Hacettepe Journal of Mathematics and Statistics
Volume 35 (2) (2006), 173 – 180

ON GENERALIZED DERIVATIONS OF PRIME NEAR-RINGS

Öznur Gölbaşı*

Received 03 : 04 : 2006 : Accepted 28 : 09 : 2006

Abstract

Let N be a 2−torsion free prime near-ring with center Z, (f, d) and (g, h) two generalized derivations on N. In this case: (i) If $f([x, y]) = 0$ or $f([x, y]) = \pm[x, y]$ or $f^2(x) \in Z$ for all $x, y \in N$, then N is a commutative ring. (ii) If $a \in N$ and $[f(x), a] = 0$ for all $x \in N$, then $d(a) \in Z$. (iii) If (fg, dh) acts as a generalized derivation on N, then $f = 0$ or $g = 0$.

Keywords: Prime near-ring, Derivation, Generalized derivation.

2000 AMS Classification: 16 Y 30, 16 N 60, 16 W 25, 16 U 80.

1. Introduction

Throughout this paper N will denote a zero symmetric left near-ring with multiplicative centre Z. Recall that a near-ring N is *prime* if $xNy = 0$ implies $x = 0$ or $y = 0$. An additive mapping $d : N \to N$ is said to be a *derivation on* N if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$ or equivalently, as noted in [3], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. Further an element $x \in N$ for which $d(x) = 0$ is called a *constant*. For $x, y \in N$ the symbol $[x, y]$ will denote the commutator $xy - yx$, while the symbol (x, y) will denote the additive-group commutator $x + y - x - y$.

Over the last two decades, a lot of work has been done on commutativity of prime rings with derivation. It is natural to look for comparable results on near-rings and this has been done [1,3,4] (where further references can be found). Recently, in [5], Bresar defined the following notation:

An additive mapping $f : R \to R$ is called a *generalized derivation* if there exits a derivation d of R such that

$$f(xy) = f(x)y + xd(y) \text{ for all } x, y \in R.$$

The concept of generalized derivation cover also the concept of a derivation. In the present paper we extend some well-known results concerning derivations of prime rings to generalized derivations of prime near-rings.

*Cumhuriyet University, Department of Mathematics, 58140, Sivas, Turkey.
E-mail: `ogolbasi@cumhuriyet.edu.tr`

2. Preliminaries

We will make use of the following lemmas.

2.1. Lemma. [4, Lemma 1] *Let d be an arbitrary derivation on a near-ring N. Then N satisfies the following partial distributive law:*

$$(ad(b) + d(a)b)c = ad(b)c + d(a)bc$$

and

$$(d(a)b + ad(b))c = d(a)bc + ad(b)c$$

for all $a, b, c \in N$.

2.2. Lemma. [4, Lemma 3] *Let N be a 3-prime near-ring.*

(i) *If $z \in Z/\{0\}$, then z is not a zero divisor in N.*
(ii) *If $Z/\{0\}$ contains an element z for which $z + z \in Z$, then $(N, +)$ is abelian.*
(iii) *Let d be a nonzero derivation on N. Then $xd(N) = \{0\}$ implies $x = 0$, and $d(N)x = \{0\}$ implies $x = 0$.*
(iv) *If N is 2-torsion free and d is a derivation on N such that $d^2 = 0$, then $d = 0$.*

2.3. Definition. [7, Definition 1] Let N be a near-ring and d a derivation of N. An additive mapping $f : N \to N$ is said to be *a right generalized derivation of N associated with d* if

(2.1) $\quad f(xy) = f(x)y + xd(y)$ for all $x, y \in R$.

and f is said to be *a left generalized derivation of N associated with d* if

(2.2) $\quad f(xy) = d(x)y + xf(y)$ for all $x, y \in R$.

Finally, f is said to be *a generalized derivation of N associated with d* if it is both a left and right generalized derivation of N associated with d.

2.4. Lemma. [7, Lemma 2]

(i) *Let f be a right generalized derivation of a near ring N associated with d. Then $f(xy) = xd(y) + f(x)y$ for all $x, y \in N$.*
(ii) *Let f be a left generalized derivation of near ring N associated with d. Then $f(xy) = xf(y) + d(x)y$ for all $x, y \in N$.*

2.5. Lemma. [7, Lemma 3]

(i) *Let f be a right generalized derivation of the near ring N associated with d. Then $(f(x)y + xd(y))z = f(x)yz + xd(y)z$ for all $x, y \in N$.*
(ii) *Let f be a left generalized derivation of the near ring N with associated d. Then $(d(x)y + xf(y))z = d(x)yz + xf(y)z$ for all $x, y \in N$.*

2.6. Lemma. [7, Lemma 4] *Let N be a prime near-ring, f a nonzero generalized derivation of N associated with the nonzero derivation d, and $a \in N$.*

(i) *If $af(N) = 0$, then $a = 0$.*
(ii) *If $f(N)a = 0$, then $a = 0$.*

2.7. Lemma. [7, Theorem 5] *Let f be a generalized derivation of N associated with the nonzero derivation d. If N is a 2-torsion free near-ring and $f^2 = 0$, then $f = 0$.*

2.8. Lemma. [7, Theorem 6] *Let N be a prime near-ring with a nonzero generalized derivation f associated with d. If $f(N) \subset Z$, then $(N, +)$ is abelian. Moreover, if N is 2-torsion free, then N is commutative ring.*

3. Results

We denote a generalized derivation $f : N \to N$ determined by a derivation d of N by (f, d). We assume that d is a nonzero derivation of N unless stated otherwise.

The following two theorems are motivated by [2, Theorem 3] and [6, Theorem 1], respectively.

3.1. Theorem. *Let (f, d) be a generalized derivation of N. If $f([x, y]) = 0$ for all $x, y \in N$, then N is commutative ring.*

Proof. Assume that $f([x, y]) = 0$ for all $x, y \in N$. Substitute xy instead of y, obtaining

$$f([x, xy]) = f(x[x, y]) = d(x)[x, y] + xf([x, y]) = 0.$$

Since the second term is zero, it is clear that

(3.1) $\quad d(x)xy = d(x)yx$ *for all* $x, y \in N$.

Replacing y by yz in (3.1) and using this equation, we get

$$d(x)N[x, z] = 0 \text{ for all } x, z \in N.$$

Hence either $x \in Z$ or $d(x) = 0$. Let $K = \{x \in N \mid x \in Z\}$ and $L = \{x \in N \mid d(x) = 0\}$. Then K and L are two additive subgroups of $(N, +) = K \cup L$. However, a group cannot be the union of proper subgroups, hence either $N = K$ or $N = L$. Since $d \neq 0$, we are forced to conclude that N is commutative ring. □

3.2. Theorem. *Let (f, d) be a generalized derivation of N. If $f([x, y]) = \pm[x, y]$ for all $x, y \in N$, then N is a commutative ring.*

Proof. Assume that $f([x, y]) = \pm[x, y]$ for all $x, y \in N$. Replacing y by xy in the hypothesis, we have

$$f([x, xy]) = \pm(x^2y - xyx) = \pm x[x, y].$$

On the other hand,

$$f([x, xy]) = f(x[x, y]) = d(x)[x, y] + xf([x, y]) = d(x)[x, y] + x(\pm[x, y]).$$

It follows from the two expressions for $f([x, xy])$ that

$$d(x)xy = d(x)yx \text{ for all } x, y \in N.$$

Using the same argument as in the proof of Theorem 3.1, we get that N is a commutative ring. □

3.3. Theorem. *Let (f, d) be a nonzero generalized derivation of N. If f acts as a homomorphism on N, then f is the identity map.*

Proof. Assume that f acts as a homomorphism on N. Then one obtains

(3.2) $\quad f(xy) = f(x)f(y) = d(x)y + xf(y)$ for all $x, y \in N$.

Replacing y by yz in (3.2), we arrive at

$$f(x)f(yz) = d(x)yz + xf(yz).$$

Since (f, d) be a generalized derivation and f acts as a homomorphism on N, we deduce that

$$f(xy)f(z) = d(x)yz + xd(y)z + xyf(z).$$

By Lemma 2.5 (ii), we get

$$d(x)yf(z) + xf(y)f(z) = d(x)yz + xd(y)z + xyf(z),$$

and so

$$d(x)yf(z) + xf(yz) = d(x)yz + xd(y)z + xyf(z).$$

That is,

$$d(x)yf(z) + xd(y)z + xyf(z) = d(x)yz + xd(y)z + xyf(z).$$

Hence, we deduce that

$$d(x)y(f(z) - z) = 0 \text{ for all } x, y, z \in N.$$

Because N is prime and $d \neq 0$, we have $f(z) = z$ for all $z \in N$. Thus, f is the identity map. □

3.4. Theorem. *Let (f, d) be a nonzero generalized derivation of N. If f acts as an anti-homomorphism on N, then f is the identity map.*

Proof. By the hypothesis, we have

$$f(xy) = f(y)f(x) = d(x)y + xf(y) \text{ for all } x, y \in N. \tag{3.3}$$

Replacing y by xy in the last equation, we obtain

$$f(xy)f(x) = d(x)xy + xf(xy).$$

Since (f, d) is a generalized derivation and f acts as an anti-homomorphism on N, we get

$$(d(x)y + xf(y))f(x) = d(x)xy + xf(y)f(x).$$

By Lemma 2.5 (ii), we conclude that

$$d(x)yf(x) + xf(y)f(x) = d(x)xy + xf(y)f(x),$$

and so

$$d(x)yf(x) = d(x)xy \text{ for all } x, y \in N.$$

Replacing y by yz and using this equation, we have

$$d(x)N[f(x), z] = 0 \text{ for all } x, z \in N.$$

Hence we obtain the following alternatives: $d(x) = 0$ or $f(x) \in Z$, for all $x \in N$. By a standard argument, one of these must hold for all $x \in N$. Since $d \neq 0$, the second possibility gives that N is commutative ring by Lemma 2.8, and so we deduce that f is the identity map by Theorem 3.3. □

3.5. Theorem. *Let (f, d) be a generalized derivation of N such that $d(Z) \neq 0$, and $a \in N$. If $[f(x), a] = 0$ for all $x \in N$, then $a \in Z$.*

Proof. Since $d(Z) \neq 0$, there exists $c \in Z$ such that $d(c) \neq 0$. Furthermore, as d is a derivation, it is clear that $d(c) \in Z$. Replacing x by cx in the hypothesis and using Lemma 2.5 (ii), we have

$$f(cx)a = af(cx)$$
$$d(c)xa + cf(x)a = ad(c)x + acf(x).$$

Since $c \in Z$ and $d(c) \in Z$, we get

$$d(c)N[y, a] = 0 \text{ for all } y \in N.$$

By the primeness of N and $0 \neq d(c) \in Z$, we obtain that $a \in Z$. □

3.6. Theorem. *Let (f, d) be a generalized derivation of N, and $a \in N$. If $[f(x), a] = 0$ for all $x \in N$, then $d(a) \in Z$.*

Proof. If $a = 0$, then there is nothing to prove. Hence, we assume that $a \neq 0$.

Replacing x by ax in the hypothesis, we have

$$f(ax)a = af(ax)$$
$$d(a)xa + af(x)a = ad(a)x + aaf(x).$$

Using $f(x)a = af(x)$, we have

$$d(a)xa = ad(a)x \text{ for all } x \in N.$$

Taking xy instead of x in the last equation and using this, we conclude that

$$d(a)N[a, y] = 0 \text{ for all } y \in N.$$

Since N is a prime near-ring, we have either $d(a) = 0$ or $a \in Z$. If $0 \neq a \in Z$, then $(N, +)$ is abelian by Lemma 2.2 (ii). Thus

$$f(xa) = f(ax)$$
$$f(x)a + xd(a) = d(a)x + af(x),$$

and so

$$[d(a), x] = 0 \text{ for all } x \in N.$$

That is, $d(a) \in Z$. Hence in either case we have $d(a) \in Z$. This completes the proof. □

3.7. Theorem. *Let (f, d) be a generalized derivation of N. If N is a 2-torsion free near-ring and $f^2(N) \subset Z$, then N is a commutative ring.*

Proof. Suppose that $f^2(N) \subset Z$. Then we get

$$f^2(xy) = f^2(x)y + 2f(x)d(y) + xd^2(y) \in Z \text{ for all } x, y \in N.$$

In particular, $f^2(x)c + 2f(x)d(c) + xd^2(c) \in Z$ for all $x \in N,\ c \in Z$. Since the first summand is an element of Z, we have

$$(3.4) \qquad 2f(x)d(c) + xd^2(c) \in Z \text{ for all } x \in N, c \in Z.$$

Taking $f(x)$ instead of x in (3.4), we obtain that

$$2f^2(x)d(c) + f(x)d^2(c) \in Z \text{ for all } x \in N,\ c \in Z.$$

Since $d(c) \in Z, f^2(x) \in Z$, and so $f^2(x)d(c) \in Z$ for all $x \in N,\ c \in Z$, we conclude

$$f(x)d^2(c) \in Z \text{ for all } x \in N,\ c \in Z.$$

Since N is prime, we get $d^2(Z) = 0$ or $f(N) \subseteq Z$. If $f(N) \subseteq Z$ then N is a commutative ring by Lemma 2.8. Hence, we assume $d^2(Z) = 0$. By (3.4), we get

$$2f(x)d(c) \in Z \text{ for all } x \in N,\ c \in Z.$$

Since N is a 2-torsion free near-ring and $d(c) \in Z$, we obtain that either $f(N) \subset Z$ or $d(Z) = 0$. If $f(N) \subset Z$, then we are already done. So, we may assume that $d(Z) = 0$. Then

$$f(cx) = f(xc)$$
$$f(c)x + cd(x) = f(x)c + xd(c),$$

and so

$$(3.5) \qquad f(c)x + cd(x) = f(x)c \text{ for all } x \in N,\ c \in Z.$$

Now replacing x by $f(x)$ in (3.5), and using the fact that $f^2(N) \subset Z$, we get

$$f(c)f(x) + cd(f(x)) = f^2(x)c \text{ for all } x \in N,\ c \in Z.$$

That is,

(3.6) $f(c)f(x) + cd(f(x)) \in Z$ for all $x \in N,\ c \in Z.$

Again taking $f(x)$ instead of x in this equation, one can obtain

$$f(c)f^2(x) + cd(f^2(x)) \in Z \text{ for all } x \in N,\ c \in Z.$$

The second term is equal to zero because of $d(Z) = 0$. Hence we have

$$f(c)f^2(x) \in Z \text{ for all } x \in N,\ c \in Z.$$

Since $f^2(N) \subset Z$ by the hypothesis, we get either $f^2(N) = 0$ or $f(Z) \subset Z$. If $f^2(N) = 0$, then the theorem holds by Lemma 2.7. If $f(Z) \subset Z$, then $f(xf(c)) = f(f(c)x)$ for all $x \in N,\ c \in Z$, and so

$$d(x)f(c) = f(c)f(x) \text{ for all } x \in N,\ c \in Z.$$

Using $f(c) \in Z$, we now have

$$f(c)(d(x) - f(x)) = 0 \text{ for all } x \in N,\ c \in Z.$$

Since $f(Z) \subset Z$, we have either $f(Z) = 0$ or $d = f$. If $d = f$, then f is a derivation of N and so N is commutative ring by Lemma 2.7.

Now assume that $f(Z) = 0$. Returning to the equation (3.5), we have

$$c(d(x) - f(x)) = 0 \text{ for all } x \in N,\ c \in Z.$$

Since $c \in Z$ we have either $d = f$ or $Z = 0$. Clearly $d = f$ implies the theorem hoşds. If $Z = 0$, then $f^2(N) = 0$ by the hypothesis, and so N is a commutative ring by Lemma 2.2 (iv). Hence, the proof is completed. □

3.8. Corollary. *Let N be a 2-torsion free near-ring, (f, d) a nonzero generalized derivation of N. If $[f(N), f(N)] = 0$, then N is a commutative ring.*

3.9. Lemma. *Let (f, d) and (g, h) be two generalized derivations of N. If h is a nonzero derivation on N and $f(x)h(y) = -g(x)d(y)$ for all $x, y \in N$, then $(N, +)$ is abelian.*

Proof. Suppose that

$$f(x)h(y) + g(x)d(y) = 0 \text{ for all } x, y \in N.$$

We substitute $y + z$ for y, thereby obtaining

$$f(x)h(y) + f(x)h(z) + g(x)d(y) + g(x)d(z) = 0.$$

Using the hypothesis, we get

$$f(x)h(y, z) = 0 \text{ for all } x, y, z \in N.$$

It follows by Lemma 2.6 (ii) that $h(y, z) = 0$ for all $y, z \in N$. For any $w \in N$, we have

$$h(wy, wz) = h(w(y, z)) = h(w)(y, z) + wh(y, z) = 0,$$

and so

$$h(w)(y, z) = 0 \text{ for all } w, y, z \in N.$$

An appeal to Lemma 2.2 (iii) yields that $(N, +)$ is abelian. □

3.10. Theorem. *Let (f, d) and (g, h) be two generalized derivations of N. If N is 2-torsion free and $f(x)h(y) = -g(x)d(y)$ for all $x, y \in N$, then $f = 0$ or $g = 0$.*

Proof. If $h = 0$ or $d = 0$, then the proof of the theorem is obvious. So, we may assume that $h \neq 0$ and $d \neq 0$. Therefore we know that $(N, +)$ is abelian by Lemma 3.9.

Now suppose that

$$f(x)h(y) + g(x)d(y) = 0 \text{ for all } x, y \in N.$$

Replacing x by uv in this equation and using the hypothesis, we get

$$f(uv)h(y) + g(uv)d(y) = uf(v)h(y) + d(u)vh(y) + ug(v)d(y) + h(u)vd(y) = 0,$$

and so

$$d(u)vh(y) = -h(u)vd(y) \text{ for all } u, v, y \in N. \tag{3.7}$$

Taking yt instead of y in the above relation, we obtain

$$d(u)vh(y)t + d(u)vyh(t) = -h(u)vd(y)t - h(u)vyd(t).$$

That is,

$$d(u)vyh(t) = -h(u)vyd(t) \text{ for all } u, v, y, t \in N. \tag{3.8}$$

Replacing y by $h(y)$ in (3.8) and using this relation, we have

$$h(u)N(d(y)h(t) - h(y)d(t)) = 0 \text{ for all } u, y, t \in N.$$

Since N is a prime near-ring and $h \neq 0$, we obtain that

$$d(y)h(t) = h(y)d(t) \text{ for all } y, t \in N. \tag{3.9}$$

Now again taking uv instead of x in the initial hypothesis, we get

$$f(u)vh(y) + ud(v)h(y) + g(u)vd(y) + uh(v)d(y) = 0.$$

Using (3.9) yields that

$$f(u)vh(y) + 2uh(v)d(y) + g(u)vd(y) = 0 \text{ for all } u, v, y \in N.$$

Taking $h(v)$ instead of v in this equation, we arrive at

$$f(u)h(v)h(y) + 2uh^2(v)d(y) + g(u)h(v)d(y) = 0.$$

By the hypothesis and (3.9), we have

$$\begin{aligned} 0 &= -g(u)d(v)h(y) + 2uh^2(v)d(y) + g(u)h(v)d(y) \\ &= -g(u)h(v)d(y) + 2uh^2(v)d(y) + g(u)h(v)d(y), \end{aligned}$$

and so

$$2uh^2(v)d(y) = 0 \text{ for all } u, v, y \in N.$$

Since N is a 2-torsion free prime near-ring, we obtain that $h^2(N)d(N) = 0$. An appeal to Lemmas 2.2 (iii) and (iv) gives that $h = 0$. This contradicts by our assumption. Thus the proof is completed. □

3.11. Theorem. *Let (f, d) and (g, h) be two generalized derivations of N. If (fg, dh) acts as a generalized derivation on N, then $f = 0$ or $g = 0$.*

Proof. By calculating $fg(xy)$ in two different ways, we see that

$$g(x)d(y) + f(x)h(y) = 0 \text{ for all } x, y \in N.$$

The proof is completed by using Theorem 3.10. □

References

[1] Bell, H. E. *On derivations in near-rings II*, Kluwer Academic Publ. Math. Appl. Dordr. **426**, 191–197, 1997.

[2] Bell, H. E. and Daif, M. N. *On derivations and commutativity in prime rings*, Acta Math. Hungar **66** (4), 337–343, 1995.

[3] Bell, H. E. and Mason, G. *On derivations in near-rings and rings*, Math. J. Okayama Univ. **34**, 135–144, 1992.

[4] Bell, H. E. and Mason G. *On derivations in near-rings* (in: Near-rings and Near-fields, North-Holland Mathematical Studies 137, 1987), 31–35.

[5] Bresar M. *On the distance of composition of two derivations to the generalized derivations*, Glasgow Math. J. **33**, 89–93, 1991.

[6] Daif, M. N. and Bell, H. E. *Remarks on derivations on semiprime rings*, Internat J. Math. and Math. Sci. **15** (1), 205–206, 1992.

[7] Gölbaşı, Ö. *Notes on prime near-rings with generalized derivation*, Southeast Asian Bull. of Math. **30**, 1–5, 2006.

[8] Hvala, B. *Generalized derivations in rings*, Comm. Algebra **26**, 1147–1166, 1998.