Hacettepe Journal of Mathematics and Statistics
Volume 35 (2) (2006), 155 – 159

INFINITESIMAL AFFINE TRANSFORMATIONS IN THE TANGENT BUNDLE OF A RIEMANNIAN MANIFOLD WITH RESPECT TO THE HORIZONTAL LIFT OF AN AFFINE CONNECTION

A. Gezer*† and K. Akbulut*

Received 28 : 02 : 2006 : Accepted 04 : 12 : 2006

Abstract

The main purpose of the present paper is to study properties of vertical infinitesimal affine transformation in the tangent bundle of a Riemannian manifold with respect to the horizontal lift of an affine connection, and to apply the results obtained to the study of fibre-preserving infinitesimal affine transformations in this setting.

Keywords: Lift, Tangent bundle, Infinitesimal affine transformation, Fibre-preserving transformation.

2000 AMS Classification: 53 B 05, 53 C 07, 53 A 45

1. Introduction

Let M_n be a Riemannian manifold with metric g whose components in a coordinate neighborhood U are g_{ji}, and denote by Γ^h_{ji} the Christoffel symbols formed with g_{ji}. If, in the neighborhood $\pi^{-1}(U)$ of the tangent bundle $T(M_n)$ over M_n, U being a neighborhood of M_n, then ${}^Hg$ has components given by

$$ {}^Hg = \begin{pmatrix} \Gamma^t_j g_{ti} + \Gamma^t_i g_{jt} & g_{ji} \\ g_{ji} & 0 \end{pmatrix} $$

with respect to the induced coordinates (x^i, y^i) in $T(M_n)$, where $\Gamma^h_i = y^j \Gamma^h_{ji}$, Γ^h_{ji} being the components of the affine connection in M_n.

Let g be a pseudo-Riemannian metric. Then the horizontal lift ${}^Hg$ of g with respect to ∇ is a pseudo-Riemannian metric in $T(M_n)$. Since ${}^Hg$ is defined by ${}^Hg = {}^C g - \gamma(\nabla g)$,

*Department of Mathematics, Faculty of Art and Sciences, Atatürk University, 25240, Erzurum, Turkey. E-mail: (A. Gezer) agezer@atauni.edu.tr (K. Akbulut) kakbulut@atauni.edu.tr

†This paper is supported by TUBITAK grant number 105T551/TBAG-HD-112

where $\gamma(\nabla g)$ is a tensor field of type $(0,2)$, which has components of the form $\gamma(\nabla g) = \begin{pmatrix} y^s\nabla_s g & 0 \\ 0 & 0 \end{pmatrix}$, we have that ${}^H g$ and ${}^C g$ coincide if and only if $\nabla g = 0$ [1, p.105].

If we write $ds^2 = g_{ji}dx^j dx^i$ the pseudo-Riemannian metric in M_n given by g, then the pseudo-Riemannian metric in $T(M_n)$ given by the ${}^H g$ of g to $T(M_n)$ with respect to an affine connection ∇ in M_n is

$$ds^2 = 2g_{ji}\tilde{\delta}y^j dx^i, \tag{1}$$

where $\tilde{\delta}y^j = dy^j + \tilde{\Gamma}^j_{lk}y^l dx^k$ and $\tilde{\Gamma}^h_{ji} = \Gamma^h_{ij}$ are components of the connection $\tilde{\nabla}$ defined by $\tilde{\nabla}_X Y = \nabla_Y X + [X,Y]$, $\forall X, Y \in T^1_0(M_n)$, [1, p.67].

We shall now define the horizontal lift ${}^H\nabla$ of the affine connection ∇ in M_n to $T(M_n)$ by the conditions

$$\begin{aligned} &{}^H\nabla_{{}^V X}{}^V Y = 0, && {}^H\nabla_{{}^V X}{}^H Y = 0, \\ &{}^H\nabla_{{}^H X}{}^V Y = (\nabla_X Y)^V, && {}^H\nabla_{{}^H X}{}^H Y = (\nabla_X Y)^H, \end{aligned} \tag{2}$$

for $X, Y \in \Im^1_0(M_n)$. From (2), the horizontal lift ${}^H\nabla$ of ∇ has components ${}^H\Gamma^K_{JI}$ such that

$$\begin{aligned} &{}^H\Gamma^k_{ij} = \Gamma^k_{ij}, {}^H\Gamma^k_{i\bar{j}} = {}^H\Gamma^k_{\bar{i}j} = {}^H\Gamma^k_{\bar{i}\bar{j}} = {}^H\Gamma^{\bar{k}}_{\bar{i}\bar{j}} = 0, \\ &{}^H\Gamma^{\bar{k}}_{ij} = y^s\partial_s\Gamma^k_{ij} - y^s R^k_{sij}, {}^H\Gamma^{\bar{k}}_{\bar{i}j} = {}^H\Gamma^{\bar{k}}_{i\bar{j}} = \Gamma^k_{ij} \end{aligned} \tag{3}$$

with respect to the induced coordinates in $T(M_n)$, where Γ^k_{ij} are the components of ∇ in M_n.

Let g and ∇ be, respectively, a pseudo-Riemannian metric and an affine connection such that $\nabla g = 0$. Then ${}^H\nabla{}^H g = 0$, where ${}^H g$ is a pseudo-Riemannian metric. The connection ${}^H\nabla$ has nontrivial torsion even for the Riemannian connection ∇ determined by g, unless g is locally flat [1, p.111].

Let there be given an affine connection ∇ and a vector field $X \in \Im^1_0(M_n)$. Then the Lie derivative $L_X\nabla$ with respect to X is, by definition, an element of $\Im^1_2(M_n)$ such that

$$(L_X\nabla)(Y,Z) = L_X(\nabla_Y Z) - \nabla_Y(L_X Z) - \nabla_{[X,Y]}Z \tag{4}$$

for any $Y, Z \in \Im^1_0(M_n)$.

In a manifold M_n with affine connection ∇, an infinitesimal affine transformation $x^{h'} = x^h + X^h(x^1,\ldots,x^n)\Delta t$ defined by a vector field $X \in \Im^1_0(M_n)$ is called an *infinitesimal affine transformation* if $L_X\nabla = 0$, [1, p.67].

The main purpose of the present paper is to study the infinitesimal affine transformation in $T(M_n)$ with affine connection ${}^H\nabla$.

2. Vertical infinitesimal affine transformations in a tangent bundle with ${}^H\nabla$

From (4) we see that, in terms of the components $\Gamma^\alpha_{\gamma\beta}$ of ∇, X is an infinitesimal affine transformation in the m-dimensional manifold M_n if and only if,

$$\partial_\gamma\partial_\beta X^\alpha + X^\lambda\partial_\lambda\Gamma^\alpha_{\gamma\beta} - \Gamma^\lambda_{\gamma\beta}\partial_\lambda X^\alpha + \Gamma^\alpha_{\lambda\beta}\partial_\gamma X^\lambda + \Gamma^\alpha_{\gamma\lambda}\partial_\beta X^\lambda = 0,\ \alpha,\beta,\ldots = 1,\ldots,m. \tag{5}$$

Let there be given in M_n with metric g an affine connection ∇ with Christoffel symbols Γ^k_{ij}. Let $\tilde{X} = \tilde{X}^i\partial_i + \tilde{X}^{\bar{\imath}}\partial_{\bar{\imath}}$, where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{\imath}} = \frac{\partial}{\partial y^i} = \frac{\partial}{\partial x^{\bar{\imath}}}$, $\bar{\imath} = n+1,\ldots,2n$ be a vector field in $T(M_n)$. Then, taking account of (3), we can easily see from (5) that $\tilde{X}$

is an infinitesimal affine transformations in $T(M_n)$ with ${}^H\nabla$ if and only if the following conditions (6)–(13) hold:

$$\partial_j\partial_i\tilde{X}^h + \tilde{X}^k\partial_k\Gamma^h_{ji} - (\Gamma^k_{ji}\partial_k\tilde{X}^h + \partial\Gamma^k_{ji}\partial_{\bar{k}}\tilde{X}^h) + \Gamma^h_{ki}\partial_j\tilde{X}^k + \Gamma^h_{jk}\partial_i\tilde{X}^k + y^s R^k_{sji}\partial_{\bar{k}}\tilde{X}^h = 0, \tag{6}$$

$$\partial_j\partial_{\bar{i}}\tilde{X}^h - \Gamma^k_{ji}\partial_{\bar{k}}\tilde{X}^h + \Gamma^h_{jk}\partial_{\bar{i}}\tilde{X}^k = 0, \tag{7}$$

$$\partial_{\bar{j}}\partial_i\tilde{X}^h - \Gamma^k_{ji}\partial_{\bar{k}}\tilde{X}^h + \Gamma^h_{ki}\partial_{\bar{j}}\tilde{X}^k = 0, \tag{8}$$

$$\partial_{\bar{j}}\partial_{\bar{i}}\tilde{X}^h = 0, \tag{9}$$

$$\begin{aligned}\partial_j\partial_i\tilde{X}^{\bar{h}} &+ (\tilde{X}^k\partial_k\partial\Gamma^h_{ji} + \tilde{X}^{\bar{k}}\partial_k\Gamma^h_{ji}) - (\Gamma^k_{ji}\partial_k\tilde{X}^{\bar{h}} + \partial\Gamma^k_{ji}\partial_{\bar{k}}\tilde{X}^{\bar{h}}) + (\partial\Gamma^h_{ki}\partial_j\tilde{X}^k \\ &+ \Gamma^h_{ki}\partial_j\tilde{X}^{\bar{k}}) + (\partial\Gamma^h_{jk}\partial_i\tilde{X}^k + \Gamma^h_{jk}\partial_i\tilde{X}^{\bar{k}}) - \tilde{X}^{\bar{k}}R^h_{kji} - y^s\tilde{X}^k\partial_k R^h_{sji} \\ &+ y^s R^k_{sji}\partial_{\bar{k}}\tilde{X}^{\bar{h}} - y^s R^h_{ski}\partial_j\tilde{X}^k - y^s R^h_{sjk}\partial_i\tilde{X}^k = 0,\end{aligned} \tag{10}$$

$$\partial_j\partial_{\bar{i}}\tilde{X}^{\bar{h}} + \tilde{X}^k\partial_k\Gamma^h_{ji} - \Gamma^k_{ji}\partial_{\bar{k}}\tilde{X}^{\bar{h}} + \Gamma^h_{ki}\partial_j\tilde{X}^k + (\partial\Gamma^h_{jk}\partial_{\bar{i}}\tilde{X}^k + \Gamma^h_{jk}\partial_{\bar{i}}\tilde{X}^{\bar{k}}) - y^s R^h_{sjk}\partial_{\bar{i}}\tilde{X}^k = 0, \tag{11}$$

$$\partial_{\bar{j}}\partial_i\tilde{X}^{\bar{h}} + \tilde{X}^k\partial_k\Gamma^h_{ji} - \Gamma^k_{ji}\partial_{\bar{k}}\tilde{X}^{\bar{h}} + (\partial\Gamma^h_{ki}\partial_{\bar{j}}\tilde{X}^k + \Gamma^h_{ki}\partial_{\bar{j}}\tilde{X}^{\bar{k}}) + \Gamma^h_{jk}\partial_i\tilde{X}^k - y^s R^h_{ski}\partial_{\bar{j}}\tilde{X}^k = 0, \tag{12}$$

$$\partial_{\bar{j}}\partial_{\bar{i}}\tilde{X}^{\bar{h}} - \Gamma^h_{ki}\partial_{\bar{j}}\tilde{X}^k + \Gamma^h_{jk}\partial_{\bar{i}}\tilde{X}^k = 0. \tag{13}$$

Let $\tilde{X}$ be a vertical infinitesimal affine transformation in $T(M_n)$. Then $\tilde{X}$ has components $\begin{pmatrix} 0 \\ \tilde{X}^{\bar{h}} \end{pmatrix}$ with respect to the induced coordinates. Thus, from (13), we have $\partial_{\bar{j}}\partial_{\bar{i}}\tilde{X}^{\bar{h}} = 0$, i.e.,

$$\tilde{X}^{\bar{h}} = C^h_i y^i + D^h, \tag{14}$$

where C^h_i and D^h depend only on the variables x^h. Since $\tilde{X}$ is a vector field in $T(M_n)$, $C = C^h_i\partial_h\otimes dx^i$ and $D = D^h\partial_h$ are defined elements of $\Im^1_1(M_n)$ and $\Im^1_0(M_n)$, respectively.

2.1. Theorem. *If $\tilde{X}$ is a vertical infinitesimal affine transformation of $T(M_n)$ with ${}^H\nabla$, then*

(a) $L_D\nabla + C(D\otimes R) = 0$, $D = \partial^h\frac{\partial}{\partial x^h}$, $D \in \Im^1_0(M_n)$ *and* $C(D\otimes R) = D^k R^h_{kji}$.
(b) *C is parallel with respect to ∇, i.e.,* $\nabla C = 0$.
(c) $C(T(Y,Z)) = T(CY,Z) = T(Y,CZ)$, *for any* $Y, Z \in \Im^1_0(M_n)$, *where T denotes the torsion tensor of ∇, i.e. T is a pure tensor with respect to C.*
(d) $C(\nabla_Z T)(Y,W) = (\nabla_{CZ}T)(Y,W)$, *for any* $Y, Z, W \in \Im^1_0(M_n)$.
(e) *Conversely, if C and D satisfy the conditions (a), (b), (c) and (d) then the vector field*

$$\tilde{X} = (C^h_i y^i + D^h)\frac{\partial}{\partial y^h} = \gamma C +^v D$$

is an infinitesimal affine transformation of $T(M_n)$ with connection ${}^H\nabla$, where γC is a vertical vector field which has components of the form $\gamma C = \begin{pmatrix} 0 \\ y^i C^h_i \end{pmatrix}$.

Proof. (a). Substituting (14) and $\tilde{X}^h = 0$ in (10), we have

$$\partial_j\partial_i C^h_s + C^k_s\partial_k\Gamma^h_{ji} - \Gamma^k_{ji}\partial_k C^h_s - \partial_s\Gamma^k_{ji}C^h_k + \Gamma^h_{ki}\partial_j C^k_s + \Gamma^h_{jk}\partial_i C^k_s - C^k_s R^h_{kji} + R^k_{sji}C^h_k = 0, \tag{15}$$

and

$$\partial_j\partial_i D^h + D^k\partial_k\Gamma^h_{ji} - \Gamma^k_{ji}\partial_k D^h + \Gamma^h_{ki}\partial_j D^k + \Gamma^h_{jk}\partial_i D^k - D^k R^h_{kji} = 0, \tag{16}$$

which means that $L_D\nabla + \mathcal{C}(D\otimes R) = 0$.

(b). Substituting (14) and $\tilde{X}^h = 0$ in (12), we obtain

$$\partial_i C_j^h - \Gamma_{ji}^k C_k^h + \Gamma_{ki}^h C_j^k = 0. \tag{17}$$

Substituting (14) and $\tilde{X}^h = 0$ in (11), we obtain

$$\partial_j C_i^h - \Gamma_{ji}^k C_k^h + \Gamma_{jk}^h C_i^k = 0, \tag{18}$$

which means that C is parallel in M_n.

(c). Interchanging i and j in (18), we have

$$\partial_i C_j^h - \Gamma_{ij}^k C_k^h + \Gamma_{ik}^h C_j^k = 0,$$

and subtracting the resulting equation from (17), we have

$$T_{ji}^k C_k^h = T_{ki}^h C_j^k, \tag{19}$$

that is,

$$C(T(Y,Z)) = T(CY,Z) \tag{20}$$

for any $Y, Z \in \Im_0^1(M_n)$. From (19), we obtain $T(Y,CZ)) = -T(CZ,X) = C(T(Z,Y)) = C(T(Y,Z))$ and hence

$$C(T(Y,Z)) = T(CY,Z) = T(Y,CZ),$$

which is the formula (c).

(d). Using (17) and (18), we eliminate all partial derivatives of C_j^h from (15). Then we obtain $C_k^h\nabla_j T_{li}^k = \nabla_k T_{li}^h C_j^k$, i.e. T is a ϕ-tensor with respect to C [3].

(e). If we assume that the conditions (a), (b), (c) and (d) are established, then we see that $\tilde{X}$, given in (e), is an infinitesimal affine transformation. Consequently, the proof is complete. □

2.2. Theorem. *Let C be as in Theorem 2.1. If X is an infinitesimal affine transformation of M_n with affine connection ∇, and $R(X,Y,Z;\xi)$ is pure with respect to X and ξ, then so is CX.*

3. Fibre-preserving infinitesimal affine transformation with ${}^H\nabla$

A transformation of $T(M_n)$ is said to be *fibre-preserving* if it sends each fibre of $T(M_n)$ into a fibre. An infinitesimal transformation of $T(M_n)$ is said to be *fibre-preserving* if it generates a local 1-parameter group of fibre-preserving transformations. An infinitesimal transformation $\tilde{X}$ with components $\begin{pmatrix}\tilde{X}^h\\ \tilde{X}^{\bar{h}}\end{pmatrix}$ is fibre-preserving if and only if $\tilde{X}^h$ $(h = 1,2,\dots,n)$ depend only on the variables $x^1,\dots,x^n$ with respect to the induced coordinates (x^h, y^h) in $T(M_n)$. From

$$\begin{cases} x^{h'} = x^h + \tilde{X}^h(x^1,\dots,x^n)\Delta t \\ x^{\bar{h}'} = x^{\bar{h}} + \tilde{X}^{\bar{h}}(x^1,\dots,x^n,x^{n+1},\dots,x^{2n})\Delta t \end{cases} \tag{21}$$

we see that a fibre-preserving infinitesimal transformation $\tilde{X}$ with components $\begin{pmatrix}\tilde{X}^h\\ \tilde{X}^{\bar{h}}\end{pmatrix}$ induces an infinitesimal transformation X with components $\tilde{X}^h$ in the base space M_n.

Since $\partial\Gamma_{ji}^k\partial_{\bar{k}}\tilde{X}^h = 0$ and $y^s R_{sji}^k \partial_{\bar{k}}\tilde{X}^h = 0$, then from (6) we have:

3.1. Theorem. *If $\tilde{X}$ is a fibre-preserving infinitesimal transformation of $T(M_n)$ with horizontal lift ${}^H\nabla$ of a affine connection ∇ in M_n to $T(M_n)$, then the infinitesimal transformation X induced on M_n by $\tilde{X}$ is also affine with respect to ∇.*

3.2. Theorem. *Let ∇ be an affine connection in M_n. Then,*

$$(L^H_{{}^CX}\nabla)({}^CY, {}^CZ) = {}^C(L_X\nabla)({}^CY, {}^CZ) + \gamma(L_XR)(\ \ , Y, Z)$$

for any $X \in \Im^1_0(M_n)$.

Proof. Our proposition follows from the following computations:

$$\begin{aligned}
(L^H_{{}^CX}\nabla)({}^CY, {}^CZ) &= L_{{}^CX}({}^H\nabla^C_{{}^CY}Z) - {}^H\nabla_{{}^CY}(L^C_{{}^CX}Z) - {}^H\nabla^C_{[{}^CX, {}^CY]}Z \\
&= L_{{}^CX}[{}^C(\nabla_Y Z) - \gamma(R(\ ,Y)Z)] - {}^H\nabla^C_{{}^CY}[X,Z] - {}^H\nabla^C_{{}^C[X,Y]}Z \\
&= [{}^CX, {}^C\nabla_X Y] - [{}^CX, \gamma(R(\ ,Y)Z)] - {}^C(\nabla_Y[X,Z]) \\
&\qquad + \gamma(R(\ \ ,Y)[X,Z]) - {}^C(\nabla_{[X,Y]}Z) + \gamma R([X,Y]Z)] \\
&= {}^C(L_X\nabla_X Y) - {}^C(\nabla_Y(L_X Z)) - {}^C(\nabla_{[X,Y]}Z) \\
&\qquad - \gamma(L_X R(\ ,Y)Z) + \gamma(R(\ ,Y)[X,Z]) + \gamma(R(\ \ ,[X,Y])Z) \\
&= {}^C(L_X\nabla)({}^CY, {}^CZ) + \gamma(-L_X R(\ \ ,Y)Z + R(\ \ ,Y)[X,Z] \\
&\qquad\qquad + R(\ \ ,[X,Y])Z) \\
&= {}^C(L_X\nabla)({}^CY, {}^CZ) + \gamma(L_X R)(\ \ ,Y,Z),
\end{aligned}$$

where $R(\ \ ,X)Y$ denotes a tensor field W of type $(1,1)$ in M_n such that $W(Z) = R(Z,X)Y$ for any $Z \in \Im^1_0(M_n)$. □

Let $\tilde{X}$ and X be as in Theorem 3.1. From Theorem 3.2 we see that, if X is an infinitesimal automorphism with respect to W [3], then ${}^cX$ is an infinitesimal affine transformation of $T(M_n)$ with ${}^H\nabla$. Since ${}^cX$ has the components $\begin{pmatrix} X^h \\ \partial X^h \end{pmatrix}$, it follows that $\tilde{X} - {}^cX$ is a vertical infinitesimal affine transformation in $T(M_n)$ with ${}^H\nabla$. Thus we have

3.3. Theorem. *If $\tilde{X}$ is a fibre-preserving infinitesimal affine transformation of $T(M_n)$ with lift ${}^H\nabla$, and X is an infinitesimal automorphism with respect to W, then $\tilde{X} = {}^cX + {}^vD + \gamma C$, where D and C are tensor fields of type $(1,0)$ and $(1,1)$, respectively, satisfying conditions (a), (b) and (c) of Theorem 2.1*

Acknowledgement. The authors are grateful to Professor A. A. Salimov for his valuable suggestions.

References

[1] Yano K. and Ishihara, S. *Tangent and cotangent bundles* (Marcel Dekker, New York, 1973).

[2] Magden A. and Salimov, A. A. *Horizontal lifts of tensor fields to sections of the tangent bundle.* (in Russian) Izv. Vyssh. Uchebn. Zaved. Mat. **3**, 77–80, 2001.

[3] Kobayashi S. and Nomizu K. *Foundations of Differential Geometry* (Wiley-Inter Science Publications, New York, 1963).