcal{H}[1,n]$, $\alpha, \beta, \gamma \in \mathbb{R}$ and n is a positive integer.

In this note, we find new condition on the complex-valued function A defined in the unit disc U such that the differential inequality

$$\operatorname{Re}[A(z)p^2(z) - \alpha(zp'(z) - 1)^2 + 2\beta(zp'(z))^2 + \gamma] > 0$$

implies $\operatorname{Re} p(z) > 0$, where $p \in \mathcal{H}[1,n]$, $\alpha, \beta \in \mathbb{C}$ and n is a positive integer.

2. Main results

2.1. Theorem. *Let $\alpha \in \mathbb{C}$ $(\operatorname{Re}\alpha \geq 0)$, $\beta \in \mathbb{C}$, $(\alpha+\beta) \in \mathbb{R}^+$, $\gamma \leq (\alpha+\beta)n + (\frac{n^2}{4}+1)\operatorname{Re}\alpha$ and let n be a positive integer. Suppose that the function $A : U \to \mathbb{C}$ satisfies:*

$$\operatorname{Re} A(z) > -\frac{n^2}{4}\operatorname{Re}\alpha - \frac{n}{2}(\alpha+\beta). \tag{2.1}$$

If $p \in \mathcal{H}[1,n]$ and

$$\operatorname{Re}[A(z)p^2(z) - \alpha(zp'(z) - 1)^2 + 2\beta(zp'(z))^2 + \gamma] > 0 \tag{2.2}$$

then

$$\operatorname{Re} p(z) > 0.$$

Proof. We let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ be defined by

$$\psi(p(z), zp'(z); z) = A(z)p^2(z) - \alpha(zp'(z) - 1)^2 + 2\beta(zp'(z))^2 + \gamma.$$

From (2.2) we have

$$\operatorname{Re}\psi(p(z), zp'(z); z) > 0 \text{ for } z \in U.$$

For $z \in U$ and $\sigma, \rho \in \mathbb{R}$ satisfying $\sigma \leq -\frac{n}{2}(1+\rho^2)$, we have $-\sigma^2 \leq -\frac{n^2}{4}(1+\rho^2)^2$ and hence, using (2.1), we obtain:

$$\begin{aligned}
\operatorname{Re}\psi(\rho i, \sigma; z) &= \operatorname{Re}[A(z)(\rho i)^2 - \alpha(\sigma-1)^2 + 2\beta\sigma + \gamma] \\
&= -\rho^2 \operatorname{Re} A(z) + (\sigma^2+1)\operatorname{Re}\alpha + 2(\alpha+\beta)\sigma + \gamma \\
&\leq -\rho^2 \operatorname{Re} A(z) - \left(\frac{n^2}{4}(1+\rho^2)^2 + 1\right)\operatorname{Re}\alpha - (\alpha+\beta)(1+\rho^2)n + \gamma \\
&= -\frac{n^2}{4}\rho^4 \operatorname{Re}\alpha - \left[\frac{n^2}{4}\operatorname{Re}\alpha + \frac{n}{2}(\alpha+\beta)\right. \\
&\qquad \left. + \operatorname{Re} A(z)\right]\rho^2 - (\alpha+\beta)n - \left(\frac{n^2}{4}+1\right)\operatorname{Re}\alpha + \gamma \\
&\leq 0.
\end{aligned}$$

By using Lemma 1.1, we have that $\operatorname{Re} p(z) > 0$. □

If $\gamma = (\alpha+\beta)n + \left(\frac{n^2}{4}+1\right)\operatorname{Re}\alpha$, then Theorem 2.1 can be rewritten as follows:

2.2. Corollary. *Let $\alpha \in \mathbb{C}$ $(\operatorname{Re}\alpha \geq 0)$, $\beta \in \mathbb{C}$, $(\alpha+\beta) \in \mathbb{R}^+$, and let n be a positive integer. Suppose that the function $A : U \to \mathbb{C}$ satisfies:*

$$\operatorname{Re} A(z) > -\frac{n^2}{4}\operatorname{Re}\alpha - \frac{n}{2}(\alpha+\beta). \tag{2.3}$$

If $p \in \mathcal{H}[1, n]$ *and*

$$\operatorname{Re}\left[A(z)p^2(z) - \alpha(zp'(z)-1)^2 + 2\beta(zp'(z))^2 + (\alpha+\beta)n + \left(\frac{n^2}{4}+1\right)\operatorname{Re}\alpha\right] > 0 \tag{2.4}$$

then

$$\operatorname{Re} p(z) > 0. \qquad \square$$

Taking $\beta = \overline{\alpha}$ in Theorem 2.1, we have

2.3. Corollary. *Let* $\alpha \in \mathbb{C}$ $(\operatorname{Re}\alpha \geq 0)$, $\gamma \leq (n^2+8n+4)\frac{\operatorname{Re}\alpha}{4}$, *and let* n *be a positive integer. Suppose that the function* $A : U \to \mathbb{C}$ *satisfies:*

$$\operatorname{Re} A(z) > -(n^2+2n)\frac{\operatorname{Re}\alpha}{4}. \tag{2.5}$$

If $p \in \mathcal{H}[1, n]$ *and*

$$\operatorname{Re}\left[A(z)p^2(z) - \alpha(zp'(z)-1)^2 + 2\overline{\alpha}(zp'(z))^2 + \gamma\right] > 0 \tag{2.6}$$

then

$$\operatorname{Re} p(z) > 0. \qquad \square$$

Taking $\alpha + \beta = 1$ in Theorem 2.1, we have

2.4. Corollary. *Let* $\alpha \in \mathbb{C}$ $(\operatorname{Re}\alpha \geq 0)$, $\gamma \leq n + \left(\frac{n^2}{4}+1\right)\operatorname{Re}\alpha$, *and let* n *be a positive integer. Suppose that the function* $A : U \to \mathbb{C}$ *satisfies:*

$$\operatorname{Re} A(z) > -\frac{n^2}{4}\operatorname{Re}\alpha - \frac{n}{2}. \tag{2.7}$$

If $p \in \mathcal{H}[1, n]$ *and*

$$\operatorname{Re}\left[A(z)p^2(z) - \alpha(zp'(z)-1)^2 + 2(1-\alpha)(zp'(z))^2 + \gamma\right] > 0 \tag{2.8}$$

then

$$\operatorname{Re} p(z) > 0. \qquad \square$$

Taking $\alpha = 0$ in Theorem 2.1, we have

2.5. Corollary. *Let* $\beta \geq 0$, $\gamma \leq \beta n$, *and let* n *be a positive integer. Suppose that the function* $A : U \to \mathbb{C}$ *satisfies:*

$$\operatorname{Re} A(z) > -\frac{n}{2}\beta. \tag{2.9}$$

If $p \in \mathcal{H}[1, n]$ *and*

$$\operatorname{Re}\left[A(z)p^2(z) + 2\beta(zp'(z))^2 + \gamma\right] > 0 \tag{2.10}$$

then

$$\operatorname{Re} p(z) > 0. \qquad \square$$

Taking $\beta = 0$ in Theorem 2.1, we have

2.6. Corollary. *Let* $\alpha \geq 0$, $\gamma \leq \alpha n + \left(\frac{n^2}{4}+1\right)\alpha$, *and let* n *be a positive integer. Suppose that the function* $A : U \to \mathbb{C}$ *satisfies:*

$$\operatorname{Re} A(z) > -\frac{n^2}{4}\alpha - \frac{n}{2}\alpha. \tag{2.11}$$

If $p \in \mathcal{H}[1, n]$ *and*

$$\operatorname{Re}\left[A(z)p^2(z) - \alpha(zp'(z)-1)^2 + \gamma\right] > 0 \tag{2.12}$$

then

$$\operatorname{Re} p(z) > 0. \qquad \square$$

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