ON ALMOST ℓ-CONTINUOUS MULTIFUNCTIONS

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Abstract

The aim of this paper is to introduce a new class of multifunctions between topological spaces, namely almost ℓ-continuous multifunctions, which properly contains the class of ℓ-continuous multifunctions introduced by Sakalova in 1989. We relate this class of multifunctions to other classes of multifunctions, and provide characterizations of related concepts especially in terms of an appropriate change of topology.

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1. Introduction and preliminaries

Various authors have extended weak forms of continuity to multifunctions, for example almost continuity and ℓ-continuity [11], c-continuity [6, 11], almost c-continuity [9], nearly continuity [1] and almost nearly continuity [2]. The notion of almost ℓ-continuous functions between topological spaces was introduced by Konstadilaki-Savvopoulou and Reilly [8]. A function $f : (X, \sigma) \to (Y, \tau)$ is defined to be almost ℓ-continuous if for each point $x \in X$ and each regular open set V in Y containing $f(x)$ and having Lindelöf complement there exists an open set U in X containing x such that $f(U) \subseteq V$. The purpose of this paper is to extend this concept to multifunctions.

Throughout this paper, the closure (resp. interior) of a subset B in (Y, τ) is denoted by cl$B$ (resp. int$B$). Then B is called *regular open* if $B = \text{int}(\text{cl}B)$. The family of all regular open sets in (Y, τ), which is denoted by $RO(Y, \tau)$, forms a base for a topology τ_S on Y, known as the *semiregularization* of τ. In general $\tau_S \subseteq \tau$, and if $\tau_S = \tau$ then (Y, τ) is called a *semiregular space.*

For a topological space (Y, τ), the *cocompact topology of τ on Y* is denoted by $c(\tau)$ and defined by $c(\tau) = \{\emptyset\} \cup \{U \in \tau : Y \setminus U$ is τ-compact$\}$. The *almost cocompact topology of τ*

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on Y is denoted by $e(\tau)$ and it has as a base, $e'(\tau) = \{U \in RO(Y, \tau) : Y \backslash U \text{ is } \tau\text{-compact}\}$. These topologies are considered by Gauld [3] and [4].

If (Y, τ) is a topological space then the *coLindelöf topology of* τ *on* Y is denoted by $\ell(\tau)$ and defined by $\ell(\tau) = \{\emptyset\} \cup \{U \in \tau : Y \setminus U \text{ is } \tau\text{-Lindelöf}\}$, considered by Gauld, Mrsevic, Reilly and Vamanamurthy [5]. The *almost coLindelöf topology of* τ *on* Y is denoted by $q(\tau)$ and it has as a base $q'(\tau) = \{U \in RO(Y, \tau) : Y \backslash U \text{ is } \tau\text{-Lindelöf}\}$. These topologies are considered by Konstadilaki-Savvopoulou and Reilly [8].

By a multifunction $F : (X, \sigma) \to (Y, \tau)$, we mean a point-to-set correspondence from (X, σ) into (Y, τ), and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For each $B \subseteq Y$,

$$F^+(B) = \{x \in X : F(x) \subseteq B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

For each $A \subseteq X$,

$$F(A) = \bigcup_{x \in A} F(x).$$

Then F is said to be a *surjection* if $F(X) = Y$, or equivalently, if for each $y \in Y$ there exists an $x \in X$ such that $y \in F(x)$.

Moreover $F : (X, \sigma) \to (Y, \tau)$ is called *upper semicontinuous*, abbreviated as *u.s.c.* (resp. *lower semicontinuous*, abbreviated as *l.s.c.*) if $F^+(V)$ (resp. $F^-(V)$) is open in (X, τ) for every open set V of (Y, τ).

2. Almost ℓ-continuous multifunctions

We now introduce a new class of multifunctions between topological spaces with the following definition.

2.1. Definition. A multifunction $F : X \to Y$ is called

(a) Upper almost ℓ-continuous, or *u.a.ℓ*-continuous at $x \in X$, if for each regular open subset V of Y with $F(x) \subseteq V$ and having Lindelöf complement, there exists an open neighbourhood U of x such that $F(U) \subseteq V$.

(b) Lower almost ℓ-continuous, or *l.a.ℓ*-continuous at $x \in X$, if for each regular open subset V of Y with $F(x) \cap V \neq \emptyset$ and having Lindelöf complement, there exists an open neighbourhood U of x such that $F(z) \cap V \neq \emptyset$ for every point $z \in U$.

(c) Almost ℓ-continuous at $x \in X$ if it is both *u.a.ℓ*-continuous and *l.a.ℓ*-continuous at $x \in X$.

(d) Almost ℓ-continuous (resp. *u.a.ℓ*-continuous, *l.a.ℓ*-continuous) if it is almost ℓ-continuous (resp. *u.a.ℓ*-continuous, *l.a.ℓ*-continuous) at each point of X.

2.2. Theorem. *The following conditions are equivalent for a multifunction* $F : (X, \sigma) \to (Y, \tau)$.

(a) *F is upper almost ℓ-continuous.*

(b) *$F^+(V)$ is open for any regular open set V having Lindelöf complement in Y.*

(c) *$F^+(V)$ is open for each $V \in q'(\tau)$.*

(d) *$F^-(V)$ is closed for any regular closed Lindelöf subset V of Y.*

(e) *For each $x \in X$ and each net (x_α) which converges to x in X, and for each regular open subset V with $Y \setminus V$ Lindelöf such that $x \in F^+(V)$, the net (x_α) is eventually in $F^+(V)$.*

Proof. (a)$\Longrightarrow$(b). Let V be any regular open subset in Y having Lindelöf complement. Let $x \in F^+(V)$. Then there exists an open U containing x such that $F(U) \subseteq V$, hence $x \in U \subseteq F^+(V)$. This shows that $F^+(V)$ is open.

(b)$\Longrightarrow$(a). Let $x \in X$ and V be any regular open subset of Y with $F(x) \subseteq V$ and having Lindelöf complement in Y. Then $x \in F^+(V)$ and $F^+(V)$ is open. Put $U = F^+(V)$, hence U is an open neighbourhood of x and $F(U) \subseteq V$.

(b) $\Longleftrightarrow$ (c). Obvious.

(b) $\Longleftrightarrow$ (d). This follows from the fact that $F^+(Y \setminus B) = X \setminus F^-(B)$ for any subset B of Y.

(a)$\Longrightarrow$(e). Let $J = (x_\alpha)$ be a net which converges to $x \in X$ and let V be a regular open set with $Y \setminus V$ Lindelöf such that $x \in F^+(V)$. From (a) there exists an open set $U \subseteq X$ containing x such that $U \subseteq F^+(V)$. Since (x_α) converges to x, it follows that there exists $x_{\alpha_0} \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. Therefore $x_\alpha \in F^+(V)$ for all $\alpha \geq \alpha_0$. Hence the net (x_α) is eventually in $F^+(V)$.

(e)$\Longrightarrow$(a). Suppose that (a) is not true. Then there exists a point $x \in X$ and a regular open subset V of Y having Lindelöf complement, with $F(x) \subseteq V$ such that $F(U) \not\subseteq V$ for each open set $U \subseteq X$ containing x. Then for the neighborhood net (x_U), $x_U \to x$, but (x_U) is not eventually in $F^+(V)$. This is a contradiction. $\qquad\square$

Similarly, we can obtain the following characterizations of lower almost ℓ-continuity for multifunctions.

2.3. Theorem. *The following conditions are equivalent for a multifuction $F : (X, \sigma) \to (Y, \tau)$.*

  (a) *F is lower almost ℓ-continuous.*
  (b) *$F^-(V)$ is open for any regular open set V having Lindelöf complement in Y.*
  (c) *$F^-(V)$ is open for each $V \in q'(\tau)$.*
  (d) *$F^+(V)$ is closed for any regular closed Lindelöf subset V of Y.*
  (e) *For each $x \in X$ and for each net (x_α) which converges to x in X, and for each regular open subset V having Lindelöf complement such that $x \in F^-(V)$, the net (x_α) is eventually in $F^-(V)$.*

From the definitions it is clear that the multifunction F is *u.s.c.* (resp. *l.s.c.*) implies that F is *u.a.ℓ-*continuous (resp. *l.a.ℓ-*continuous). The following example shows that these implications are not reversible in general.

2.4. Example. Let X and Y be the set $\mathbb{R}$ of real numbers, σ and τ be the usual and cocountable topologies on X and Y, respectively. Let $\mathbb{Q}$ be the set of all rational numbers. Define the multifunction $F : (X, \sigma) \to (Y, \tau)$ as follows:

$$F(x) = \begin{cases} \{x\}, & x \text{ is irrational} \\ \mathbb{Q} & x \text{ is rational} \end{cases}$$

Then the multifunction F is *u.a.ℓ-*continuous by Theorem 2.2 since $q'(\tau) = \{Y, \emptyset\}$. In fact F is *a.ℓ-*continuous. However F is not *u.s.c.* or *l.s.c.* since $V = \mathbb{R} \setminus \mathbb{Q}$ is open in (Y, τ) but $F^+(V)$ and $F^-(V)$ are not open in (X, σ).

The next result shows that there exists an obvious change of topology which reduces *l.a.ℓ-*continuity to lower semi-continuity.

2.5. Theorem. *Let $F : (X, \sigma) \to (Y, \tau)$ be a multifunction. Then $F : (X, \sigma) \to (Y, \tau)$ is l.a.ℓ-continuous if and only if $F : (X, \sigma) \to (Y, q(\tau))$ is l.s.c.*

Proof. $\Longrightarrow$. Let $V \in q(\tau)$. We can write $V = \bigcup_{\alpha \in \Lambda} V_\alpha$, where V_α is a regular open set having Lindelöf complement for $\alpha \in \Lambda$. We have $F^-(\bigcup_{\alpha \in \Lambda} V_\alpha) = \bigcup_{\alpha \in \Lambda} F^-(V_\alpha)$. From Theorem 2.3, $F^-(V_\alpha)$ is an open set for $\alpha \in \Lambda$, so $F^-(V)$ is an open set. Hence $F : (X, \sigma) \to (Y, q(\tau))$ is *l.s.c.*

$\Longleftarrow$. Obvious. $\qquad\square$

A result analogous to Theorem 2.5 does not hold for upper almost ℓ-continuity, as the following example shows.

2.6. Example. Let $X = Y = \{1, 2, 3, 4\}$ and let $\sigma = \{\{1\}, \{1, 3, 4\}, X, \emptyset\}$ be the topology on X and $\tau = \{\{1\}, \{2\}, \{1, 2\}, Y, \emptyset\}$ the topology on Y. Let F be defined as

$$F(1) = \{4\}, \ F(2) = \{1, 2\}, \ F(3) = \{3\}, \ F(4) = \{4\}.$$

The family $q'(\tau) = \{\{1\}, \{2\}, Y, \emptyset\}$ is a base consisting of regular open sets having Lindelöf complement in Y for $q(\tau) = \tau$. Then for any $V \in q'(\tau)$ we have $F^+(V) \in \sigma$. Therefore $F : (X, \sigma) \to (Y, \tau)$ is $u.a.\ell$-continuous by Theorem 2.2. The topology $q(\tau)$ contains the set $\{1, 2\}$ but $F^+(\{1, 2\}) = \{2\} \notin \sigma$. Hence $F : (X, \sigma) \to (Y, q(\tau))$ is not $u.s.c.$

2.7. Proposition. ([8]) *If (Y, τ) is Lindelöf and semiregular, then $q(\tau) = \tau$.*

2.8. Theorem. *Let X, Y be topological spaces with Y Lindelöf and semiregular, and let $F : (X, \sigma) \to (Y, \tau)$ be a multifunction. Then F is $l.a.\ell$-continuous if and only if F is $l.s.c.$*

Proof. This follows directly from Theorem 2.5 and Proposition 2.7. $\qquad\square$

A result analogous to Theorem 2.8 does not hold for $u.a.\ell$-continuity, as Example 2.6 shows.

The following definition and characterization are given by Kucuk [9].

2.9. Definition. A multifunction $F : X \to Y$ is called

(a) Almost c-upper semi continuous, or $a.c.u.s.c.$, at $x \in X$ if for each compact set C with $F(x) \cap C = \emptyset$, there exists an open neighbourhood U of x such that $F(z) \cap \operatorname{cl}(\operatorname{int}(C)) = \emptyset$ for $z \in U$.

(b) Almost c-lower semi continuous, or $a.c.l.s.c.$, at $x \in X$ if whenever $Y \setminus V$ is compact and $F(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x such that $F(z) \cap \operatorname{int}(\operatorname{cl}(V)) \neq \emptyset$ for $z \in U$.

(c) Almost c-continuous at $x \in X$ if it is both $a.c.u.s.c.$ and $a.c.l.s.c.$ at $x \in X$.

(d) Almost c-continuous if it is almost c-continuous at each point of X.

2.10. Theorem. *Let $F : (X, \sigma) \to (Y, \tau)$ be a multifuction and Y a Hausdorff space. Then:*

(a) *F is $a.c.u.s.c.$ at x if and only if for each open subset V of Y with compact complement and satisfying $F(x) \subseteq V$, there exists an open neighborhood U of x such that $F(z) \subseteq \operatorname{int}(\operatorname{cl}(V))$ for $z \in U$.*

(b) *F is $a.c.l.s.c.$ at x if and only if for each open subset V of Y with compact complement and satisfying $F(x) \cap V \neq \emptyset$, there exists an open neighbourhood U of x such that $F(z) \cap \operatorname{int}(\operatorname{cl}(V) \neq \emptyset$ for $z \in U$.*

The following example shows that the multifunction F being $u.s.c.$ ($l.s.c.$) does not imply that F is $a.c.u.s.c.$ ($a.c.l.s.c.$). Hence F being $u.a.\ell$-continuous ($l.a.\ell$-continuous) does not imply that F is $a.c.u.s.c.$ ($a.c.l.s.c.$).

2.11. Example. Let us redefine the topology σ in Example 2.6 as follows

$$\sigma = \{\{1\}, \{2\}, \{1, 2\}, \{1, 3, 4\}, X, \emptyset\}.$$

No other changes are made to Example 2.6. It is obvious that F is $u.s.c.$ ($l.s.c.$) and hence $u.a.\ell$-continuous ($l.a.\ell$-continuous). However F is not almost c-continuous. For example, x is not $a.c.u.s.c.$ at 4 since $C = \{1, 2, 3\}$ is compact and $F(4) \cap C = \emptyset$ but there is no σ-open U containing 4 such that $F(x) \cap \operatorname{cl}(\operatorname{int}(C)) = \emptyset$ for each $x \in U$. Similarly F is not $a.c.l.s.c.$ at $x = 4$.

Now we introduce a new class of multifunctions which is larger than the class of almost ℓ-continuous multifunctions.

2.12. Definition. A multifunction $F : X \to Y$ is called

(a) Upper K-almost c-continuous, or $u.K$-$a.c$-continuous, at $x \in X$ if for each regular open subset V of Y with $F(x) \subseteq V$ and having compact complement, there exists an open neighbourhood U of x such that $F(U) \subseteq V$.

(b) Lower K-almost c-continuous, or $l.K$-$a.c$-continuous, at $x \in X$ if for each regular open subset V of Y with $F(x) \cap V \neq \emptyset$ and having compact complement, there exists an open neighbourhood U of x such that $F(z) \cap V \neq \emptyset$ for every point $z \in U$.

(c) K-almost c-continuous at $x \in X$ if it is both $u.K$-$a.c$-continuous and $l.K$-$a.c$-continuous at $x \in X$.

(d) K-almost c-continuous (resp. $u.K$-$a.c$-continuous, $l.K$-$a.c$-continuous) if it is K-almost c-continuous (resp. $u.K$-$a.c$-continuous, $l.K$-$a.c$-continuous) at each point of X.

From the definitions it is clear that F is almost ℓ-continuous implies that F is K-almost c-continuous. But the reverse implication does not hold in general. A simple example with single-valued functions may be given (see Example 2 of [8]).

2.13. Proposition. *Let $F : (X, \sigma) \to (Y, \tau)$ be a multifunction. If F is a.c.u.s.c. (a.c.l.s.c.), then F is $u.K$-$a.c$-continuous ($l.K$-$a.c$-continuous).*

Proof. Let F be *a.c.u.s.c.* at x. Let V be a regular open subset of Y having compact complement and satisfing $F(x) \subseteq V$. Then $Y \setminus V$ is compact and $F(x) \cap (Y \setminus V) = \emptyset$. Since F is *a.c.u.s.c.*, there exists an open neighborhood U of x satisfying $F(z) \cap \mathrm{cl}(\mathrm{int}(Y \setminus V)) = \emptyset$ such that $z \in U$. Thus $F(U) \subseteq \mathrm{int}(\mathrm{cl}(V))$. Hence F is $u.K$-$a.c$-continuous.

The proof for $l.K$-$a.c$-continuity is similar. $\qquad\square$

The reverse implication of Proposition 2.13 does not hold in general, as Example 2.11 shows.

2.14. Proposition. *Let $F : (X, \sigma) \to (Y, \tau)$ be a multifunction and Y a Hausdorff space. Then F is $u.K$-$a.c$-continuous ($l.K$-$a.c$-continuous) if and only if F is a.c.u.s.c. (a.c.l.s.c.).*

Proof. This follows from Theorem 2.10. $\qquad\square$

The following two results are analogous to Theorem 2.2 and Theorem 2.3, respectively, so we omit their proofs.

2.15. Theorem. *The following conditions are equivalent for a multifunction $F : (X, \sigma) \to (Y, \tau)$.*

(a) *F is upper K-almost c-continuous.*

(b) *$F^+(V)$ is open for any regular open set V having compact complement in Y.*

(c) *$F^+(V)$ is open for each $V \in e'(\tau)$.*

(d) *$F^-(V)$ is closed for any regular closed compact subset V of Y.*

(e) *For each $x \in X$, for each net (x_α) which converges to x in X, and for each regular open subset V with $Y \setminus V$ compact such that $x \in F^+(V)$, the net (x_α) is eventually in $F^+(V)$.*

2.16. Theorem. *The following conditions are equivalent for a multifuction $F : (X, \sigma) \to (Y, \tau)$.*

(a) *F is lower K-almost c-continuous.*

(b) $F^-(V)$ *is open for any regular open set V having compact complement in Y.*
(c) $F^-(V)$ *is open for each $V \in e'(\tau)$.*
(d) $F^+(V)$ *is closed for any regular closed compact subset V of Y.*
(e) *For each $x \in X$, for each net (x_α) which converges to x in X, and for each regular open subset V having compact complement such that $x \in F^-(V)$, the net (x_α) is eventually in $F^-(V)$.*

2.17. Theorem. *Let $F : (X, \sigma) \to (Y, \tau)$ be a multifunction. Then $F : (X, \sigma) \to (Y, \tau)$ is l.K-a.c-continuous if and only if $F : (X, \sigma) \to (Y, e(\tau))$ is l.s.c.*

Proof. $\Longrightarrow$. Let $V \in e(\tau)$. We can write $V = \bigcup_{\alpha \in \Lambda} V_\alpha$, where $V_\alpha \in e'(\tau)$ for $\alpha \in \Lambda$. We have $F^-(\bigcup_{\alpha \in \Lambda} V_\alpha) = \bigcup_{\alpha \in \Lambda} F^-(V_\alpha)$. From Theorem 2.16, $F^-(V_\alpha)$ is an open set for $\alpha \in \Lambda$. So $F^-(V)$ is an open set. Hence $F : (X, \sigma) \to (Y, e(\tau))$ is l.s.c.

$\Longleftarrow$. Obvious. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A result analogous to Theorem 2.17 for upper K-almost c-continuity does not hold as Example 2.6 shows.

2.18. Proposition. *If (Y, τ) is compact and semiregular, then $e(\tau) = \tau$.*

2.19. Theorem. *Let X, Y be topological spaces with Y compact and semiregular, and let $F : (X, \sigma) \to (Y, \tau)$ be a multifunction. Then F is l.K-a.c-continuous if and only if F is l.s.c.*

Proof. This follows directly from Theorem 2.17 and Proposition 2.18. $\qquad\qquad\square$

A result analogous to Theorem 2.19 for $u.K$-$a.c$-continuity does not hold in general as Example 2.6 shows.

2.20. Proposition. *Let $F : (X, \sigma) \to (Y, \tau)$ be a multifunction. If $F : (X, \sigma) \to (Y, q(\tau))$ is u.s.c. (l.s.c.), then $F : (X, \sigma) \to (Y, e(\tau))$ is u.s.c. (l.s.c.).*

Proof. This follows from the fact that $e(\tau) \subseteq q(\tau)$. $\qquad\qquad\qquad\qquad\qquad\square$

Sakalova [11] introduced several variations of continuity for multifunctions. Three of these notions are important for our discussion, and we recall their definitions below.

2.21. Definition. Let $F : (X, \sigma) \to (Y, \tau)$ be a multifunction.

(a) F is upper (lower) c-continuous if and only if for any $V \in \tau$ such that the complement of V is compact we have $F^+(V) \in \sigma$ ($F^-(V) \in \sigma$).
(b) F is upper (lower) ℓ-continuous if and only if for any $V \in \tau$ such that the complement of V is Lindelöf we have $F^+(V) \in \sigma$ ($F^-(V) \in \sigma$).
(c) F is upper (lower) almost continuous if and only if for any $V \in \tau$ such that $V = \text{int}(\text{cl}(V))$ we have $F^+(V) \in \sigma$ ($F^-(V) \in \sigma$).

From the definitions it is clear that the class of almost ℓ-continuous multifunctions lies between the class of almost continuous multifunctions and the class of K-almost c-continuous multifunctions. Also, the multifunction F is ℓ-continuous implies that F is almost ℓ-continuous. However, these implications are not reversible in general. Simple examples with single-valued functions may be given (see Examples 1, 2 and 3 of [8]).

2.22. Corollary.

(a) *If $F : (X, \sigma) \to (Y, \tau)$ is l.a.ℓ-continuous, then $F : (X, \sigma) \to (Y, q(\tau))$ is lower ℓ-continuous.*
(b) *If $F : (X, \sigma) \to (Y, \tau)$ is l.K-a.c-continuous, then $F : (X, \sigma) \to (Y, e(\tau))$ is lower c-continuous.*

Proof. (a) This follows from Theorem 2.5.

(b) This follows from Theorem 2.17. □

Example 2.6 shows that results analogous to Corollary 2.22 for upper almost ℓ-continuity and for upper K-almost c-continuity do not hold.

2.23. Theorem.
   (a) *If $F : (X, \sigma) \to (Y, \tau)$ is u.a.ℓ-continuous (l.a.ℓ-continuous) and (Y, τ) is Lindelöf, then $F : (X, \sigma) \to (Y, \tau)$ is upper almost continuous (lower almost continuous).*
   (b) *If $F : (X, \sigma) \to (Y, \tau)$ is u.K-a.c-continuous (l.K-a.c-continuous) and (Y, τ) is compact, then $F : (X, \sigma) \to (Y, \tau)$ is upper almost continuous (lower almost continuous).*

Proof. (a) Let F be u.a.ℓ-continuous and V a regular open subset of (Y, τ). Then $Y \setminus V$ is a Lindelöf subset of (Y, τ), therefore $V \in q'(\tau)$. Since F is u.a.ℓ-continuous, $F^+(V) \in \sigma$ by Theorem 2.2. Hence F is upper almost continuous.

The proof for lower almost continuity is similar.

(b) Let F be u.K-a.c-continuous and V a regular open subset of (Y, τ). Then $Y \setminus V$ is a compact subset of (Y, τ), therefore $V \in e'(\tau)$. Since F is u.K-a.c-continuous, $F^+(V) \in \sigma$ by Theorem 2.15. Hence F is upper almost continuous.

The proof for lower almost continuity is similar. □

Once again, a change of topology is crucial to the discussion.

The proof of the following proposition is straightforward, so we omit it.

2.24. Proposition.
   (a) *The multifunction $F : (X, \sigma) \to (Y, \tau)$ is upper ℓ-continuous (lower ℓ-continuous) if and only if $F : (X, \sigma) \to (Y, \ell(\tau))$ is u.s.c. (l.s.c).*
   (b) *The multifunction $F : (X, \sigma) \to (Y, \tau)$ is upper c-continuous (lower c-continuous) if and only if $F : (X, \sigma) \to (Y, c(\tau))$ is u.s.c. (l.s.c).*

2.25. Proposition. [3] *If (Y, τ) is a compact space, then $c(\tau) = \tau$.*

2.26. Proposition. [5]
   (a) *If (Y, τ) is a Lindelöf space, then $\ell(\tau) = \tau$.*
   (b) *If (Y, τ) is any topological space, then $c(\tau) \subseteq \ell(\tau)$.*

2.27. Proposition. *If the multifunction $F : (X, \tau) \to (Y, \ell(\tau))$ is u.s.c. (l.s.c.), then the multifunction $F : (X, \sigma) \to (Y, c(\tau))$ is u.s.c. (l.s.c.).*

Proof. This follows from Proposition 2.26 (b). □

2.28. Corollary. *If the multifunction $F : (X, \sigma) \to (Y, \tau)$ is upper ℓ-continuous (lower ℓ-continuous), then the multifunction $F : (X, \sigma) \to (Y, \tau)$ is upper c-continuous (lower c-continuous).*

Proof. This follows from Propositions 2.24 and 2.27. □

The reverse implication of Corollary 2.28 does not hold in general. A simple example with single-valued functions may be given (see [7]).

We have the following corollary by Propositions 2.24, 2.25 and 2.26.

2.29. Corollary. *Let $F : (X, \sigma) \to (Y, \tau)$ be a multifunction.*

(a) *Let (Y, τ) be a compact space. Then $F : (X, \sigma) \to (Y, \tau)$ is upper c-continuous (lower c-continuous) if and only if $F : (X, \sigma) \to (Y, \tau)$ is u.s.c. (l.s.c.).*

(b) *Let (Y, τ) be a Lindelöf space. Then $F : (X, \sigma) \to (Y, \tau)$ is upper ℓ-continuous (lower ℓ-continuous) if and only if $F : (X, \sigma) \to (Y, \tau)$ is u.s.c (l.s.c.).*

Recall that a topological space whose Lindelöf subsets are closed is called an an *LC-space* by Mukherji and Sarkar [10], and by Gauld, Mrsevic, Reilly and Vamanamurthy [5].

2.30. Theorem. *A topological space (Y, τ) is an LC-space if and only if the following is true. For any topological space X and any multifunction $F : X \to Y$ we have F is upper ℓ-continuous (lower ℓ-continuous) if and only if $F^-(L)$ $(F^+(L))$ is closed for any Lindelöf $L \subseteq Y$.*

Proof. $\Longrightarrow$. Let F be upper ℓ-continuous and let L be any Lindelöf subset of Y. Since Y is an LC-space, L is closed. Since F is upper ℓ-continuous, $F^-(L)$ is closed. Let V be an open set having Lindelöf complement in Y. By hypothesis $F^-(Y \setminus V)$ is closed. But $F^+(V) = X \setminus F^-(V)$. So $F^+(V)$ is open. Hence F is upper ℓ-continuous. The proof for lower ℓ-continuity is similar.

$\Longleftarrow$. Suppose (Y, τ) is not an LC-space. It is sufficient to find a single-valued function $f : X \to Y$ such that f is ℓ-continuous but $f^{-1}(L)$ is not closed for some Lindelöf set $L \subseteq Y$. Let X and Y be the set $\mathbb{R}$ of real numbers, σ and τ be the usual and cofinite topologies on X and Y, respectively. Let $f : X \to Y$ be the identity function and $\mathbb{Q}$ the set of all rational numbers. Then f is an ℓ-continuous function and $\mathbb{Q}$ a Lindelöf subset of Y, but $f^{-1}(\mathbb{Q})$ is not closed. This is a contradiction. Hence (Y, τ) is an LC-space. $\square$

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