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COMMON FIXED POINT THEOREMS FOR MAPPINGS SATISFYING AN IMPLICIT RELATION WITHOUT DECREASING ASSUMPTION

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Abstract

The authors prove common fixed point theorems in metric spaces for four mappings satisfying an implicit relation, without decreasing assumption, using the concept of weak compatibility. These generalize two theorems of V. Popa, a theorem of M. Imdad, S. Kumar and M. S. Khan, a theorem of H. Bouhadjera and a theorem of A. Djoudi and A. Aliouche, respectively.

Keywords: Weakly compatible mappings, Common fixed point, Metric space.

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1. Introduction

Let S and T be self-mappings of a metric space (X, d).

S and T are *commuting* if $STx = TSx$ for all $x \in X$.

Sessa [16] defined S and T to be *weakly commuting* if for all $x \in X$,

(1.1) $\quad d(STx, TSx) \leq d(Tx, Sx)$.

Jungck [5] defined S and T to be *compatible*, as a generalization of weakly commuting, if

(1.2) $\quad \lim_{n\to\infty} d(STx_n, TSx_n) = 0$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

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It is easy to show that commuting implies weakly commuting implies compatible, and there are examples in the literature verifying that the inclusions are proper, see [5] and [16].

Jungck et al [6] defined S and T to be *compatible mappings of type* (A) if

$$\lim_{n\to\infty} d(STx_n, T^2x_n) = 0 \text{ and } \lim_{n\to\infty} d(TSx_n, S^2x_n) = 0 \tag{1.3}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Examples are given to show that the two concepts of compatibility are independent, see [6].

Recently, Pathak and Khan [11] defined S and T to be *compatible mappings of type* (B), as a generalization of compatible mappings of type (A), if

$$\begin{aligned} &\lim_{n\to\infty} d(TSx_n, S^2x_n) \le \frac{1}{2}\big[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d(Tt, T^2x_n)\big], \text{ and} \\ &\lim_{n\to\infty} d(STx_n, T^2x_n) \le \frac{1}{2}\big[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d(St, S^2x_n)\big] \end{aligned} \tag{1.4}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Clearly, compatible mappings of type (A) are compatible mappings of type (B), but the converse is not true, see [11]. However, compatibility, compatibility of type (A) and compatibility of type (B) are equivalent if S and T are continuous, see [11].

Pathak *et al* [12] defined S and T to be *compatible mappings of type* (P) if

$$\lim_{n\to\infty} d(S^2x_n, T^2x_n) = 0 \tag{1.5}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

However, compatibility, compatibility of type (A) and compatibility of type (P) are equivalent if S and T are continuous, see [12].

Pathak *et al* [13] defined S and T to be *compatible mappings of type* (C), as a generalization of compatible mappings of type (A), if

$$\begin{aligned} &\lim_{n\to\infty} d(TSx_n, S^2x_n) \le \frac{1}{3}\big[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d(Tt, S^2x_n) + \\ &\qquad\qquad \lim_{n\to\infty} d(Tt, T^2x_n)\big], \text{ and} \\ &\lim_{n\to\infty} d(STx_n, T^2x_n) \le \frac{1}{3}\big[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d(St, T^2x_n) + \\ &\qquad\qquad \lim_{n\to\infty} d(St, S^2x_n)\big] \end{aligned} \tag{1.6}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

However, compatibility, compatibility of type (A) and compatibility of type (C) are equivalent if S and T are continuous, see [13].

Pant [10] defined S and T to be *reciprocally continuous* if

$$\lim_{n\to\infty} STx_n = St \quad \text{and} \quad \lim_{n\to\infty} TSx_n = Tt \tag{1.7}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t in X.

It is evident that if S and T are both continuous, then they are reciprocally continuous, but the converse is not true. Moreover, it has been proved in [10] that in the setting of common fixed point theorems for compatible mappings satisfying contractive conditions, the continuity of one of the mappings S or T implies their reciprocal continuity, but not conversely.

2. Preliminaries

2.1. Definition. [7] S and T are said to be *weakly compatible* if they commute at their coincidence points; i.e., if $Su = Tu$ for some $u \in X$, then $STu = TSu$.

2.2. Lemma. [5, 6, 11, 12, 13] *If S and T are compatible, or compatible of type (A), or compatible of type (P), or compatible of type (B), or compatible of type (C), then they are weakly compatible.*

The converse is not true in general, see [1].

2.3. Definition. [8] S and T are said to be *R–weakly commuting* if there exists an $R > 0$ such that

$$d(STx, TSx) \leq Rd(Tx, Sx) \text{ for all } x \in X. \tag{2.1}$$

2.4. Definition. [8] S and T are said to be *pointwise R–weakly commuting* if for all $x \in X$, there exists an $R > 0$ such that (2.1) holds.

It was proved in [8] and [9] that R–weak commutativity is equivalent to commutativity at coincidence points; i.e., S and T are pointwise R–weakly commuting if and only if they are weakly compatible.

Let $\mathbb{R}_+$ be the set of all non-negative reals numbers and F_6 the family of all continuous mappings $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(F_1) : F is decreasing in the variables t_5 and t_6.

(F_2) : There exists $0 \leq h < 1$ such that for all $u, v \geq 0$ with
  (F_a) : $F(u, v, v, u, u + v, 0) \leq 0$, or
  (F_b) : $F(u, v, u, v, 0, u + v) \leq 0$,
 we have $u \leq hv$.

(F_3) : $F(u, u, 0, 0, u, u) > 0$ for all $u > 0$.

The following Theorems have been proved in [14] and [4], respectively.

2.5. Theorem. *Let S, T, I and J be self-mappings of a complete metric space (X, d) satisfying the conditions:*

(a) *$S(X) \subset J(X)$ and $T(X) \subset I(X)$.*
(b) *One of S, T, I and J is continuous,*
(c) *The pairs (S, I) and (T, J) are compatible,*

$$F(d(Sx, Ty),\ d(Ix, Jy),\ d(Ix, Sx),\ d(Jy, Ty),\ d(Ix, Ty),\ d(Sx, Jy)) \leq 0, \tag{2.2}$$
for all $x, y \in X$ and $F \in F_6$.

Then, S, T, I and J have a unique common fixed point in X.

2.6. Theorem. *Let S, T, I and J be self-mappings of a metric space (X, d) which satisfy (a) and (2.2). If one of $S(X)$, $T(X)$, $I(X)$ and $J(X)$ is a complete subspace of X, then*

(e) *S and I have a coincidence point,*
(f) *T and J have a coincidence point.*

Moreover, if the pairs (S, I) and (T, J) are weakly compatible, then S, T, I and J have a unique common fixed point.

Let $\mathcal{F}$ be the set of all continuous functions $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(F_1) : F is decreasing in the variables t_5 and t_6.
(F_2') : There exists $0 < \alpha < 1$ such that for all $u, v \geq 0$ with
(F_a) : $F(u, v, u, v, u + v, 0) \leq 0$, or
(F_b) : $F(u, v, v, u, 0, u + v) \leq 0$
we have $u \leq \alpha v$.
(F_3) : $F(u, u, 0, 0, u, u) > 0$ for all $u > 0$.

It will be noted that the condition F_2' involved here differs slightly from the previous condition F_2. The following Theorem has been proved in [3].

2.7. Theorem. *Let $\{A_i\}$, $i = 1, 2, \ldots$, S and T be self-mappings of a complete metric space (X, d) satisfying:*

(2.3) $A_1(X) \subset T(X)$ *and* $A_i(X) \subset S(X)$, $i > 1$,

(2.4) $F(d(A_1x,\ A_iy),\ d(Sx, Ty),\ d(A_1x, Sx),\ d(A_iy, Ty),\ d(A_1x, Ty),\ d(Sx, A_iy)) \leq 0,$

for all $x, y \in X$ and $F \in \digamma$. Let S be compatible with A_1 and T compatible with A_k for some $k > 1$. If the mappings in one of the compatible pairs (A_1, S) and (A_k, T) are reciprocally continuous, then $\{A_i\}$, S and T have a unique common fixed point in X.

It is our purpose in this paper to prove common fixed point theorems in metric spaces for weakly compatible mappings satisfying an implicit relation without decreasing assumption which generalize Theorem 2.5 of [14], a Theorem of [15], Theorem 2.6 of [4], a Theorem of [2] and Theorem 2.7 of [3].

3. Implicit relation

Throughout the remainder of this paper C_6 will denote the family of all continuous mappings $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$.

For $F \in C_6$ consider the following conditions:

(C_1) : There exists $0 \leq h < 1$ such that for all $u, v, w \geq 0$ with
(C_a) : $F(u, v, v, u, w, 0) \leq 0$, or
(C_b) : $F(u, v, u, v, 0, w) \leq 0$
we have $u \leq hv$.
(C_2) : $F(u, u, 0, 0, u, u) > 0$ for all $u > 0$.

3.1. Example. Let $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - h \max\{t_2, t_3, t_4\} + b(t_5 + t_6)$, $0 \leq h < 1$ and $b > 0$.

(C_1). Let $u, v, w \geq 0$. For C_a we have

$$F(u, v, v, u, w, 0) = u - h \max\{v, u\} + bw \leq 0.$$

If $v \leq u$, then $u < u$, which is a contradiction. Therefore, $u \leq hv$. Similarly, if $F(u, v, u, v, 0, w) \leq 0$, then $u \leq hv$.

(C_2). $F(u, u, 0, 0, u, u) = (1 - h)u + 2bu > 0$ for all $u > 0$.

3.2. Example. Let $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - h \max\{t_2, t_3, t_4\} + bt_5t_6$, $0 \leq h < 1$ and $b > 0$.

(C_1) and (C_2) follow as in Example 3.1.

3.3. Example. Let $F(t_1, t_2, t_3, t_4, t_5, t_6) = (1+pt_2)t_1 - pt_3t_4 - h \max\{t_2, t_3, t_4\} + b(t_5+t_6)$, $0 \leq h < 1$, $b > 0$ and $p \geq 0$.

(C_1) and (C_2) follow as in Example 3.1.

3.4. Example. Let $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - at_2^2 - b\dfrac{t_3^2+t_4^2}{t_5+t_6+1}$, $0 < a,b < 1$ and $a+2b < 1$.

(C_1). Let $u,v,w \geq 0$ and $F(u,v,v,u,w,0) = u^2 - av^2 - b\dfrac{(u^2+v^2)}{w+1} \leq 0$.

Then, $u^2 \leq \dfrac{a+b}{1-b}v^2$. Hence, $u \leq hv$, $h = \left(\dfrac{a+b}{1-b}\right)^{\frac{1}{2}} < 1$.

Similarly, if $F(u,v,u,v,0,w) \leq 0$ then $u \leq hv$.

(C_2). For all $u > 0$, $F(u,u,0,0,u,u) = (1-a)u^2 > 0$.

3.5. Example. Let $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - at_2^2 - b\dfrac{t_3^2+t_4^2}{t_5t_6+1}$, $0 < a,b < 1$, $a+2b < 1$.

(C_1) and (C_2) are established as in Example 3.4.

3.6. Example. Let $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^3 - a\dfrac{t_3^2t_4^2}{t_2+t_5+t_6+1}$, $0 \leq a < 1$.

(C_1). Let $u,v,w \geq 0$ and $F(u,v,v,u,w,0) = u^3 - a\dfrac{u^2v^2}{v+w+1} \leq 0$.

Then, $u \leq a\left(\dfrac{v^2}{v+w+1}\right) < av$.

Similarly, if $F(u,v,u,v,0,w) \leq 0$, then $u \leq hv$.

(C_2). $F(u,u,0,0,u,u) = u^3 > 0$ for all $u > 0$.

3.7. Example. Let $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^3 - a\dfrac{t_3^2t_4^2}{t_2+t_5t_6+1}$, $0 \leq a < 1$.

(C_1) and (C_2) follow as in Example 3.6.

3.8. Example. Let $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - bt_3 - c\dfrac{t_4t_5}{t_5+t_6+1}$, $0 < a,b,c < 1$ and $a+b+c < 1$.

(C_1). Let $u,v,w \geq 0$ and $F(u,v,v,u,w,0) = u - av - bv - c\dfrac{uw}{w+1} \leq 0$.

Then, $u \leq \dfrac{a+b}{1-c}v$, $h = \dfrac{a+b}{1-c} < 1$.

Similarly, if $F(u,v,u,v,0,w) \leq 0$, then $u \leq hv$.

(C_2). $F(u,u,0,0,u,u) = (1-a)u > 0$ for all $u > 0$.

3.9. Example. Let $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - b\dfrac{t_3t_6}{t_5+t_6+1} - ct_4$, $0 < a,b,c < 1$ and $a+b+c < 1$.

(C_1) and (C_2) follow as in Example 3.8.

4. Main Results

4.1. Theorem. *Let f, g, S and T be self-mappings of a metric space (X,d) satisfying the following conditions:*

(4.1) $S(X) \subset g(X)$ *and* $T(X) \subset f(X)$,

(4.2) $F(d(Sx,Ty),\ d(fx,gy),\ d(fx,Sx),\ d(gy,Ty),\ d(fx,Ty),\ d(Sx,gy)) \leq 0$ *for all* $x,y \in X$, *where* $F \in C_6$ *satisfies* (C_1) *and* (C_2).

Suppose that one of $S(X)$, $T(X)$, $f(X)$ and $g(X)$ is a complete subspace of X, and that the pairs (S,f) and (T,g) are weakly compatible. Then, f, g, S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. By (4.1), we can define inductively a sequence $\{y_n\}$ in X such that:

$$y_{2n} = Sx_{2n} = gx_{2n+1} \text{ and } y_{2n+1} = fx_{2n+2} = Tx_{2n+1} \tag{4.3}$$

for all $n = 0, 1, 2, \ldots$. Using (4.2) and (4.3) we have

$$\begin{aligned} &F\big(d(Sx_{2n},\ Tx_{2n+1}),\ d(fx_{2n},\ gx_{2n+1}),\ d(fx_{2n},\ Sx_{2n}),\\ &\qquad d(gx_{2n+1}, Tx_{2n+1}),\ d(fx_{2n},\ Tx_{2n+1}),\ d(Sx_{2n},\ gx_{2n+1})\big)\\ &= F\big(d(y_{2n},\ y_{2n+1}),\ d(y_{2n-1},\ y_{2n}),\ d(y_{2n-1},\ y_{2n}),\\ &\qquad d(y_{2n},\ y_{2n+1}),\ d(y_{2n-1},\ y_{2n+1}), 0\big)\\ &\leq 0. \end{aligned}$$

By (C_a) we get

$$d(y_{2n},\, y_{2n+1}) \leq hd(y_{2n-1},\, y_{2n}).$$

Similarly, we obtain

$$d(y_{2n+1},\, y_{2n+2}) \leq hd(y_{2n},\, y_{2n+1}).$$

Therefore,

$$d(y_n,\, y_{n+1}) \leq hd(y_{n-1},\, y_n).$$

Then, $\{y_n\}$ is a Cauchy sequence in X, hence the subsequence $\{y_{2n}\} = \{gx_{2n+1}\} \subset g(X)$ is a Cauchy sequence in $g(X)$. Assume that $g(X)$ is complete. Therefore, $\{y_n\}$ converges to a point $z = gv$ for some $v \in X$. Hence, the sequence $\{y_n\}$ converges also to z and the subsequences $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$, $\{fx_{2n+2}\}$ converge to z.

If $z \neq Tv$, using (4.2) we have,

$$F(d(Sx_{2n}, Tv),\ d(fx_{2n}, gv),\ d(fx_{2n}, Sx_{2n}),\ d(gv, Tv),\ d(fx_{2n}, Tv),\ d(Sx_{2n}, gv)) \leq 0.$$

Letting $n \to \infty$, and using the continuity of F, we obtain:

$$F(d(z, Tv), 0, 0, d(z, Tv), d(z, Tv), 0) \leq 0.$$

By (C_a), we get $z = Tv = gv$.

Since $T(X) \subset f(X)$, there exists $u \in X$ such that $z = fu = Tv$.

If $z \neq Su$, using (4.2) we have:

$$\begin{aligned} &F\big(d(Su, Tv),\ d(fu, gv),\ d(fu, Su),\ d(gv, Tv),\ d(fu, Tv),\ d(Su, gv))\\ &\quad = F(d(Su, z), 0, d(z, Su), 0, 0, d(Su, z))\\ &\quad \leq 0. \end{aligned}$$

By (C_b), we get $z = Su = fu$. Since the pairs (S, f) and (T, g) are weakly compatible, we get $fz = Sz$ and $gz = Tz$. If $z \neq Sz$, using (4.2) we have:

$$\begin{aligned} &F\big(d(Sz, Tv), d(fz, gv), d(fz, Sz), d(gv, Tv), d(fz, Tv), d(Sz, gv)\big)\\ &\quad = F(d(Sz, z), d(Sz, z), 0, 0, d(Sz, z), d(Sz, z))\\ &\quad \leq 0, \end{aligned}$$

which is a contradiction to (C_2). Therefore, $z = Sz = fz$.

Similarly, we can prove that $z = gz = Tz$. Hence, z is a common fixed point of f, g, S and T.

The proof is similar if we suppose that one of $S\,(X)$, $T\,(X)$ or $f(X)$ is complete instead of $g(X)$.

The uniqueness of z follows from (4.2) and (C_2). □

Theorem 4.1 generalizes Theorem 2.5 of [14], a Theorem of [15], Theorem 2.6 of [4] and a Theorem of [2].

Now consider the following conditions on $F \in C_6$:

(C_1') : there exists $0 \leq h < 1$ such that for all $u, v, w \geq 0$ with
(C_a') : $F(u, v, v, u, 0, w) \leq 0$, or
(C_b') : $F(u, v, u, v, w, 0) \leq 0$
we have $u \leq hv$.

(C_2) : $F(u, u, 0, 0, u, u) > 0$ for all $u > 0$.

It is easy to see that the functions F defined in Examples 3.1–3.9 satisfy (C_1').

4.2. Theorem. *Let $\{A_i\}$, $i = 1, 2, \ldots$, S and T be self-mappings of a metric space (X, d) satisfying* (2.3) *and* (2.4)*, and let $F \in C_6$ satisfy (C_1') and (C_2). Suppose that S is weakly compatible with A_1 and T is weakly compatible with A_k for some $k > 1$, and that one of $S(X)$ and $T(X)$ is a complete subspace of X. Then, $\{A_i\}$, S and T have a unique common fixed point in X.*

Proof. Let x_0 be an arbitrary point in X. Then by (2.3), we can define inductively a sequence $\{y_n\}$ in X such that

$$y_{2n} = A_1 x_{2n} = T x_{2n+1},\ y_{2n+1} = S x_{2n+2} = A_i x_{2n+1},\ i > 1,$$

for all $n = 0, 1, 2, \ldots$. As in the proof of Theorem 4.1, $\{y_n\}$ is a Cauchy sequence in X. Therefore, the sequence $\{y_{2n+1}\} = \{S x_{2n+2}\} \subset S(X)$ is a Cauchy sequence in $S(X)$. Assume that $S(X)$ is complete. Then, $\{y_n\}$ converges to a point $z = Su$ for some $u \in X$. Hence, the subsequences $\{A_1 x_{2n}\}, \{A_i x_{2n+1}\}, \{T x_{2n+1}\}$ converge also to z.

If $z \neq A_1 u$, then using (2.4) we get

$$F\big(d(A_1 u, A_k x_{2n+1}),\ d(Su, T x_{2n+1}),\ d(A_1 u, Su),\ d(A_k x_{2n+1} T x_{2n+1}),\ d(A_1 u, T x_{2n+1}),\ d(Su, A_k x_{2n+1})\big) \leq 0.$$

Letting $n \to \infty$, we obtain

$$F(d(A_1 u, z),\ 0,\ d(A_1 u, z),\ 0,\ d(A_1 u, z),\ 0) \leq 0.$$

By (C_b') we have $z = A_1 u = Su$.

Since $A_1(X) \subset T(X)$, there exists $v \in X$ such that $A_1 u = Tv = z$.

If $z \neq A_k v$, then using (2.4) we get

$$\begin{aligned} &F\big(d(A_1 u, A_k v),\ d(Su, Tv),\ d(A_1 u, Su),\\ &\qquad d(A_k v, Tv),\ d(A_1 u, Tv),\ d(Su, A_k v)\big)\\ &= F\big(d(z, A_k v),\ 0,\ 0,\ d(z, A_k v),\ 0,\ d(z, A_k v)\big)\\ &\leq 0. \end{aligned}$$

By (C_a') we have $z = A_k v = Tv$. Since the pair (A_1, S) is weakly compatible we have $A_1 z = Sz$.

If $A_1 z \neq z$, then using (2.4) we get

$$\begin{aligned} &F\big(d(A_1 z, A_k v),\ d(Sz, Tv),\ d(A_1 z, Sz),\ d(A_k v, Tv),\\ &\qquad d(A_1 z, Tv),\ d(Sz, A_k v)\big)\\ &= F\big(d(A_1 z, z),\ d(A_1 z, z),\ 0,\ 0,\ d(A_1 z, z),\ d(A_1 z, z)\big)\\ &\leq 0, \end{aligned}$$

which is a contradiction to (C_2). Then, $A_1 z = Sz = z$.

Since the pair (A_k, T) is weakly compatible we have $A_k z = Tz$.

If $A_k z \neq z$, then using (2.4) we get

$$\begin{aligned} &F\big(d(A_1z, A_kz),\ d(Sz, Tz),\ d(A_1z, Sz),\ d(A_kz, Tz),\ d(A_1z, Tz),\ d(Sz, A_kz)\big)\\ &\quad = F\big(d(z, A_kz), d(z, A_kz), 0, 0, d(z, A_kz), d(z, A_kz)\big)\\ &\quad \leq 0, \end{aligned}$$

which is a contradiction to (C_2). Then, $A_kz = Tz = z$.

Similarly, we can prove that $A_iz = z$ for all $i > 1$. Therefore, $A_1z = Sz = A_iz = Tz = z$, $i > 1$. Hence, $\{A_i\}$, S and T have a common fixed point z in X.

The proof is similar if we assume that $T(X)$ is complete instead of $S(X)$. The uniqueness of z follows from (2.4) and (C_2). □

Theorem 4.2 generalizes Theorem 2.7 of [3].

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