Hacettepe Journal of Mathematics and Statistics
Volume 36 (1) (2007), 75 – 78

THE GEOMETRICAL STRUCTURE OF A COMPLEXIFIED COMPLETE SET OF PAIRWISE ORTHOGONAL LATIN SQUARES

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Received 27 : 01 : 2006 : Accepted 21 : 05 : 2007

Abstract

In this paper, the complexification of a Latin square, a complexified set of pairwise orthogonal Latin squares of complex order n, and the complexified complete set are defined. In addition, the complexified projective plane corresponding to the complete set of pairwise orthogonal Latin squares of complex order 2 is obtained.

Keywords: Latin square, Projective plane, Complexification.

2000 AMS Classification: Primary: 05 B 15. Secondary: 05 B 25.

1. Introduction

A Latin square of side n is an arrangement of n symbols into n^2 sub-squares of a square in such a way that every row and every column contains each symbol exactly once.

Two Latin squares are orthogonal if when one is superimposed upon the other, every ordered pair of symbols occurs exactly once in the resulting square. The number n is known as the *order* of the Latin square.

A Latin square is said to be in *standard form* if the symbols in the first row are in natural order. Orthogonal Latin squares are assumed to have this property (Raghavarao, [4]).

If in a set of Latin squares every pair is orthogonal, the set is called a set of pairwise orthogonal Latin squares (POLS).

A set of pairwise orthogonal Latin squares (POLS) of order n has at most $n - 1$ members. It is called a complete set if it has exactly $n - 1$ members.

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In a projective plane of order n, $PG(2,n)$, there are n^2+n+1 points and the same number of lines. Furthermore, on each line, there are $n+1$ points and through each point there pass $n+1$ lines. A single line passes through every pair of points and every pair of lines intersects at a single point.

There is a $1-1$ correspondence between a complete set of (POLS) and the finite projective plane $PG(2,n)$ of order n (Stevensen, [3]).

The process of complexifying a real object is well known in mathematics; as Hadamard said, the shortest path between two real results often goes through complex terrain (Berger, [2]).

Complexifying a real vector space consists of embedding it in the smallest possible complex vector space, as one does in passing from $\mathbb{R}$ to $\mathbb{C}$. This is accomplished as follows:

1.1. Definition. Let E be a real vector space. The complexification of E, denoted by E^c, is the product $E \times E$ endowed with the complex vector structure defined by

$$(x,y)+i(x^{'},y')=(x+x',y+y')$$
$$(\lambda+i\mu)(x,y)=(\lambda x-\mu y,\lambda y+\mu x).$$

The embedding $E \to E^c$ is given by $x \to (x,0)$. The involution

$$\sigma:(x,y)\to(x,-y)$$

from E^c into itself is called the *conjugation* in E^c.

One can observe that $i(y,0)=(0,y)\cong iy$, so it can be written as $(x,y)=x+iy$ and $E^c=E\oplus iE$ (the real direct sum). Then $\sigma(x+iy)=x-iy$, which justifies the word conjugation (Berger, [2]).

In [5] the authors studied real projective planes and investigated the corresponding real complete set of pairwise orthogonal Latin squares (POLS) of 2^{nd}, 3^{rd} and 4^{th} real order. In [5] they showed that the t-structures are the geometrical structures of the real complete sets of pairwise orthogonal Youden squares (POYS) of real order $(3,2)$ and $(4,3)$.

A complete set of (POLS) is a well known concept in the real case. In the present paper, this concept is generalized to the complex case by defining the complexification of a Latin square, the orthogonality of two complexified Latin squares, a complexified set of pairwise orthogonal Latin squares of complex order n, and the complexified complete set of such Latin squares.

We also study complexified projective planes and investigate the corresponding complete set of pairwise orthogonal Latin squares $(\text{POLS})^C$ of complex order 2.

2. Complexification of a Projective Space.

2.1. Definition. Let $\wp=P(E)$ be a real projective space. The *complexification* of $\wp$, denoted by $\wp^c$, is the complex projective space $\wp^c=P(E^c)$. We identify $\wp$ with a subset of $\wp^c$ by taking the quotient of the inclusion $E\setminus 0\to E^c\setminus 0$. The map $\wp^c\to\wp^c$ is obtained from $\sigma:E^c\setminus 0\to E^c\setminus 0$ by passing to the quotient in $\wp^c$.

Generally, $\wp$ is not a projective subspace of $\wp^c$ any more then E is a complex vector subspace of E^c. Clearly, $\sigma:\wp^c\to\wp^c$ takes collinear points into collinear points. Also, $\wp$ has the same real dimension as the complex dimension of $\wp^c$ (Berger, [2]).

3. The Complexified Set of Pairwise Orthogonal Latin Squares $(\mathbf{POLS})^{\mathbf{c}}$

Latin square designs are normally used in experiments to remove the heterogeneity of experimental material. Any real Latin square obtained from an experiment results is a real matrix. But the results of an experiment are not always real numbers. So, the concept of complexified Latin square will be very useful in combinatorial problems of the design of experiments.

3.1. Definition. The complexification of a Latin square is the complexified matrix corresponding to it.

3.2. Definition. Two complexified Latin squares are called *orthogonal* if when one is superimposed upon the other, every ordered pair of symbols occurs exactly once in the resulting square.

The complex order of the complexified Latin square is n. In other words, $2n$ is real order of the complexified Latin square.

3.3. Definition. A set of Latin squares is called a *complexified set of pairwise orthogonal Latin squares*, $(\mathrm{POLS})^C$, if in a set of pairwise complexified Latin squares, every pair of which is orthogonal.

3.4. Definition. A complexified set of pairwise orthogonal Latin squares $(POLS)^C$ of complex order n is called a *complexified complete set* if it contains exactly $n-1$ complexified Latin squares.

4. The geometrical structure corresponding to the complete set of pairwise orthogonal Latin squares of complex order 2

The complete set of (POLS) of real order 2 is

$$\pounds = \left\{ L = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \right\}$$

According to Definition 3.4, the complexified complete set of $(\mathrm{POLS})^C$ is

$$\pounds^C = \left\{ L^C = \begin{bmatrix} 1+i & 2+2i \\ 2+2i & 1+i \end{bmatrix} \right\} \tag{1}$$

The first and the second treatments in the Latin square L occur in the second row of L. But different conditions may be tied to the same treatments, and then the results obtained from these experiments will be different. Hence, the Latin square L is expressed as follows:

$$L = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}.$$

Hence, the complexified Latin square of the Latin square L is

$$L^c = \begin{bmatrix} N_1 + iN_1 & N_2 + iN_2 \\ N_3 + iN_3 & N_4 + iN_4 \end{bmatrix}. \tag{2}$$

In this paper, we will adopt the convention of (Bayrak and Turgut Vanlı [1, 5]). Thus, each of the rows and each of columns in the complexified Latin square L^c is considered as a line. The following lines are obtained.

$$l_1^c = \{N_1 + iN_1, N_2 + iN_2\}$$
$$l_2^c = \{N_3 + iN_3, N_4 + iN_4\}$$
$$l_3^c = \{N_1 + iN_1, N_3 + iN_3\}$$
$$l_4^c = \{N_2 + iN_2, N_4 + iN_4\}\,.$$

The element $N_1 + iN_1$ in (2) corresponds to the symbol $(1 + i)$, which is in the first row and first column of L^c. The element $N_4 + iN_4$ in (2) corresponds to the symbol $(1 + i)$, which is in the second row and second column in (1). Then,

$$l_5^c = \{N_1 + iN_1, N_4 + iN_4\}\,.$$

Similarly, the element $N_1 + iN_1$ in (2) corresponds to the symbol $(2 + 2i)$ in the first row and second column of L^c. The element $N_3 + iN_3$ of L^c corresponds to the symbol $2 + 2i$ in the second row and the first row in (1). Hence,

$$l_6^c = \{N_2 + iN_2, N_3 + iN_3\}\,.$$

So, l_1^c and l_2^c are parallel lines. The point of intersection of l_1^c and l_2^c will be denoted by $N_5 + iN_5$. Similarly, the remaining intersection points are denoted by $N_6 + iN_6$ and $N_7 + iN_7$. Then the ideal line is,

$$l_7^c = \{N_6 + iN_6, N_7 + iN_7\}\,.$$

Consequently, the system obtained has 7 point and 7 lines. It is the complex projective plane of complex order 2.

The geometrical structure obtained is the complexified projective plane of the real projective plane corresponding to the complete set of pairwise orthogonal Latin square of real order two.

5. Conclusion

The concept of complexified Latin square will be very useful in the combinatorial problem of the design of experiments. We have obtained the complexified projective plane of the real projective plane corresponding the complete set of pairwise orthogonal Latin square of real order two.

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