Hacettepe Journal of Mathematics and Statistics
Volume 36 (1) (2007), 49 – 51

ALGEBRAIC MODELS OF SMOOTH MANIFOLDS AND NON-ZERO HARMONICITY

Ali Öztürk*

Received 15 : 03 : 2007 : Accepted 01 : 05 : 2007

Abstract

In this note we give an obstruction in terms of $\operatorname{Im} H^k(X, \mathbb{Z})$ and the Euler characteristic $\chi(X)$, to the harmonicity of products of harmonic forms representing cohomology classes on $X_{\mathbb{C}}$, where X is a real algebraic variety.

Keywords: Real algebraic varieties, Algebraic homology, Harmonic forms.

2000 AMS Classification: Primary: 14 P 25, Secondary: 14 F 25.

1. Introduction

In this work, by a real algebraic variety we mean a complex algebraic variety X with an anti-holomorphic involution $\tau : X \to X$ such that $X^\tau = \{x \in X \mid \tau(x) = x\}$ is the set of real points of X. We will denote X^τ by $X(\mathbb{R})$ and the set of complex points by $X(\mathbb{C})$.

All real algebraic varieties under consideration in this note are nonsingular. It is well known that real projective varieties are affine ([1, Proposition 2.4.1] or [2, Theorem 3.4.4]). Moreover, compact affine real algebraic varieties are projective [1, Corollary 2.5.14] and, therefore, we will not distinguish between real compact affine varieties and real projective varieties.

For real algebraic varieties $X \subseteq \mathbb{R}^r$ and $Y \subseteq \mathbb{R}^s$, a map $F : X \to Y$ is said to be *entire rational* if there exist $f_i, g_i \in \mathbb{R}[x_1, \dots, x_r]$, $i = 1, \dots, s$, such that each g_i vanishes nowhere on X and $F = (f_1/g_1, \dots, f_s/g_s)$. We say X and Y are *isomorphic* if there are entire rational maps $F : X \to Y$ and $G : Y \to X$ such that $F \circ G = id_Y$ and $G \circ F = id_X$. Isomorphic algebraic varieties will be regarded as being the same.

An algebraic homology group $H_k^{alg}(X, R)$ ($R = \mathbb{Z}$ or $\mathbb{Z}_2$) is defined as the subgroup of $H_k(X, R)$ generated by the classes represented by real algebraic cycles. For a compact nonsingular real algebraic variety X of dimension n, let $H^k_{alg}(X, R)$ be the Poincaré dual of the group $H^{alg}_{n-k}(X, R)$.

*Department of Mathematics, Abant İzzet Baysal University, Gölköy Kampüsü, Bolu, Turkey. E-mail: ozturkali@ibu.edu.tr

Let R be a commutative ring with unity. For an R-orientable nonsingular compact real algebraic variety X, we define $KH_*(X,R)$ to be the kernel of the induced map $i_* : H_*(X,R) \to H_*(X_{\mathbb{C}},R)$ on homology where $i : X \to X_{\mathbb{C}}$ is a projective non-singular complexification map.

In [4], Y. Ozan has proved that $KH_*(X,R)$ is independent of the choice of the projective complexification $i : X \to X_{\mathbb{C}}$ of X and thus it is an isomorphism invariant of X. Dually, if we denote the image of the homomorphism

$$i^* : H^*(X_{\mathbb{C}},R) \to H^*(X,R)$$

by $\operatorname{Im} H^*(X,R)$ then this is also an isomorphism invariant. We refer the reader to [1, 2] for the basic definitions and facts about real algebraic geometry.

2. Results

Let X be a n-dimensional compact oriented non-singular real algebraic variety and $X_{\mathbb{C}}$ a non-singular projective complexification of X. Theorem 2.6 describes an obstruction in terms of $\operatorname{Im} H^k(X,\mathbb{Z})$ and the Euler characteristic $\chi(X)$, to the harmonicity of products of harmonic forms representing cohomology classes on $X_{\mathbb{C}}$.

2.1. Definition. Let M be an n-dimensional closed, oriented, Riemannian manifold. Define $H^p := \{w \in E^p(M) \mid \triangle w = 0\}$, where $E^p(M)$ is the vector space of all smooth p-forms on M and $\triangle$ on $E^p(M)$ is the Laplace-Beltrami operator. The elements of H^p are called *harmonic p-forms.*

We know that there is a relation between harmonic forms and de Rham cohomology classes. For this, we give the following classical theorem from [5].

2.2. Theorem. *A de Rham cohomolgy class on a compact oriented Riemannian manifold M contains a unique harmonic representative.*

2.3. Remark. If $\mu \neq 0$ is a harmonic form then $[\mu] \neq 0$.

2.4. Definition. A Riemannian manifold is called (*metrically*) *formal* if all wedge products of harmonic forms are also harmonic on the manifold. A closed manifold is called *geometrically formal* if it admits a formal Riemannian metric.

Compact globally symmetric spaces are metrically formal, as are Riemannian metrics on a rational homology sphere. In [3], D. Kotschick proved the following theorem which gives us the existence of a non-formal Riemannian metric globally.

2.5. Theorem. *A closed oriented manifold admits a non-formal Riemannian metric if and only if it is not a rational homology sphere.*

We now state our result which is about harmonicity of real algebraic varieties.

2.6. Theorem. *Let X^n be a non-singular compact oriented real algebraic variety with $\chi(X) \neq 0$. Let $i : X \to X_{\mathbb{C}}$ be a nonsingular projective complexification. Let $i^*(a) \neq 0 \in \operatorname{Im} H^k(X,\mathbb{Z})$, $0 \leq k \leq 2n$, for some $a \in H^k(X_{\mathbb{C}},\mathbb{Z})$ and $x = D([X]) \in H^n(X_{\mathbb{C}},\mathbb{Z})$ be the Poincaré dual of $[X]$. If the product $u \wedge v$ is a non-zero harmonic form, where u and v are harmonic representatives for $x \in H^n(X_{\mathbb{C}},\mathbb{Z})$ and $a \in H^k(X_{\mathbb{C}},\mathbb{Z})$ respectively, then $i^*([\mu]) \neq 0$ in $H^{n-k}(X,\mathbb{Z})$, where $\mu = *(u \wedge v)$ and $*$ is the Hodge star operator.*

Proof. The Poincaré dual of $i^*(a)$ is represented by $X^n \pitchfork L^{2n-k}$ where $L = D(a) \in H_{2n-k}(X_{\mathbb{C}},\mathbb{Z})$. But $X^n \pitchfork L^{2n-k}$ is also $D(x \cup a)$. By assumption, $u \wedge v$ is non-zero harmonic and therefore so is $\mu = *(u \wedge v)$, and

$$\int_{X_{\mathbb{C}}} \mu \wedge u \wedge v = \int_{X_{\mathbb{C}}} *(u \wedge v) \wedge (u \wedge v) = \|u \wedge v\| \neq 0.$$

So, $[\mu \wedge u \wedge v]([X_{\mathbb{C}}]) = \mu(D([u \wedge v]))$ is not zero. Finally, since $D([u \wedge v])$ is represented by $X^n \pitchfork L^{2n-k}$, a class in $H_{n-k}(X, \mathbb{Z})$, we get $i^*([\mu]) \neq 0$. □

2.7. Example. Let X be an algebraic model for $\mathbb{CP}^3$ with $\operatorname{Im} H^2(X, \mathbb{Z}) = 0$. We can construct such an algebraic model in the following way: Let $T^2 \subset \mathbb{CP}^3$ be a smoothly embedded submanifold realizing the homology class of $[\mathbb{CP}^1] \in H_2(\mathbb{CP}^3, \mathbb{Z})$. Such a T^2 can be obtained by attaching a one-handle to $\mathbb{CP}^1$ in a disc neighbourhood of a point $p \in \mathbb{CP}^1$ in $\mathbb{CP}^3$.

Embed $\mathbb{CP}^3$ smoothly into some Euclidian space $\mathbb{R}^n$, so that the submanifold T^2 maps diffeomorphically onto $S^1 \times S^1 \subseteq \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^{n-4}$, where S^1 is the standard unit circle. Now recall the following theorem:

> Let $L \subseteq M \subseteq \mathbb{R}^k$, where L is a nonsingular real algebraic variety and M an embedded closed smooth manifold. Then there is a smooth embedding $g : M \to \mathbb{R}^k \times \mathbb{R}^l$ such that $X = g(M)$ is a nonsingular real algebraic variety with $g(x) = x$ for all $x \in L$ if and only if the normal bundle $N_M(L)$ of L in M has a strongly algebraic structure.

For the proof of this fact we refer the reader to [1, Theorem 2.8.4]. By [2, Corollary 12.5.4 and Remark 12.6.8], if L is a nonsingular real algebraic variety of dimension less than or equal to 3 such that L has totally algebraic homology then any smooth vector bundle over L is strongly algebraic. In our case $L = T^2$ has totally algebraic homology. Hence its normal bundle in $\mathbb{CP}^3$ has a strongly algebraic structure. Moreover, $ImH^2(X, \mathbb{Z}) = 0$ because S^1 bounds in its complexification $S^1_{\mathbb{C}} = \mathbb{CP}^1 = S^2$ and we have the following commutative diagram:

$$\begin{array}{ccc} S^1 \times S^1 & \xrightarrow{\;e\;} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ S^1_{\mathbb{C}} \times S^1_{\mathbb{C}} & \xrightarrow{\;e_{\mathbb{C}}\;} & X_{\mathbb{C}} \end{array}$$

It then follows that $i^* : H^2(X_{\mathbb{C}}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is the zero homomorphism. Thus we have found an algebraic model X of $\mathbb{CP}^3$ such that $\operatorname{Im} H^2(X, \mathbb{Z}) = 0$. Since $\chi(X) = 4 \neq 0$ and $p_1(X) = 2c_2 - c_1^2 = -4 \neq 0$, we can find non-zero harmonic forms representing $x = D([X])$ such that $a = p_1(X) \in \operatorname{Im} H^4(X, \mathbb{Z})$. Let u and v be such representatives, respectively. If the product of these harmonic forms were harmonic then $\mu = *(u \wedge v) \in H^2(X_{\mathbb{C}}, \mathbb{Z})$ would satisfy $i^*([\mu]) \neq 0$ in $H^2(X, \mathbb{Z})$, which is a contradiction.

References

[1] Akbulut, S and King, H. *Topology of real algebraic sets*, M. S. R. I. Book Series (Springer, New York, 1992).

[2] Bochnak, J., Coste, M. and Roy, M.-F. *Real Algebraic Geometry*, (Springer, Berlin and New York, 1998).

[3] Kotschick, D. *On products of harmonic forms*, Duke Math. Journal, **107**, 521–531, 2001.

[4] Ozan, Y. *On homology of real algebraic varieties*, Proc. Amer. Math. Soc., **129**, 3167–3175, 2000.

[5] Warner, F. W. *Foundations of differentiable manifolds and Lie groups*, (Scott, Foresman, Glenview, Ill., 1971).