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p-CLOSED TOPOLOGICAL SPACES IN TERMS OF GRILLS

M. N. Mukherjee* and B. Roy*†

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Abstract

The concept of p-closedness, a kind of covering property for topological spaces, has already been studied with meticulous care from different angles and via different approaches. In this paper, we continue the said investigation in terms of a different concept viz. grills. The deliberations in the article include certain characterizations and a few necessary conditions for the p-closedness of a space, the latter conditions are also shown to be equivalent to p-closedness in a pre-almost regular space. All these and the associated discussions and results are done with grills as the prime supporting tool.

Keywords: Preopen set, Preclosure, Preconvergence, Grill, $p(\theta)$-convergence and $p(\theta)$-adherence of a grill, p-closed space.

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1. Introduction

It is seen from the literature that a wide variety of topological properties neighbouring on compactness have been introduced and studied in detail by many researchers. For example, mention may be of H-closedness, near compactness, S-closedness, different variant forms of paracompactness etc. The notion of p-closedness, a kind of covering property akin to quasi H-closedness, was first introduced by Abo-Khadra [1], by using preopen sets and preclosure operator. Detailed study in this regard by many investigators during the last fifteen years or so has enriched this particular field to a considerable extent (see [3] for some details). Our intention in the present article to make a further investigation of the notion of p-closedness via a new approach with the notion of grill as the chief appliance.

The idea of grill was initiated by G. Choquet [2] in 1947, and since then it has been observed in connection with many mathematical investigations such as the theories of

*Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata–700019, India. E-mail: (M. N. Mukherjee) mukherjeemn@yahoo.co.in

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proximity spaces, compactifications etc, that grills as a tool (like filters) are extremely useful and convenient for many situations. A grill $\mathcal{G}$ on a topological space X is defined to be a collection of nonempty subsets of X such that (i)$A \in \mathcal{G}$ and $A \subseteq B \subseteq X \Rightarrow B \in \mathcal{G}$ and (ii) $A, B \subseteq X$ and $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

In section 2, we shall define the $p(\theta)$-adherence and $p(\theta)$-convergence of a grill, and develop the concept to some extent so that the results we derive here may support our subsequent deliberations.

Section 3 is meant for the discussion of the actual intended part of the paper, where first we characterize p-closedness in terms of grills. Certain neighbouring forms of p-closedness are then considered and that too via grills, and it is shown that in the presence of a suitable separation axiom (viz. pre-almost regularity), p-closedness is equivalent to such forms.

In what follows by a space X we shall mean a topological space X. A subset A of a space X is called preopen (preclosed) [1] if $A \subseteq$int cl$A$ (resp. cl int$A \subseteq A$), where int$A$ and cl$A$ respectively denote the interior and closure of A in the space X. The set of all preopen sets of X will be denoted by $PO(X)$, and the set of all those members of $PO(X)$, which contain a given point x of X will be designated by $PO(x)$. It is known [4] that a subset A in X is preopen iff $X \setminus A$ is preclosed. The union (intersection) of all preopen (resp. preclosed) sets in X, which are contained in (resp. contain) a given set $A(\subseteq X)$ is called the preinterior (resp. preclosure)of A, to be denoted by pint$A$ (resp. pcl$A$). It is known that for $x \in X$ and $A \subseteq X$, $x \in$pcl$A$ iff $U \cap A \neq \varnothing$, for all $U \in PO(x)$. Again, for any set A in X, $p(\theta)$-closure of A, denoted by $p(\theta)$-cl$A$, is defined as $p(\theta)\text{-cl}A = \{x \in X : \text{pcl}U \cap A \neq \varnothing, \text{ for all } U \in PO(x)\}$.

2. Grills: $p(\theta)$-convergence and $p(\theta)$-adherence

2.1. Definition. A grill $\mathcal{G}$ on a topological space X is said to:

(a) *$p(\theta)$-adhere at* $x \in X$ if for each $U \in PO(x)$ and each $G \in \mathcal{G}$, $\text{pcl}\, U \cap G \neq \varnothing$,

(b) *$p(\theta)$-converge to a point* $x \in X$ if for each $U \in PO(x)$, there is some $G \in \mathcal{G}$ such that $G \subseteq \text{pcl}\, U$ (in this case we shall also say that $\mathcal{G}$ is *$p(\theta)$-convergent to* x).

2.2. Remark. It at once follows that a grill $\mathcal{G}$ is $p(\theta)$-convergent to a point $x \in X$ iff $\mathcal{G}$ contains the collection $\{\text{pcl}\, U : U \in PO(x)\}$.

The concepts of $p(\theta)$-adherence and $p(\theta)$-convergence of filters were defined in [5] in the same way as above,

2.3. Definition. A filter $\mathcal{F}$ on a topological space X is said to *$p(\theta)$-adhere at* $x \in X$ (*$p(\theta)$-converge to* $x \in X$) if for each $F \in \mathcal{F}$ and each $U \in PO(x)$, $F \cap \text{pcl}\, U \neq \varnothing$ (resp. to each $U \in PO(x)$, there corresponds $F \in \mathcal{F}$ such that $F \subseteq \text{pcl}\, U$).

We recall the following notation from [8].

2.4. Definition. If $\mathcal{G}$ is a grill (or a filter) on a space X, then the *section of* $\mathcal{G}$, denoted by sec$\mathcal{G}$, is given by

$$\text{sec}\mathcal{G} = \{A \subseteq X : A \cap G \neq \varnothing, \text{ for all } G \in \mathcal{G}\}.$$

The following results will be used in the sequel.

2.5. Theorem. [8] (*a*) *For any grill (filter)* $\mathcal{G}$ *on a space* X*,* sec$\mathcal{G}$ *is a filter (resp. grill) on* X*.*

(*b*) *If* $\mathcal{F}$ *and* $\mathcal{G}$ *are respectively a filter and a grill on a space* X *with* $\mathcal{F} \subseteq \mathcal{G}$*, then there is an ultrafilter* $\mathcal{U}$ *on* X *such that* $\mathcal{F} \subseteq \mathcal{U} \subseteq \mathcal{G}$*.*

We note at this stage that unlike the case of filters, the notion of $p(\theta)$-adherence of a grill is strictly stronger than that of $p(\theta)$-convergence. In fact, we have

2.6. Theorem. *If a grill $\mathcal{G}$ on a topological space X, $p(\theta)$-adheres at some point $x \in X$, then $\mathcal{G}$ is $p(\theta)$-convergent to x.*

Proof. Let a grill $\mathcal{G}$ on X, $p(\theta)$-adhere at $x \in X$. Then for each $U \in PO(x)$ and each $G \in \mathcal{G}$, $\operatorname{pcl} U \cap G \neq \varnothing$ so that $\operatorname{pcl} U \in \sec\mathcal{G}$, for each $U \in PO(x)$, and hence $X \setminus \operatorname{pcl} U \notin \mathcal{G}$. Then $\operatorname{pcl} U \in \mathcal{G}$ (as $\mathcal{G}$ is a grill and $X \in \mathcal{G}$), for each $U \in PO(x)$. Hence $\mathcal{G}$ must $p(\theta)$-converge to x (see Remark 2.2). □

The following example shows that a $p(\theta)$-convergent grill need not $p(\theta)$-adhere at any point of the space even if the space is finite.

2.7. Example. Let $X = \{a, b, c\}$, $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. It is easy to verify that (X, τ) is a topological space such that $PO(X) = \tau$. Let

$$\mathcal{G} = \{\{a\}, \{b\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}.$$

Then $\mathcal{G}$ is a grill on X. Since for any open set $A \subseteq X$, $\operatorname{pcl} A = \operatorname{cl} A$ [3], we have $\operatorname{pcl}\{a\} = \{a, c\}$, $\operatorname{pcl}\{b\} = \{b\}$, $\operatorname{pcl}\{a, c\} = \{a, c\}$, and $\operatorname{pcl}\{a, b\} = \operatorname{pcl} X = X$. It can then be verified that the grill $\mathcal{G}$ is $p(\theta)$-convergent to each $x \in X$, but does not $p(\theta)$-adhere at any $x \in X$.

2.8. Notation. Let X be a topological space . Then for any $x \in X$, we adopt the following notation:

$$\mathcal{G}(p(\theta), x) = \{A \subseteq X : x \in p(\theta)\text{-}\operatorname{cl} A\},$$
$$\sec\mathcal{G}(p(\theta), x) = \{A \subseteq X : A \cap G \neq \varnothing, \text{ for all } G \in \mathcal{G}(p(\theta), x)\}.$$

In the next two theorems, we characterize the $p(\theta)$-adherence and $p(\theta)$-convergence of grills in terms of the above notations.

2.9. Theorem. *A grill $\mathcal{G}$ on a space X, $p(\theta)$-adheres to a point $x \in X$ iff $\mathcal{G} \subseteq \mathcal{G}(p(\theta), x)$.*

Proof. A grill $\mathcal{G}$ on a space X $p(\theta)$-adheres at $x \in X$

$$\begin{aligned} &\Longrightarrow \operatorname{pcl} U \cap G \neq \varnothing, \text{ for all } U \in PO(x) \text{ and all } G \in \mathcal{G} \\ &\Longrightarrow x \in p(\theta)\text{-}\operatorname{cl} G, \text{ for all } G \in \mathcal{G} \\ &\Longrightarrow G \in \mathcal{G}(p(\theta), x), \text{ for all } G \in \mathcal{G} \\ &\Longrightarrow \mathcal{G} \subseteq \mathcal{G}(p(\theta), x). \end{aligned}$$

Conversely, let $\mathcal{G} \subseteq \mathcal{G}(p(\theta), x)$. Then for all $G \in \mathcal{G}$, $x \in p(\theta)\text{-}\operatorname{cl} G$, so that for all $U \in PO(x)$ and for all $G \in \mathcal{G}$, $\operatorname{pcl} U \cap G \neq \varnothing$. Hence $\mathcal{G}$ $p(\theta)$-adheres at x. □

2.10. Theorem. *A grill $\mathcal{G}$ on a topological space X is $p(\theta)$-convergent to a point x of X iff $\sec\mathcal{G}(p(\theta), x) \subseteq \mathcal{G}$.*

Proof. Let $\mathcal{G}$ be a grill on X, $p(\theta)$-converging to $x \in X$. Then for each $U \in PO(x)$ there exists $G \in \mathcal{G}$ such that $G \subseteq \operatorname{pcl} U$, and hence

(1) $\operatorname{pcl} U \in \mathcal{G}$ for each $U \in PO(x)$.

Now, $B \in \sec\mathcal{G}(p(\theta), x) \Longrightarrow X \setminus B \notin \mathcal{G}(p(\theta), x) \Longrightarrow x \notin p(\theta)\text{-}\operatorname{cl}(X \setminus B) \Longrightarrow$ there exists $U \in PO(x)$ such that $\operatorname{pcl} U \cap (X \setminus B) = \varnothing \Longrightarrow \operatorname{pcl} U \subseteq B$, where $U \in PO(x) \Longrightarrow B \in \mathcal{G}$ (by (1)).

Conversely, let if possible, $\mathcal{G}$ not to $p(\theta)$-converge to x. Then for some $U \in PO(x)$, $\operatorname{pcl} U \notin \mathcal{G}$ and hence $\operatorname{pcl} U \notin \sec\mathcal{G}(p(\theta), x)$. Thus for some $A \in \mathcal{G}(p(\theta), x)$,

(2) $A \cap \operatorname{pcl} U = \varnothing$.

But $A \in \mathcal{G}(p(\theta), x) \implies x \in p(\theta)\text{-cl}\, A \implies \text{pcl}\, U \cap A \neq \varnothing$, contradicting (2). □

We now introduce two types of grill, which will facilitate the development in the next section.

2.11. Definition. A grill $\mathcal{G}$ on a space X is said to be:

(a) $p(\theta)$-*linked* if for any two members $A, B \in \mathcal{G}$, $p(\theta)\text{-cl}\, A \cap p(\theta)\text{-cl}\, B \neq \varnothing$,

(b) $p(\theta)$-*conjoint* if for every finite subfamily $A_1, A_2, \ldots, A_n$ of $\mathcal{G}$,

$$\text{pint}\, [\bigcap_{i=1}^{n} p(\theta)\text{-cl}\, A_i] \neq \varnothing.$$

It follows from the definitions that every $p(\theta)$-conjoint grill is $p(\theta)$-linked. That the converse is false is exhibited by the following example.

2.12. Example. Let $X = \{a, b, c\}$ and $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}\}$. Then (X, τ) is a topological space such that $PO(X) = \tau$. Let

$$\mathcal{G} = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}.$$

Then $\mathcal{G}$ is a grill on X. It is easy to show that $p(\theta)\text{-cl}\, \{a\} = \{a, c\}$, $p(\theta)\text{-cl}\, \{b\} = \{b, c\}$, $p(\theta)\text{-cl}\, \{a, b\} = p(\theta)\text{-cl}\, \{b, c\} = p(\theta)\text{-cl}\, \{a, c\} = X$. It is obvious that $\mathcal{G}$ is a $p(\theta)$-linked grill on X but it is not $p(\theta)$-conjoint as $\text{pint}\, \left[\bigcap_{A \in \mathcal{G}} p(\theta)\text{-cl}\, A\right] = \text{pint}\, \{c\} = \varnothing$.

3. p-closedness and grills

As proposed earlier, in this section we investigate p-closedness of a topological space in terms of grills. We begin by recalling the definition of p-closedness from [3].

3.1. Definition. A non-empty subset A of a topological space X is called *p-closed relative to X* if for every cover $\mathcal{U}$ of A by preopen sets of X, there exists a finite subset $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subseteq \cup\{\text{pcl}\, U : U \in \mathcal{U}_0\}$. If, in addition, $A = X$, then X is called a *p-closed space.*

Some of the known characterizations of a p-closed space, which we shall need here, are given below:

3.2. Theorem. [3] *For any topological space X, the following are equivalent:*

(a) *X is p-closed.*

(b) *For every family $\{U_\alpha : \alpha \in \Lambda\}$ of non-void preclosed sets in X with the property $\bigcap\{U_\alpha : \alpha \in \Lambda\} = \varnothing$, there is a finite subset Λ_0 of Λ such that $\bigcap\{\text{pint}\, U_\alpha : \alpha \in \Lambda_0\} = \varnothing$.*

(c) *Every filterbase on X, $p(\theta)$-adheres at some $x \in X$.*

(d) *Every maximal filterbase on X, $p(\theta)$-converges to some $x \in X$.*

We shall now derive new characterizations of p-closedness in terms of grills and the associated concepts developed so far.

3.3. Theorem. *A topological space X is p-closed iff every grill on X is $p(\theta)$-convergent in X.*

Proof. Let $\mathcal{G}$ be any grill on a p-closed space X. Then by Theorem 2.5, sec$\mathcal{G}$ is a filter on X. Let $B \in \text{sec}\mathcal{G}$, then $X \setminus B \notin \mathcal{G}$ and hence $B \in \mathcal{G}$ (as $\mathcal{G}$ is a grill). Thus sec$\mathcal{G} \subseteq \mathcal{G}$. Then by Theorem 2.5(b), there exists an ultrafilter $\mathcal{U}$ on X such that sec$\mathcal{G} \subseteq \mathcal{U} \subseteq \mathcal{G}$. Now as X is p-closed, in view of Theorem 3.2 the ultrafilter $\mathcal{U}$ is $p(\theta)$-convergent to some point $x \in X$. Then for each $U \in PO(x)$, there exists $F \in \mathcal{U}$ such that $F \subseteq \text{pcl}\, U$. Consequently, $\text{pcl}\, U \in \mathcal{U} \subseteq \mathcal{G}$, i.e., $\text{pcl}\, U \in \mathcal{G}$, for each $U \in PO(x)$. Hence $\mathcal{G}$ is $p(\theta)$-convergent to x.

Conversely, let every grill on X be $p(\theta)$-convergent to some point of X. By virtue of Theorem 3.2 it is enough to show that every ultrafilter on X is $p(\theta)$-converges in X, which is immediate from the fact that an ultrafilter on X is also a grill on X. □

3.4. Theorem. *A subset A of a topological space X is p-closed relative to X iff every grill $\mathcal{G}$ on X with $A \in \mathcal{G}$, $p(\theta)$-converges to a point in A.*

Proof. Let A be p-closed relative to X and $\mathcal{G}$ a grill on X satisfying $A \in \mathcal{G}$ such that $\mathcal{G}$ does not $p(\theta)$-converge to any $a \in A$. Then to each $a \in A$, there corresponds some $U_a \in PO(a)$ such that $\operatorname{pcl} U_a \notin \mathcal{G}$. Now $\{U_a : a \in A\}$ is a cover of A by preopen sets of X. Then $A \subseteq \bigcup_{i=1}^{n} \operatorname{pcl} U_{a_i} = U$ (say) for some positive integer n. Since $\mathcal{G}$ is a grill, $U \notin \mathcal{G}$ and hence $A \notin \mathcal{G}$, which is a contradiction.

Conversely, let A be not p-closed relative to X. Then for some cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of A by preopen sets of X, $\mathcal{F} = \{A \setminus \bigcup_{\alpha \in \Lambda_0} \operatorname{pcl} U_\alpha : \Lambda_0 \text{ is a finite subset of } \Lambda\}$ is a filterbase on X. Then the family $\mathcal{F}$ can be extended to an ultrafilter $\mathcal{F}^*$ on X. Then $\mathcal{F}^*$ is a grill on X with $A \in \mathcal{F}^*$ (as each F of $\mathcal{F}$ is a subset of A). Now for each $x \in A$, there must exist $\beta \in \Lambda$ such that $x \in U_\beta$, as $\mathcal{U}$ is a cover of A. Then for any $G \in \mathcal{F}^*$, $G \cap (A \setminus \operatorname{pcl} U_\beta) \neq \varnothing$, so that $G \nsubseteq \operatorname{pcl} U_\beta$, for all $G \in \mathcal{G}$. Hence $\mathcal{F}^*$ cannot $p(\theta)$-converge to any point of A. The contradiction proves the desired result. □

3.5. Theorem. *Let X be any topological space such that every grill $\mathcal{G}$ on X, with the property that $\bigcap_{i=1}^{n} p(\theta)\text{-cl}\, G_i \neq \varnothing$ for every finite subfamily $\{G_1, G_2, \ldots, G_n\}$ of $\mathcal{G}$, $p(\theta)$-adheres in X, then X is a p-closed space.*

Proof. Let $\mathcal{U}$ be any ultrafilter on X. Then $\mathcal{U}$ is a grill on X and also for each finite subcollection $\{U_1, U_2, \ldots, U_n\}$ of $\mathcal{U}$, $\bigcap_{i=1}^{n} p(\theta)\text{-cl}\, U_i \supseteq \bigcap_{i=1}^{n} U_i \neq \varnothing$, so that $\mathcal{U}$ is a grill on X with the given condition. Hence by hypothesis, $\mathcal{U}$, $p(\theta)$-adheres. Consequently, by Theorem 3.2, the space X is p-closed. □

3.6. Theorem. [6] *For any $A \subseteq X$, $p(\theta)\text{-cl}\, A = \bigcap\{\operatorname{pcl} U : A \subseteq U \in PO(X)\}$.*

3.7. Theorem. *In a p-closed space X, every $p(\theta)$-conjoint grill $p(\theta)$-adheres in X.*

Proof. Consider any $p(\theta)$-conjoint grill $\mathcal{G}$ on a p-closed space X. We first note from Theorem 3.5 that for $A \subseteq X$, $p(\theta)\text{-cl}\, A$ is preclosed (as an arbitrary intersection of preclosed sets is preclosed). Thus $\{p(\theta)\text{-cl}\, A : A \in \mathcal{G}\}$ is a collection of preclosed sets in X such that $\operatorname{pint}\big[\bigcap_{i=1}^{n} p(\theta)\text{-cl}\, A_i\big] \neq \varnothing$ for any finite subcollection $A_1, A_2, \ldots, A_n$ of $\mathcal{G}$. Then $\operatorname{pint} \bigcap_{i=1}^{n} [p(\theta)\text{-cl}\, A_i] \neq \varnothing$ for any finite subcollection $A_1, A_2, \ldots, A_n$ of $\mathcal{G}$. Thus by Theorem 3.2, $\bigcap_{\alpha \in \Lambda} \{p(\theta)\text{-cl}\, A : A \in \mathcal{G}\} \neq \varnothing$, i.e., there exists $x \in X$ such that $x \in p(\theta)\text{-cl}\, A$ for all $A \in \mathcal{G}$. Hence $\mathcal{G} \subseteq \mathcal{G}(p(\theta), x)$ so that by Theorem 2.9, $\mathcal{G}$, $p(\theta)$-adheres at $x \in X$. □

3.8. Definition. [6] A subset A of a topological space X is called *pre-regular open* if $A = \operatorname{pint} \operatorname{pcl} A$. The complement of such sets are known as *pre-regular closed*.

3.9. Definition. [6] A topological space X is called *pre-almost regular* if for each $x \in X$ and each pre-regular open set V in X with $x \in V$, there is a pre-regular open set U in X such that $x \in U \subseteq \operatorname{pcl} U \subseteq V$.

3.10. Theorem. [6] *A topological space X is pre-almost regular iff for all $A \subseteq X$ we have $p(\theta)\text{-cl}\,[p(\theta)\text{-cl}\, A] = p(\theta)\text{-cl}\, A$.*

3.11. Theorem. *In a pre-almost regular p-closed space X, every grill $\mathcal{G}$ on X with the property $\bigcap_{i=1}^{n} p(\theta)\text{-cl}\, G_i \neq \varnothing$ for every finite subfamily $\{G_1, G_2, \ldots, G_n\}$ of $\mathcal{G}$, $p(\theta)$-adheres in X.*

Proof. Let X be a pre-almost regular p-closed space and $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$ a grill on X with the property that $\bigcap_{\alpha\in\Lambda_0} p(\theta)\text{-cl}\, G_\alpha \neq \varnothing$ for every finite subset Λ_0 of Λ. We consider

$$\mathcal{F} = \{ \bigcap_{\alpha\in\Lambda_0} p(\theta)\text{-cl}\, G_\alpha : \Lambda_0 \text{ is a finite subfamily of } \Lambda\}.$$

Then $\mathcal{F}$ is a filterbase on X. By the p-closedness of X, $\mathcal{F}$, $p(\theta)$-adheres at some $x \in X$, i.e., $x \in p(\theta)\text{-cl}\,(p(\theta)\text{-cl}\, G)$, for all $G \in \mathcal{G}$, i.e., $\mathcal{G} \subseteq \mathcal{G}(p(\theta), x)$. Hence by Theorem 2.9, $\mathcal{G}$ $p(\theta)$-adheres at $x \in X$. □

In view of Theorems 3.5, 3.7 and 3.11 we have,

3.12. Corollary. *In a pre-almost regular space X, the following are equivalent:*

(a) *Every grill $\mathcal{G}$ on X with the property that $\bigcap_{i=1}^{n} p(\theta)\text{-cl}\, G_i \neq \varnothing$ for every finite subfamily $\{G_1, G_2, \ldots, G_n\}$ of $\mathcal{G}$, $p(\theta)$-adheres in X.*
(b) *X is p-closed.*
(c) *Every $p(\theta)$-conjoint grill $p(\theta)$-adheres in X.*

In the next theorem we establish another condition equivalent to the condition (a) of the above Corollary.

3.13. Theorem. *Every grill $\mathcal{G}$ on a space X with the property that $\bigcap\{p(\theta)\text{-cl}\, G : G \in \mathcal{G}_0\} \neq \varnothing$ for every finite subset $\mathcal{G}_0$ of $\mathcal{G}$, $p(\theta)$-adheres in X iff for every family $\mathcal{F}$ of subsets of X for which the family $\{p(\theta)\text{-cl}\, F : F \in \mathcal{F}\}$ has the finite intersection property, we have $\bigcap\{p(\theta)\text{-cl}\, F : \mathcal{F}\} \neq \varnothing$.*

Proof. Let every grill on a topological space X satisfying the given condition, $p(\theta)$-adhere in X, and suppose that $\mathcal{F}$ is a family of subsets of X such that the family $\mathcal{F}^* = \{p(\theta)\text{-cl}\, F : F \in \mathcal{F}\}$ has the finite intersection property. Let $\mathcal{U}$ be the collection of all those families $\mathcal{G}$ of subsets of X for which $\mathcal{G}^* = \{p(\theta)\text{-cl}\, G : G \in \mathcal{G}\}$ has the finite intersection property and $\mathcal{F} \subseteq \mathcal{G}$. Then $\mathcal{F} \in \mathcal{U}$ and $\mathcal{U}$ is a partially ordered set under set inclusion in which every chain clearly has an upper bound. By Zorn's lemma, $\mathcal{F}$ is then contained in a maximal family $\mathcal{U}^* \in \mathcal{U}$. It is easy to verify that $\mathcal{U}^*$ is a grill with the stipulated property. Hence

$$\bigcap\{p(\theta)\text{-cl}\, F : F \in \mathcal{F}\} \supseteq \bigcap\{p(\theta)\text{-cl}\, U : U \in \mathcal{U}^*\} \neq \varnothing.$$

Conversely, if $\mathcal{F}$ is a grill on X with the given property, then for every finite subfamily $\mathcal{F}_0$ of $\mathcal{F}$, $\bigcap\{p(\theta)\text{-cl}\, F : F \in \mathcal{F}_0\} \neq \varnothing$. So, by hypothesis,

$$\bigcap\{p(\theta)\text{-cl}\, F : F \in \mathcal{F}\} \neq \varnothing.$$

Hence $\mathcal{F}$, $p(\theta)$-adheres in X. □

We now define a subclass of the class of all p-closed spaces and show that the two classes coincide in a pre-almost regular space.

3.14. Definition. A topological space X is called $p(\theta)$-linkage p-closed if every $p(\theta)$-linked grill on X, $p(\theta)$-adheres.

It follows at once from Corollary 3.12 that

3.15. Theorem. *Every $p(\theta)$-linkage p-closed space is p-closed.*

3.16. Theorem. *In the class of pre-almost regular spaces, the concepts of p-closedness and $p(\theta)$-linkage p-closedness become identical.*

Proof. In view of Theorem 3.15, it is enough to show that a pre-almost regular p-closed space is $p(\theta)$-linkage p-closed. Let $\mathcal{G}$ be any $p(\theta)$-linked grill on a pre-almost regular p-closed space X such that $\mathcal{G}$ does not $p(\theta)$-adhere in X. Then for each $x \in X$, there exists $G_x \in \mathcal{G}$ such that $x \notin p(\theta)\text{-cl}\, G_x = p(\theta)\text{-cl}\,[p(\theta)\text{-cl}\, G_x]$ (by Theorem 3.10). Then there exists $U_x \in PO(x)$ such that $\text{pcl}\, U_x \cap p(\theta)\text{-cl}\, G_x = \varnothing$, which gives

$$p(\theta)\text{-cl}\, U_x \cap p(\theta)\text{-cl}\, G_x = \varnothing$$

(it is known from [6] that for any preopen set U in any space X, $p(\theta)\text{-cl}\, U = \text{pcl}\, U$). Since $p(\theta)\text{-cl}\, G_x \in \mathcal{G}$, and $\mathcal{G}$ is a $p(\theta)$-linked grill on X, $p(\theta)\text{-cl}\, U_x = \text{pcl}\, U_x \notin \mathcal{G}$. Now, $\{U_x : x \in X\}$ is a cover of X by preopen sets of X. So by p-closedness of X, $X = \bigcup\{\text{pcl}\, U_{x_i} :\ i = 1, 2, \ldots, n\}$, for a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X. It then follows that $X \notin \mathcal{G}$ (since $\text{pcl}\, U_{x_i} \notin \mathcal{G}$ for $i = 1, 2, \ldots, n$), which is a contradiction. Hence $\mathcal{G}$ must $p(\theta)$-adhere in X, proving X to be $p(\theta)$-linkage p-closed. □

In [4], Jankovic, Reilly and Vamanamurty introduced strongly compact topological spaces in the following way:

3.17. Definition. A topological space X is said to be strongly compact if every cover $\mathcal{U}$ of X by preopen sets of X has a finite subcover.

Clearly every strongly compact space is p-closed, also it is a known result that a pre-regular p-closed space is strongly compact, where pre-regularity is defined in [7] in the usual way as follows.

3.18. Definition. A topological space X is *pre-regular* if for each $x \in X$ and each $U \in PO(x)$, there exists $V \in PO(x)$ such that $\text{pcl}\, V \subseteq U$.

It is now our intention to define a separation axiom strictly weaker than pre-regularity, in terms of grills, and to establish an improved version of the above stated result - namely that in the presence of such a separation axiom, the classes of p-closed spaces and strongly compact spaces become identical. Our proposed definition goes as follow.

3.19. Definition. A topological space X is called *$p(\theta)$-regular* if every grill on X which $p(\theta)$-converges must preconverge (not necessarily to the same point), where preconvergence of a grill is defined in the usual way, i.e., a grill $\mathcal{G}$ on X is said to preconverge to a point $x \in X$ if $PO(x) \subseteq \mathcal{G}$.

3.20. Theorem. *A topological space X is strongly compact iff every grill preconverges.*

Proof. Let $\mathcal{G}$ be a grill on a strongly compact space such that $\mathcal{G}$ does not preconverge to any point $x \in X$. Then for each $x \in X$, there exists $U_x \in PO(x)$ with

(3) $\quad U_x \notin \mathcal{G}$.

As $\{U_x : x \in X\}$ is a cover of the strongly compact space X by preopen sets, there exist finitely many points $\{x_1, x_2, \ldots, x_n\}$ in X such that $X = \bigcup_{i=1}^{n} U_{x_i}$. Since $X \in \mathcal{G}$, for some i, $(1 \leqslant i \leqslant n)$, $U_{x_i} \in \mathcal{G}$, which goes against (3).

Conversely, let every grill on X preconverge and if possible, let X be not strongly compact. Then there exists a cover $\mathcal{U}$ of X by preopen sets of X, having no finite subcover. Then

$$\mathcal{F} = \{X \setminus \bigcup \mathcal{U}_0 : \mathcal{U}_0 \text{ is a finite subcollection of } \mathcal{U}\}$$

is a filterbase on X. Then $\mathcal{F}$ is contained in an ultrafilter $\mathcal{G}$, and then $\mathcal{G}$ is a grill on X. By hypothesis, $\mathcal{G}$ preconverges to some point x of X. Then for some $U \in \mathcal{U}$, $x \in U$, and hence $U \in \mathcal{G}$. But $X \setminus U \in \mathcal{F} \subseteq \mathcal{U}$. Thus U and $X \setminus U$ both belong to $\mathcal{U}$, which is a filter, so giving a contradiction. □

It now follows from Theorems 3.3 and 3.20 that

3.21. Theorem. *A strongly compact space X is p-closed, while the converse is also true if X is $p(\theta)$-regular.*

We now need to show that $p(\theta)$-regularity is strictly a weaker version of pre-regularity.

3.22. Theorem. *Every pre-regular space is $p(\theta)$-regular.*

Proof. Let $\mathcal{G}$ be a grill on a pre-regular space X, $p(\theta)$-converging to a point x of X. For each $U \in PO(x)$, there exists, by pre-regularity of X, a $V \in PO(x)$ such that $\operatorname{pcl} V \subseteq U$. By hypothesis, $\operatorname{pcl} V \in \mathcal{G}$. Hence $\mathcal{G}$ preconverges to x, proving X to be $p(\theta)$-regular. □

The following example shows that the converse of the above result is false.

3.23. Example. Let $X = \{a, b, c\}$ and $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Then (X, τ) is a topological space such that $PO(X) = \tau$. Clearly X is strongly compact (X being a finite set). Hence by Theorem 3.20, every grill on X must preconverge in X. Thus X is $p(\theta)$-regular. But it is easy to check that X is not pre-regular.

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