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CHARACTERIZATIONS OF REAL DIFUNCTIONS

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Abstract

The notion of difunction between textures has proved to be of considerable interest and importance. In this paper the authors consider real difunctions, that is difunctions from a given texture $(S, \mathcal{S})$ to the real texture $(\mathbb{R}, \mathcal{R})$, and seek representations of such difunctions in terms of ordinary point functions. It is shown that in general real difunctions cannot be represented in terms of real-valued point functions on S, but that they can be represented by real-valued point functions on the core $S^\flat$ of S. Equivalently, it is shown that instead of restricting to the core of S, the real texture $(\mathbb{R}, \mathcal{R})$ may be replaced by the extended real texture $(\mathbb{R}^+, \mathcal{R}^+)$ and representations obtained in terms of point functions from S to $\mathbb{R}^+$.

Keywords: Texture, Ditopology, Real texture, Real difunction, Point function, Interval-valued function, Bicontinuity, Extended real texture.

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1. Introduction

There is now a considerable literature on the theory of ditopological texture spaces, and an adequate introduction to this theory and the motivation for its study may be obtained from [2, 3, 4, 5, 6, 7].

For a texture $(S, \mathcal{S})$, most properties are conveniently defined in terms of the *p-sets* $P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}$ and the *q-sets*, $Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\}$. However, as noted in [1] we may associate with $(S, \mathcal{S})$ the C-space (core-space) [9, 10, 11, 13, 14] $(S, \mathcal{S}^c)$, and then the frequently occurring relationship $P_{s'} \not\subseteq Q_s$, $s, s' \in S$, is equivalent to $s\,\omega_S\,s'$, where ω_S is the *interior relation* for $(S, \mathcal{S}^c)$. In this paper we will use whichever notation seems to be the more convenient in each particular instance.

We will be especially interested in the *real texture* $(\mathbb{R}, \mathcal{R})$, where $\mathbb{R}$ denotes the set of real numbers and $\mathcal{R}$ is the texturing $\{(-\infty, r] \mid r \in \mathbb{R}\} \cup \{(-\infty, r) \mid r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$.

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The natural ditopology on $(\mathbb{R}, \mathcal{R})$ is $(\tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$, where $\tau_{\mathbb{R}} = \{(-\infty, r) \mid r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$ and $\kappa_{\mathbb{R}} = \{(-\infty, r] \mid r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. Clearly for $r \in \mathbb{R}$ we have $P_r = (-\infty, r]$, $Q_r = (-\infty, r)$, so $P_r \not\subseteq Q_{r'} \iff r' \leq r \iff r' \, \omega_{\mathbb{R}} \, r$.

The notion of difunction [5, Definition 2.22] is of particular importance, and one of the main categories of textures considered to date is the category **dfTex** of textures and difunctions. Likewise, **dfDitop** denotes the category of ditopological texture spaces and bicontinuous difunctions. **dfDitop** is known to be topological over **dfTex** [6, Theorem 3.6]. However, we will not be considering categorical aspects of our work in this paper.

Our main object of study in this paper will be the family $\mathrm{DF}(S)$ of real difunctions on a texture $(S, \mathcal{S})$, that is the family of difunctions $(f, F) : (S, \mathcal{S}) \to (\mathbb{R}, \mathcal{R})$. This family was first studied in [8], where it was shown that the operations of meet, join and sum on $\mathbb{R}$ extend to corresponding operations on $\mathrm{DF}(S)$, the result being an additive lattice. In the case where $(S, \mathcal{S})$ has a ditopology, (τ, κ), and $(\mathbb{R}, \mathcal{R})$ is given its natural ditopology, it is also shown in [8] that the above operations on $\mathrm{DF}(S)$ restrict naturally to the family $\mathrm{BDF}(S)$ of bicontinuous real difuctions on $(S, \mathcal{S}, \tau, \kappa)$. We will interested in the way that the representations we obtain for the elements of $\mathrm{DF}(S)$, and more particularly of $\mathrm{BDF}(S)$, relate with the operations defined in [8].

In general difunctions are not directly related to ordinary (point) functions between the base sets, but we recall from [5, Lemma 3.4] that if $(S, \mathcal{S})$, $(T, \mathcal{T})$ are textures and $\varphi : S \to T$ a point function satisfying the compatibility condition

(a) $P_s \not\subseteq Q_{s'} \implies P_{\varphi(s)} \not\subseteq Q_{\varphi(s')}$,

then the formulae

$$(1.1) \qquad \begin{aligned} f_\varphi &= \bigvee \{\overline{P}_{(s,t)} \mid \exists\, u \in S \text{ with } P_s \not\subseteq Q_u \text{ and } P_{\varphi(u)} \not\subseteq Q_t\}, \\ F_\varphi &= \bigcap \{\overline{Q}_{(s,t)} \mid \exists\, v \in S \text{ with } P_v \not\subseteq Q_s \text{ and } P_t \not\subseteq Q_{\varphi(v)}\}, \end{aligned}$$

define a difunction (f_φ, F_φ) from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Moreover, it is easy to verify that for each $B \in \mathcal{T}$ we have $f_\varphi^{\leftarrow} B = \varphi^{\leftarrow} B = F_\varphi^{\leftarrow} B$, where

$$(1.2) \qquad \varphi^{\leftarrow} B = \bigvee \{P_u \mid \varphi(u) \in B\} = \bigcap \{Q_v \mid \varphi(v) \notin B\}.$$

It will be noted that the compatibility condition (a) merely expresses the fact that φ preserves the interior relation, so a point function satisfying this condition could be referred to as *ω-preserving*, although for compatibility with [8] we will continue to say that it satisfies condition (a) throughout this paper.

Conversely, if $(S, \mathcal{S})$ is plain or $(T, \mathcal{T})$ is simple, then each difunction may be represented as in (1.1) by a unique point function φ satisfying (a) and the additional condition

(b) $P_{\varphi(s)} \not\subseteq B$, $B \in \mathcal{T} \implies \exists\, s' \in S$ with $P_s \not\subseteq Q_{s'}$ for which $P_{\varphi(s')} \not\subseteq B$

of [5, Propositions 3.6,3.7]). In the general case, however, it is known that there may be no function $\varphi : S \to T$ satisfying (a) for which $(f, F) = (f_\varphi, F_\varphi)$ (see [6, Example 2.14]). Example 2.1 below shows that even for real difunctions $(f, F) : (S, \mathcal{S}) \to (\mathbb{R}, \mathcal{R})$ we may again have no such function $\varphi : S \to \mathbb{R}$. On the other hand we do show that we may represent such difunctions in terms of suitable point functions $\varphi : S^\flat \to \mathbb{R}$, where $S^\flat = \{s \in S \mid Q_s \neq S\}$ is the core of S, and the study of representations based on this fact occupy § 2.

The representations of real difunctions based on real point functions on the core of S have the potential disadvantage that in general the core of S need not belong to the texturing $\mathcal{S}$, and is therefore external to the texture $(S, \mathcal{S})$. For this reason, alternative characterizations based on extended-real point functions $\varphi : S \to \mathbb{R}^+$, where $\mathbb{R}^+ = \mathbb{R} \cup \{\infty\}$, are considered in § 3.

The reader is referred to [12] for terms from lattice theory not defined here.

2. Characterization of real difunctions

In the present paper we will be interested in difunctions between an arbitrary texture $(S, \mathcal{S})$ and the real texture $(\mathbb{R}, \mathcal{R})$. Since we are not assuming that $(S, \mathcal{S})$ is plain, and since $(\mathbb{R}, \mathcal{R})$ is plain but not simple, the results mentioned in the introduction do not guarantee the existence of a point function $\varphi : S \to \mathbb{R}$ satisfying (a) for which $(f, F) = (f_\varphi, F_\varphi)$. The following example shows that indeed such a function may not exist.

2.1. Example. Consider the texture $(L, \mathcal{L})$ of [5, Examples 1.1 (3)] and define

$$f = \{(l, r) \mid 0 < l < 1, r < \frac{l}{1-l}\} \cup \{(1, r) \mid r \in \mathbb{R}\},$$
$$F = \{(l, r) \mid 0 < l < 1, r \leq \frac{l}{1-l}\} \cup \{(1, r) \mid r \in \mathbb{R}\}.$$

It is straightforward to check that f is a relation and F a corelation from $(L, \mathcal{L})$ to $(\mathbb{R}, \mathcal{R})$. To show that (f, F) is a difunction we must verify conditions *DF1* and *DF2* of [5, Definition 2.22].

Take $l, l' \in L$ with $P_l \not\subseteq Q_{l'}$. Then $l' < l$ and we may take $u \in L$ with $l' < u < l$. If we set $t = \dfrac{u}{1-u}$ it is easy to verify that $f \not\subseteq \overline{Q}_{(l,t)}$ and $\overline{P}_{(l',t)} \not\subseteq F$, which establishes *DF1*.

For *DF2* take $l \in L$ and $r, r' \in \mathbb{R}$ with $f \not\subseteq \overline{Q}_{(l,r)}$, $\overline{P}_{(l,r')} \not\subseteq F$. By the definition of F the second condition gives $l \neq 1$, so $0 < l < 1$ and $r' > \frac{l}{1-l}$. On the other hand the first condition now gives $r < \frac{l}{1-l}$, and we deduce $r < r'$, that is $P_{r'} \not\subseteq Q_r$ as required.

Suppose now that the difunction $(f, F) : (L, \mathcal{L}) \to (\mathbb{R}, \mathcal{R})$ can be obtained from a function $\varphi : L \to \mathbb{R}$ as in [5, Lemma 3.4]. Then

$$F = F_\varphi = \bigcap \{\overline{Q}_{(l,r)} \mid \exists\, u \in L \text{ satisfying } P_u \not\subseteq Q_l,\ P_r \not\subseteq Q_{\varphi(u)}\},$$

and we obtain $F \subseteq \overline{Q}_{(l,\varphi(1))} = (L \times (-\infty, \varphi(1))) \cup ((0, l] \times \mathbb{R})$ for any $l < 1$. However $(1, \varphi(1) + 1) \in F \setminus \overline{Q}_{(l,\varphi(1))}$, which is a contradiction.

The difficulty in the above example occurs at a point which is not in the core $S^\flat$ of the base set S, and this suggests that we should consider instead functions $\varphi : S^\flat \to \mathbb{R}$. We shall also need to weaken the condition (b), and in the following definition we also give a corresponding weakening of the condition (c) of [6, Lemma 3.8] for functions satisfying (a), as well as restating (a) in a form suitable for our current investigation.

2.2. Definition. Let $(S, \mathcal{S})$ be a texture. The conditions (a), (b*) and (c*) for a point function $\varphi : S^\flat \to \mathbb{R}$ are defined as follows:

(a) $s, s' \in S^\flat$, $s' \,\omega_S\, s \implies \varphi(s') \leq \varphi(s)$.
(b*) $s \in S^\flat, r \in \mathbb{R},\ r < \varphi(s) \implies \exists\, s' \in S^\flat$ with $s' \,\omega_S\, s,\ r < \varphi(s')$.
(c*) $s \in S^\flat, r \in \mathbb{R},\ \varphi(s) < r \implies \exists\, s' \in S^\flat$ with $s \,\omega_S\, s',\ \varphi(s') < r$.

It will be noted that if $\varphi : S \to \mathbb{R}$ satisfies (b), and $r < \varphi(s)$ for $s \in S^\flat$, then $P_{\varphi(s)} \not\subseteq P_r$ and so we have $s' \in S$ with $s' \,\omega_S\, s$, $P_{\varphi(s')} \not\subseteq P_r$. This gives $r < \varphi(s')$, while clearly $Q_{s'} \neq S$ so $s' \in S^\flat$. Hence (b*) is indeed a weakening of (b), and likewise, in the presence of (a), (c*) is a weakening of (c).

In order to present our basic characterization of real difunctions in terms of real-valued point functions we will make essential use of the following result, which is closely related to [5, Lemma 3.4].

2.3. Lemma. *Let $\varphi : S^\flat \to \mathbb{R}$ be a point function. Then:*

(1) *The equalities*

$$f_\varphi = \bigvee\{\overline{P}_{(s,\varphi(u))} \mid s \in S,\ u\,\omega_S\, s,\ u \in S^\flat\},$$

$$F_\varphi = \bigcap\{\overline{Q}_{(s,\varphi(u))} \mid s \in S,\ s\,\omega_S\, u,\ u \in S^\flat\},$$

define a difunction from $(S, \mathcal{S})$ to $(\mathbb{R}, \mathcal{R})$ if and only if φ satisfies (a).

(2) *If φ satisfies condition (a) then $f_\varphi^\leftarrow B = F_\varphi^\leftarrow B = \varphi^\leftarrow B$ for all $B \in \mathcal{R}$, where $\varphi^\leftarrow B = \bigvee\{P_s \mid s \in \varphi^{-1}[B]\} = \bigcap\{Q_s \mid s \notin \varphi^{-1}[B]\}$.*

(3) *If φ satisfies (a) and $(b*)$ then $(\varphi^\leftarrow P_r) \cap S^\flat = \varphi^{-1}[P_r]$ for all $r \in \mathbb{R}$.*

Proof. (1) Let $\varphi : S^\flat \to \mathbb{R}$ be a point function, and define f_φ, F_φ as above. It is trivial to verify that (f_φ, F_φ) is a direlation. Moreover, if $P_s \not\subseteq Q_{s'}$ for $s, s' \in S$ and we take $u\,\omega_S\, s$, $s'\,\omega_S\, u$, then $u \in S^\flat$ and setting $t = \varphi(u)$ leads easily to $f_\varphi \not\subseteq \overline{Q}_{(s,t)}$, $\overline{P}_{(s',t)} \not\subseteq F_\varphi$, so (f_φ, F_φ) satisfies *DF1*. It remains to prove that (f_φ, F_φ) satisfies *DF2* if and only if φ satisfies (a).

First let (f_φ, F_φ) satisfy *DF2* and take $s, s' \in S^\flat$ with $s'\,\omega_S\, s$. Let $r = \varphi(s)$, $r' = \varphi(s')$ and take $u\,\omega_S\, s$, $s'\,\omega_S\, u$. Then $f_\varphi \not\subseteq \overline{Q}_{(u,r')}$, $\overline{P}_{(u,r)} \not\subseteq F_\varphi$, so by *DF2* we have $P_r \not\subseteq Q_{r'}$, that is $\varphi(s') = r' \leq r = \varphi(s)$. Hence φ satisfies (a).

Now let φ satisfy (a), and take $s \in S$, $t, t' \in \mathbb{R}$ with $f_\varphi \not\subseteq \overline{Q}_{(s,t)}$, $\overline{P}_{(s,t')} \not\subseteq F_\varphi$. Then for some $u, u' \in S^\flat$ with $u\,\omega_S\, s$, $s\,\omega_S\, u'$ we have $P_{\varphi(u)} \not\subseteq Q_t$ and $P_{t'} \not\subseteq Q_{\varphi(u')}$. On the other hand, by (a), $P_{\varphi(s)} \not\subseteq Q_{\varphi(u)}$ and $P_{\varphi(u')} \not\subseteq Q_{\varphi(s)}$, so $P_{t'} \not\subseteq Q_t$ which verifies *DF2*.

(2) The equality of the two expressions is straightforward, and is omitted. Since $f_\varphi^\leftarrow B = F_\varphi^\leftarrow B$ it will be sufficient to show that $f_\varphi^\leftarrow B \subseteq \varphi^\leftarrow B \subseteq F_\varphi^\leftarrow B$. We establish the first inclusion, leaving the dual proof of the second inclusion to the interested reader. Suppose $f_\varphi^\leftarrow B \not\subseteq \varphi^\leftarrow B$ and take $s \in S$ with $f_\varphi^\leftarrow B \not\subseteq Q_s$, $P_s \not\subseteq \varphi^\leftarrow B$. By the definition of $f_\varphi^\leftarrow B$ we have $s' \in S$ with $s\,\omega_S\, s'$ and

$$f_\varphi \not\subseteq \overline{Q}_{(s',t)} \implies P_t \subseteq B \ \forall t \in \mathbb{R}. \tag{2.1}$$

Since clearly $s \in S^\flat$ we deduce from the definition of f_φ that $f_\varphi \not\subseteq \overline{Q}_{(s',\varphi(s))}$, whence $P_{\varphi(s)} \subseteq B$ by implication (2.1). But now $s \in \varphi^{-1}[B]$, which gives the contradiction $P_s \subseteq \varphi^\leftarrow B$.

(3) Let φ satisfy (a) and $(b*)$. Since we clearly have $\varphi^{-1}[P_r] \subseteq \varphi^\leftarrow P_r$ it remains to show that $(\varphi^\leftarrow P_r) \cap S^\flat \subseteq \varphi^{-1}[P_r]$. Suppose this is not so and take $s \in (\varphi^\leftarrow P_r) \cap S^\flat$ with $s \notin \varphi^{-1}[P_r]$. Then $\varphi(s) > r$, so by condition $(b*)$ there exists $u \in S^\flat$ with $P_s \not\subseteq Q_u$ and $\varphi(u) > r$. On the other hand $P_s \subseteq \varphi^\leftarrow P_r$ gives $\varphi^\leftarrow P_r \not\subseteq Q_u$ and hence $\varphi(u) \in P_r$, which gives the contradiction $\varphi(u) \leq r$. □

The interested reader may easily verify that the difunction defined in Example 2.1 may be represented by the function $\varphi : L^\flat = (0, 1) \to \mathbb{R}$ given by

$$l \mapsto \frac{l}{1 - l}\,.$$

The following example illustrates some of the results in Lemma 2.3.

2.4. Example. Take $(S, \mathcal{S}) = (L, \mathcal{L})$ and for $0 \leq \alpha \leq 1$ consider the function $\vartheta_\alpha : L^\flat = (0, 1) \to \mathbb{R}$ defined by

$$\vartheta_\alpha(s) = \begin{cases} 0 & 0 < s < \frac{1}{2}, \\ \alpha & s = \frac{1}{2}, \\ 1 & \frac{1}{2} < s < 1. \end{cases}$$

It is clear that ϑ_α satisfies (a) for each α. Moreover, it is straightforward to verify that ϑ_α satisfies $(b*)$ if and only if $\alpha = 0$, and it satisfies $(c*)$ if and only if $\alpha = 1$ (see also Example 2.10 below). Since $s \in \vartheta_\alpha^{-1}[P_r] \iff \vartheta_\alpha(s) \le r$ we have

$$\vartheta_\alpha^{-1}[P_r] = \begin{cases} \emptyset & \text{for } r < 0, \\ (0, \frac{1}{2}) & \text{for } 0 \le r < \alpha, \\ (0, \frac{1}{2}] & \text{for } \alpha \le r < 1, \\ (0, 1) & \text{for } 1 \le r. \end{cases}$$

On the other hand $\vartheta_\alpha^{\leftarrow} P_r = \bigvee\{P_s \mid s \in \vartheta_\alpha^{-1}[P_r]\}$, so by the above

$$\vartheta_\alpha^{\leftarrow} P_r = \begin{cases} \emptyset & \text{for } r < 0, \\ (0, \frac{1}{2}] & \text{for } 0 \le r < 1, \\ L & \text{for } 1 \le r. \end{cases}$$

We note that $\vartheta_\alpha^{-1}[P_r] = (\vartheta_\alpha^{\leftarrow} P_r) \cap L^\flat$ for any α if $r \ge 1$. On the other hand, if we restrict our attention to α satisfying $0 < \alpha \le 1$ we have $\vartheta_\alpha^{-1}[P_r] \ne (\vartheta_\alpha^{\leftarrow} P_r) \cap L^\flat$ for all $r \in \mathbb{R}$ with $0 \le r < \alpha$. This shows that condition (b^*) cannot be removed from (3). In particular ϑ_1 shows that in (3) one cannot replace $(b*)$ by $(c*)$.

In just the same way we have:

$$\vartheta_\alpha^{-1}[Q_r] = \begin{cases} \emptyset & \text{for } r \le 0, \\ (0, \frac{1}{2}) & \text{for } 0 < r \le \alpha, \\ (0, \frac{1}{2}] & \text{for } \alpha < r \le 1, \\ (0, 1) & \text{for } 1 < r, \end{cases} \qquad \vartheta_\alpha^{\leftarrow} Q_r = \begin{cases} \emptyset & \text{for } r \le 0, \\ (0, \frac{1}{2}] & \text{for } 0 < r \le 1, \\ L & \text{for } 1 < r. \end{cases}$$

Again $\vartheta_\alpha^{-1}[Q_r] = (\vartheta_\alpha^{\leftarrow} Q_r) \cap L^\flat$ for any α if $r \ge 1$. On the other hand if we consider $0 < r \le \alpha \le 1$ we have $\vartheta_\alpha^{-1}[Q_r] \ne (\vartheta_\alpha^{\leftarrow} Q_r) \cap L^\flat$. This shows that ϑ_1 is an example of a function satisfying (a) and $(c*)$ for which these sets are unequal.

If $\varphi : S^\flat \to \mathbb{R}$ satisfies (a) then for $s \in S^\flat$, $u\,\omega_S\,s$ we have $u \in S^\flat$ and $\varphi(u) \le \varphi(s)$ so $\sup\{\varphi(u) \mid u\,\omega_S\,s\} \in \mathbb{R}$. Likewise, $\inf\{\varphi(v) \mid v \in S^\flat, s\,\omega_S\,v\} \in \mathbb{R}$ and we may make the following definition.

2.5. Definition. Let $\varphi : S^\flat \to \mathbb{R}$ satisfy (a). Then the point functions $\varphi_*,\ \varphi^* : S^\flat \to \mathbb{R}$ are given by

$$\varphi_*(s) = \sup\{\varphi(u) \mid u \in S^\flat,\ u\,\omega_S\,s\},\ s \in S^\flat,$$

$$\varphi^*(s) = \inf\{\varphi(v) \mid v \in S^\flat,\ s\,\omega_S\,v\},\ s \in S^\flat.$$

2.6. Lemma. *Let $\varphi : S^\flat \to \mathbb{R}$ satisfy (a). Then*

(i) $\varphi_*(s) \le \varphi(s) \le \varphi^*(s)$ *for all* $s \in S^\flat$.
(ii) *For* $s, s' \in S^\flat$ *with* $P_s \not\subseteq Q_{s'}$ *we have* $\varphi(s') \le \varphi_*(s)$ *and* $\varphi^*(s') \le \varphi(s)$.

Proof. Clear from the definitions. □

2.7. Proposition. *Let $\varphi : S^\flat \to \mathbb{R}$ satisfy (a). Then:*

(1) φ_* *satisfies* (a) *and* $(b*)$. *In particular,* φ *satisfies* $(b*)$ *if and only if* $\varphi = \varphi_*$.
(2) φ^* *satisfies* (a) *and* $(c*)$. *In particular,* φ *satisfies* $(c*)$ *if and only if* $\varphi = \varphi^*$.

Proof. We prove (1), leaving the dual proof of (2) to the reader.

Firstly take $s, s' \in S^\flat$ with $s'\,\omega_S\,s$ and suppose that $\varphi_*(s') > \varphi_*(s)$. Then we have $u\,\omega_S\,s'$ with $\varphi_*(s) < \varphi(u)$. However, Lemma 2.6 (ii) now gives $\varphi(u) \le \varphi_*(s)$, which leads to a contradiction. Hence $\varphi_*(s') \le \varphi_*(s)$, so φ_* satisfies (a).

Now take $s \in S^\flat$ and $r \in \mathbb{R}$ with $r < \varphi_*(s)$. As above we have $u\,\omega_S\,s$ with $r < \varphi(u)$. Take $s' \in S$ satisfying $s'\,\omega_S\,s$ and $u\,\omega_S\,s'$. By Lemma 2.6 (ii) we have $r < \varphi(u) \leq \varphi_*(s')$, whence φ_* satisfies $(b*)$.

Finally, it remains to show that if φ satisfies $(b*)$ then $\varphi = \varphi_*$. By Lemma 2.6 (i) we have $\varphi_* \leq \varphi$, so assume φ satisfies $(b*)$ but that $\varphi_*(s) < \varphi(s)$ for some $s \in S^\flat$. We now have $s'\,\omega_S\,s$ with $\varphi_*(s) < \varphi(s')$. On the other hand, $\varphi(s') \leq \varphi_*(s)$ by Lemma 2.6 (ii), and this is a contradiction. □

We now prove a result which implies, in particular, that when $\varphi : S^\flat \to \mathbb{R}$ satisfies (a), the point-functions φ_*, φ and φ^* all give rise to the same difunction in the sense of Lemma 2.3. We begin with the following definition.

2.8. Definition. Let $(S, \mathcal{S})$ be a texture. We denote by $\mathrm{A}(S^\flat)$ the set of point-functions $\varphi : S^\flat \to \mathbb{R}$ satisfying condition (a), and by $\sim$ the equivalence relation on $\mathrm{A}(S^\flat)$ defined by $\varphi \sim \psi \iff (f_\varphi, F_\varphi) = (f_\psi, F_\psi)$.

Note that by [5, Proposition 2.27] we have $\varphi \sim \psi \iff f_\varphi = f_\psi \iff F_\varphi = F_\psi$.

2.9. Theorem. *For $\varphi \in \mathrm{A}(S^\flat)$ the equivalence class $\widetilde{\varphi}$ of φ under the equivalence relation $\sim$ is given by*

$$\widetilde{\varphi} = [\varphi_*,\ \varphi^*] = \{\psi \mid \psi \in \mathrm{A}(S^\flat),\ \varphi_* \leq \psi \leq \varphi^*\},$$

where the ordering $\leq$ in $\mathrm{A}(S^\flat)$ is defined pointwise.

Proof. Take $\psi \in \widetilde{\varphi}$. We show that $\varphi_* \leq \psi \leq \varphi^*$. Suppose the first inequality is false. Then for some $s \in S^\flat$ we have $\psi(s) < \varphi_*(s)$. By the definition of φ_* we have $u\,\omega_S\,s$ with $\psi(s) < \varphi(u)$, whence $\overline{P}_{(s,\varphi(u))} \subseteq f_\varphi = f_\psi$. Since the texture $(S \times \mathbb{R}, \mathcal{P}(S) \otimes \mathcal{R})$ is plain this gives $f_\psi \not\subseteq \overline{Q}_{(s,\varphi(u))}$, so we have $v\,\omega_S\,s$ with $\overline{P}_{(s,\psi(v))} \not\subseteq \overline{Q}_{(s,\varphi(u))}$. Now $\varphi(u) \leq \psi(v)$, and $\psi(v) \leq \psi(s)$ since ψ satisfies (a), so we obtain the contradiction $\varphi(u) \leq \psi(s)$. This gives $\varphi_* \leq \psi$, and the proof of $\psi \leq \varphi^*$ is dual and in omitted. Hence

$$\widetilde{\varphi} \subseteq [\varphi_*,\ \varphi^*]. \tag{2.2}$$

Conversely, take $\psi \in [\varphi_*,\ \varphi^*]$. We must show $\psi \sim \varphi$, and as noted above it will be sufficient to prove $f_\varphi = f_\psi$.

To prove that $f_\varphi \subseteq f_\psi$ it will be sufficient to show that for $s \in S^\flat$ and $u\,\omega_S\,s$ we have $\overline{P}_{(s,\varphi(u))} \subseteq f_\psi$. However, if we take $s' \in S$ with $s'\,\omega_S\,s$, $u\,\omega_S\,s'$ and note that $\varphi(u) \leq \varphi_*(s')$ by Lemma 2.6 (ii), and $\varphi_*(s') \leq \psi(s')$ as $\varphi_* \leq \psi$, we obtain $\overline{P}_{(s,\varphi(u))} \subseteq \overline{P}_{(s,\psi(s'))} \subseteq f_\psi$, as required. Hence $f_\varphi \subseteq f_\psi$, and the proof of $f_\psi \subseteq f_\varphi$ is dual to this, and is omitted. Hence

$$[\varphi_*,\ \varphi^*] \subseteq \widetilde{\varphi}. \tag{2.3}$$

The result now follows from the inclusions (2.2) and (2.3). □

2.10. Example. Consider again the function $\vartheta_\alpha \in \mathrm{A}(L^\flat)$ defined in Example 2.4. Clearly

$$(\vartheta_\alpha)_*(s) = \sup\{\vartheta_\alpha(u) \mid u < s\} = \left\{ \begin{array}{ll} 0 & \text{for } s \leq \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} < s \end{array} \right\} = \vartheta_0(s)$$

for all $s \in L$. Hence $(\vartheta_\alpha)_* = \vartheta_0$, and likewise $(\vartheta_\alpha)^* = \vartheta_1$, for $0 \leq \alpha \leq 1$. We now see that the statements in Example 2.4 about which of the functions ϑ_α satisfy $(b*)$, $(c*)$ are just a special case of Proposition 2.7. Moreover, by Theorem 2.9 we deduce easily that in $\mathrm{A}(L^\flat)$ we have

$$\widetilde{\vartheta_\alpha} = [\vartheta_0, \vartheta_1] = \{\vartheta_\beta \mid 0 \leq \beta \leq 1\}$$

for all α, $0 \le \alpha \le 1$. In particular, the difunction corresponding to this equivalence class is independent of α.

Our next aim is to show that every difunction $(f,F) : (S,\mathcal{S}) \to (\mathbb{R},\mathcal{R})$ has the form (f_φ, F_φ) for some $\varphi \in \mathrm{A}(S^\flat)$. Firstly, however, it will be convenient to give a direct characterization of φ_*, φ^* in terms of (f_φ, F_φ).

2.11. Proposition. *For $\varphi \in \mathrm{A}(S^\flat)$ and $s \in S^\flat$ we have $\varphi_*(s) = \sup f_\varphi^{\to} P_s$ and $\varphi^*(s) = \inf(F_\varphi^{\to} Q_s)^c$.*

Proof. Clearly $r \in f_\varphi^{\to} P_s \iff f_\varphi^{\to} P_s \not\subseteq Q_r \iff f_\varphi \not\subseteq \overline{Q}_{(s,r)}$ by [5, Lemma 2.6 (1)]. Hence $r \in f_\varphi^{\to} P_s \iff \exists\, u\, \omega_S\, s$ with $r \le \varphi(u) \le \varphi_*(s)$, from which the equality $\varphi_*(s) = \sup f_\varphi^{\to} P_s$ follows immediately. The proof of the second equality is dual to this, and is omitted. $\square$

Since by Theorem 2.9 the point functions φ_*, φ^* also generate the difunction (f_φ, F_φ), the above proposition suggests the following:

2.12. Theorem. *Let $(f,F) : (S,\mathcal{S}) \to (\mathbb{R},\mathcal{R})$ be a difunction.*

1. *For $s \in S^\flat$ let $\lambda(s) = \sup f^{\to} P_s$. Then the point function $\lambda : S^\flat \to \mathbb{R}$ satisfies the conditions (a) and $(b*)$, and the equalities $(f,F) = (f_\lambda, F_\lambda)$ and $\lambda = \lambda_*$.*
2. *For $s \in S^\flat$ let $\xi(s) = \inf(F^{\to} Q_s)^c$. Then the point function $\xi : S^\flat \to \mathbb{R}$ satisfies the conditions (a) and $(c*)$, and the equalities $(f,F) = (f_\xi, F_\xi)$ and $\xi = \xi^*$.*

Proof. We prove (1), leaving the essentially dual proof of (2) to the interested reader.

We must first show that the set $f^{\to} P_s \subseteq \mathbb{R}$ is bounded above. Since $s \in S^\flat$ we have $Q_s \ne S$ and so we may choose $s' \in S$ with $s\, \omega_S\, s'$. By *DF1* there exists $t \in \mathbb{R}$ satisfying $f \not\subseteq \overline{Q}_{(s',t)}$ and $\overline{P}_{(s,t)} \not\subseteq F$. As in the proof of Proposition 2.11 we have $r \in f^{\to} P_s \iff f \not\subseteq \overline{Q}_{(s,r)}$, and in this case *DF2* gives $r\, \omega_{\mathbb{R}}\, t$, that is $r \le t$, so $f^{\to} P_s$ is bounded above by t. Hence λ is well defined.

To establish (a) take $s, s' \in S^\flat$. Then

$$s'\, \omega_S\, s \implies P_{s'} \subseteq P_s \implies f^{\to} P_{s'} \subseteq f^{\to} P_s \implies \lambda(s') \le \lambda(s),$$

as required. For $(b*)$, take $s \in S^\flat$ and $r \in \mathbb{R}$ with $r < \lambda(s)$. Then there exists $r' \in f^{\to} P_s$ with $r < r'$, and as noted above $f \not\subseteq \overline{Q}_{(s,r')}$. Since f is a relation, by *R2* we have $s' \in S$ with $s'\, \omega_S\, s$ and $f \not\subseteq \overline{Q}_{(s',r')}$. Then $s' \in S^\flat$, $r' \in f^{\to} P_{s'}$, so $r < r' \le \lambda(s')$ which proves $(b*)$.

Since $\lambda = \lambda_*$ follows from Proposition 2.7 (1), it remains to prove that $(f,F) = (f_\lambda, F_\lambda)$. As λ satisfies (a), we know by Lemma 2.3 (1) that (f_λ, F_λ) is a difunction, so by [5, Proposition 2.27] it will be sufficient to show that $f = f_\lambda$.

Suppose that $f_\lambda \not\subseteq f$ and take $s \in S$, $r \in \mathbb{R}$ with $f_\lambda \not\subseteq \overline{Q}_{(s,r)}$, $\overline{P}_{(s,r)} \not\subseteq f$. Now from the definition of f_λ we have $u \in S$, $u\, \omega_S\, s$ with $\overline{P}_{(s,\lambda(u))} \not\subseteq \overline{Q}_{(s,r)}$, and so

(2.4) $\quad r \le \lambda(u).$

Applying condition *DF1* for (f,F) to $P_s \not\subseteq Q_u$ gives $t \in \mathbb{R}$ satisfying

(2.5) $\quad f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(u,t)} \not\subseteq F.$

Now for $\epsilon > 0$ we have $r - \epsilon < \lambda(u)$ by (2.4), so by the definition of λ there exists $r_\epsilon \in f^{\to} P_u$ with $r - \epsilon < r_\epsilon$. As noted earlier, $r_\epsilon \in f^{\to} P_u$ is equivalent to $f \not\subseteq \overline{Q}_{(u,r_\epsilon)}$, and this together with the second result in (7) gives $P_t \not\subseteq Q_{r_\epsilon}$ by *DF2* for (f,F). Hence $r_\epsilon \le t$, and we obtain $r < t + \epsilon$ for all $\epsilon > 0$. Hence $r \le t$ and from the first result in (2.57) we have the contradiction $\overline{P}_{(s,r)} \subseteq \overline{P}_{(s,t)} \subseteq f$. Hence $f_\lambda \subseteq f$.

Finally suppose $f \not\subseteq f_\lambda$ and take $s \in S$, $r \in \mathbb{R}$ with $f \not\subseteq \overline{Q}_{(s,r)}$, $\overline{P}_{(s,r)} \not\subseteq f_\lambda$. Since f is a relation we have $u \in S$ with $u\,\omega_S\,s$ and $f \not\subseteq \overline{Q}_{(u,r)}$. But now $r \in f^{\to}P_u$, and so $r \leq \lambda(u)$. This gives $\overline{P}_{(s,r)} \subseteq \overline{P}_{(s,\lambda(u))} \subseteq f_\lambda$, which is a contradiction. Hence $f \subseteq f_\lambda$, and the proof is complete. □

Where necessary we denote λ, ξ by $\lambda_{(f,F)}, \xi_{(f,F)}$, respectively. Then for $\varphi \in \mathrm{A}(S^\flat)$, the above theorem implies that $\lambda_{(f_\varphi,F_\varphi)} = \varphi_*$, $\xi_{(f_\varphi,F_\varphi)} = \varphi^*$, whence the mapping

$$\widetilde{\varphi} \mapsto (f_\varphi, F_\varphi),$$

which is injective by definition, is also a surjective mapping from the quotient set $\mathrm{A}(S^\flat)/\!\sim$ to the set $\mathrm{DF}(S)$ of real difunctions on $(S, \mathcal{S})$.

In [8] a partial order is defined on $\mathrm{BDF}(S)$ by setting $(f,F) \leq (g,G)$ if and only if $(f,F) = (f,F)(\curlywedge, \wedge)(g,G)$, where $(f,F)(\curlywedge, \wedge)(g,G) = (f \curlywedge g, F \wedge G)$ is given by

$$f \curlywedge g = \bigvee\{\overline{P}_{(s,r_1\wedge r_2)} \mid \exists u\,\omega_S\,s \text{ with } f \not\subseteq \overline{Q}_{(u,r_1)} \text{ and } g \not\subseteq \overline{Q}_{(u,r_2)}\},$$

$$F \wedge G = \bigcap\{\overline{Q}_{(s,r_1\wedge r_2)} \mid \exists s\,\omega_S\,u \text{ with } \overline{P}_{(u,r_1)} \not\subseteq F \text{ and } \overline{P}_{(u,r_2)} \not\subseteq G\}.$$

First we note the following result.

2.13. Lemma. *The following are equivalent for difunctions* $(f,F),(g,G) : (S,\mathcal{S}) \to (\mathbb{R},\mathcal{R})$*:*

(1) $(f,F) \leq (g,G)$.
(2) $f \subseteq g$.
(3) $F \subseteq G$.

Proof. The equivalence of (2) and (3) is just [5, Proposition 2.27], so we prove (1) $\iff$ (2).

Suppose (1) holds but that $f \not\subseteq g$. Then we have $s \in S$, $t \in \mathbb{R}$ with $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t)} \not\subseteq g$. Since $f = f \curlywedge g$ we have $f \curlywedge g \not\subseteq \overline{Q}_{(s,t)}$, so for some $r_1, r_2 \in \mathbb{R}$ with $\overline{P}_{(s,r_1\wedge r_2)} \not\subseteq \overline{Q}_{(s,t)}$ we have $u \in S$ with $u\,\omega_S\,s$, $f \not\subseteq \overline{Q}_{(u,r_1)}$ and $g \not\subseteq \overline{Q}_{(u,r_2)}$. Now $t \leq r_1 \wedge r_2 \leq r_2$, so $\overline{Q}_{(u,t)} \subseteq \overline{Q}_{(u,r_2)}$ and hence $g \not\subseteq \overline{Q}_{(u,t)}$. Since g is a relation we may apply condition *R1* to give the contradiction $g \not\subseteq \overline{Q}_{(s,t)}$.

The converse is proved in a similar way, and the details are omitted. □

2.14. Theorem. *The following are equivalent for difunctions* $(f,F),(g,G) : (S,\mathcal{S}) \to (\mathbb{R},\mathcal{R})$*:*

(1) $(f,F) \leq (g,G)$.
(2) $\lambda_{(f,F)} \leq \lambda_{(g,G)}$.
(3) $\xi_{(f,F)} \leq \xi_{(g,G)}$.

Proof. To prove (1) $\iff$ (2) it is sufficient, in view of Lemma 2.13, to prove that $\lambda_{(f,F)} \leq \lambda_{(g,G)} \iff f \subseteq g$.

If $f \subseteq g$ then for $s \in S^\flat$ we have $f^{\to}P_s \subseteq g^{\to}P_s$ by [5, Lemma 2.7 (1)], and so $\lambda_{(f,F)}(s) = \sup f^{\to}P_s \leq \sup g^{\to}P_s = \lambda_{(g,G)}(s)$. Hence, $\lambda_{(f,F)} \leq \lambda_{(g,G)}$.

Now let $\lambda_{(f,F)} \leq \lambda_{(g,G)}$. Then for s, $u \in S^\flat$, $u\,\omega_S\,s$, we have $\overline{P}_{(s,\lambda_{(f,F)}(u))} \subseteq \overline{P}_{(s,\lambda_{(g,G)}(u))}$ and so $f = f_{\lambda_{(f,F)}} \subseteq f_{\lambda_{(g,G)}} = g$ by Theorem 2.12 (1) and the first equality in Lemma 2.3 (1).

(1) $\iff$ (3) is proved likewise by establishing $\xi_{(f,F)} \leq \xi_{(g,G)} \iff F \subseteq G$. The details are left to the interested reader. □

By using the bijection between $\mathrm{A}(S^\flat)/\!\sim$ and $\mathrm{DF}(S)$ given above we may transfer the partial ordering on $DF(S)$ to $\mathrm{A}(S^\flat)/\!\sim$, and Theorem 2.14 now takes the following form:

2.15. Corollary. *For $\varphi, \psi \in \mathrm{A}(S^\flat)$ the following are equivalent:*

(1) $\widetilde{\varphi} \le \widetilde{\psi}$.
(2) $\varphi_* \le \psi_*$.
(3) $\varphi^* \le \psi^*$.

Since $\mathrm{DF}(S)$ is a distributive lattice [8], the same is true of $\mathrm{A}(S^\flat)/{\sim}$.

Clearly if $\varphi, \psi \in \mathrm{A}(S^\flat)$ then $\varphi \wedge \psi$, $\varphi \vee \psi \in \mathrm{A}(S^\flat)$ so $\mathrm{A}(S^\flat)$ is a lattice under the pointwise ordering. Let us relate the lattice structure on $\mathrm{A}(S^\flat)/{\sim}$ with that of $\mathrm{A}(S^\flat)$ by noting that:

$$\widetilde{\varphi \wedge \psi} = \widetilde{\varphi} \wedge \widetilde{\psi} \quad \text{and} \quad \widetilde{\varphi \vee \psi} = \widetilde{\varphi} \vee \widetilde{\psi}.$$

Indeed $\varphi \wedge \psi \le \varphi$ and $\varphi \wedge \psi \le \psi$ give $\widetilde{\varphi \wedge \psi} \le \widetilde{\varphi} \wedge \widetilde{\psi}$, so take $\mu \in \mathrm{A}(S^\flat)$ with $\widetilde{\mu} \le \widetilde{\varphi}$ and $\widetilde{\mu} \le \widetilde{\psi}$. Then $\mu_* \le \varphi_* \le \varphi$, $\mu_* \le \psi_* \le \psi$ by Corollary 2.15 and Lemma 2.6 (i), so $\mu_* \le \varphi \wedge \psi$ which gives $\widetilde{\mu} = \widetilde{\mu_*} \le \widetilde{\varphi \wedge \psi}$ by Theorem 2.9 and Corollary 2.15. This proves the first equality, and the proof of the second is dual and is omitted.

We deduce from Theorem 2.9 and the equalities above that

$$\widetilde{\varphi} \wedge \widetilde{\psi} = [(\varphi \wedge \psi)_*,\ (\varphi \wedge \psi)^*],$$

where

$$(\varphi \wedge \psi)_* = (\varphi_* \wedge \psi_*)_* \le \varphi_* \wedge \psi_* \le \varphi \wedge \psi \le \varphi^* \wedge \psi^* \le (\varphi^* \wedge \psi^*)^* = (\varphi \wedge \psi)^*.$$

Likewise

$$\widetilde{\varphi} \vee \widetilde{\psi} = [(\varphi \vee \psi)_*,\ (\varphi \vee \psi)^*],$$

where

$$(\varphi \vee \psi)_* = (\varphi_* \vee \psi_*)_* \le \varphi_* \vee \psi_* \le \varphi \vee \psi \le \varphi^* \vee \psi^* \le (\varphi^* \vee \psi^*)^* = (\varphi \vee \psi)^*.$$

We now turn to the operation $(+, +)$ of addition on $\mathrm{DF}(S)$, which is compatible with the lattice structure in the sense that meet and join distribute over addition [8]. We recall from [8] that

$$f + g = \bigvee \{\overline{P}_{(s, r_1 + r_2)} \mid \exists\, s \in S^\flat, u\, \omega_S\, s \text{ with } f \not\subseteq \overline{Q}_{(u, r_1)}, g \not\subseteq \overline{Q}_{(u, r_2)}\},$$

with a dual formula for $F + G$. Then:

2.16. Theorem. *For the real difunction (f, F) we have*

$$\lambda_{(f,F)(+,+)(g,G)} \le \lambda_{(f,F)} + \lambda_{(g,G)} \le \xi_{(f,F)} + \xi_{(g,G)} \le \xi_{(f,F)(+,+)(g,G)}.$$

Proof. To prove the first inequality suppose there exists $s \in S^\flat$ with $\lambda_{(f,F)}(s) + \lambda_{(g,G)}(s) < \lambda_{(f,F)(+,+)(g,G)}(s)$. Then we have $t \in \mathbb{R}$ with $\lambda_{(f,F)}(s) + \lambda_{(g,G)}(s) < t \in (f+g)^{\rightarrow} P_s$, whence $(f+g)^{\rightarrow} P_s \not\subseteq Q_t$ and so $f + g \not\subseteq \overline{Q}_{(s,t)}$. Now we have $r_1, r_2 \in \mathbb{R}$ with $\overline{P}_{(s, r_1 + r_2)} \not\subseteq \overline{Q}_{(s,t)}$, and $u \in S^\flat$ with $u\, \omega_S\, s$ and $f \not\subseteq \overline{Q}_{(u, r_1)}$, $g \not\subseteq \overline{Q}_{(u, r_2)}$. Now $f^{\rightarrow} P_u \not\subseteq Q_{r_1}$, $g^{\rightarrow} P_u \not\subseteq Q_{r_2}$ so we obtain the contradiction

$$t \le r_1 + r_2 \le \lambda_{(f,F)}(u) + \lambda_{(g,G)}(u) \le \lambda_{(f,F)}(s) + \lambda_{(g,G)}(s)$$

since $\lambda_{(f,F)}$, $\lambda_{(g,G)}$ satisfy (a). The middle inequality is clear, and the third inequality dual to the above, so the proof is complete. □

In terms of the corresponding sum on $\mathrm{A}(S^\flat)/{\sim}$ we note that $\mathrm{A}(S^\flat)$ is closed under pointwise addition and that from the above

$$\widetilde{\varphi} + \widetilde{\psi} = \widetilde{\varphi + \psi} = [(\varphi + \psi)_*,\ (\varphi + \psi)^*],$$

where

$$(\varphi+\psi)_* = (\varphi_*+\psi_*)_* \le \varphi_*+\psi_* \le \varphi+\psi \le \varphi^*+\psi^* \le (\varphi^*+\psi^*)^* = (\varphi+\psi)^*.$$

Now let (τ,κ) be a ditopology on $(S,\mathcal{S})$, and $(\tau_{\mathbb{R}},\kappa_{\mathbb{R}})$ the usual ditopology on $(\mathbb{R},\mathcal{R}\}$. It is shown in [8] that the subset $\mathrm{BDF}(S)=\{(f,F)\in \mathrm{DF}(S)\mid (f,F)$ bicontinuous$\}$ of $\mathrm{DF}(S)$ is closed under meet, join and addition. We consider briefly the situation regarding the representation of elements of $\mathrm{BDF}(S)$ described above.

We call $\varphi \in \mathrm{A}(S^\flat)$ *bicontinuous* if $\varphi^{\leftarrow}B \in \tau$ for all $B \in \tau_{\mathbb{R}}$ and $\varphi^{\leftarrow}B\in\kappa$ for all $B\in\kappa_{\mathbb{R}}$. By Lemma 2.3 (1), φ is bicontinuous if and only if (f_φ,F_φ) is bicontinuous. We denote by $\mathrm{BA}(S^\flat)$ the set of bicontinuous elements of $\mathrm{A}(S^\flat)$. For $\varphi,\psi\in\mathrm{BA}(S^\flat)$ we have

$$(\varphi\wedge\psi)^{\leftarrow}B = (\varphi^{\leftarrow}B)\cup(\psi^{\leftarrow}B),\ (\varphi\vee\psi)^{\leftarrow}B = (\varphi^{\leftarrow}B)\cap(\psi^{\leftarrow}B)$$

for all $B\in\mathcal{R}$. It follows that $\varphi\wedge\psi,\ \varphi\vee\psi\in\mathrm{BA}(S^\flat)$. The first result may also be obtained by noting that $\widetilde{\varphi\wedge\psi}=\widetilde{\varphi}\wedge\widetilde{\psi}$ leads to $(f_\varphi,F_\varphi)(\curlywedge,\curlywedge)(f_\psi,F_\psi)=(f_{\varphi\wedge\psi},F_{\varphi\wedge\psi})$, whence for $B\in\mathcal{R}$,

$$(f_\varphi \curlywedge f_\psi)^{\leftarrow}B = f^{\leftarrow}_{\varphi\wedge\psi}B = (\varphi\wedge\psi)^{\leftarrow}B = F^{\leftarrow}_{\varphi\wedge\psi}B = (F_\varphi\curlywedge F_\psi)^{\leftarrow}B.$$

The bicontinuity of $\varphi\wedge\psi$ now follows from that of $(f_\varphi,F_\varphi)(\curlywedge,\curlywedge)(f_\psi,F_\psi)$, and a similar argument gives the bicontinuity of $\varphi\vee\psi$, and also that of $\varphi+\psi$. These results show that when considering the representation of elements of $\mathrm{BDF}(S)$ it suffices to restrict ones attention to the point functions in $\mathrm{BA}(S^\flat)$.

3. An alternative characterization

In this section we show that we may obtain an alternative characterization of real difunctions on an arbitrary texture $(S,\mathcal{S})$ by extending $(\mathbb{R},\mathcal{R})$ rather than restricting the domain of the functions to the core $S^\flat$.

Denote by $\mathbb{R}^+$ the real numbers extended by adding a point at infinity ∞, where we define $r<\infty$ for all $r\in\mathbb{R}$. The family $\mathcal{R}^+=\{(-\infty,r],(-\infty,r)\mid r\in\mathbb{R}\}\cup\{\mathbb{R}^+,\emptyset\}$ is easily seen to a texturing of $\mathbb{R}^+$ for which P_r, Q_r are the same as for $(\mathbb{R},\mathcal{R})$ when $r\in\mathbb{R}$, and $P_\infty=Q_\infty=\mathbb{R}^+$. We note for future reference that for $\rho,\mu\in\mathbb{R}^+$ we have $\mu\,\omega_{\mathbb{R}^+}\,\rho \iff P_\rho\not\subseteq Q_\mu \iff \mu\in\mathbb{R}$ and $\mu\le\rho$. The texture $(\mathbb{R}^+,\mathcal{R}^+)$ is neither plain nor simple.

The inclusion $\epsilon:\mathbb{R}\hookrightarrow\mathbb{R}^+$ is easily seen to satisfy the conditions (a), (b) and (c) of [6, Lemma 3.8], so the corresponding difunction (e,E) is given by

$$e=f_\epsilon=\bigvee\{\overline{P}^{-,+}_{(r,r)}\mid r\in\mathbb{R}\},\ E=F_\epsilon=\bigcap\{\overline{Q}^{-,+}_{(r,r)}\mid r\in\mathbb{R}\},$$

where $\overline{P}^{-,+}_{(,)}$, $\overline{Q}^{-,+}_{(,)}$ denote the p-sets and q-sets for the texture $(\mathbb{R}\times\mathbb{R}^+,\mathcal{P}(\mathbb{R})\otimes\mathcal{R}^+)$. Moreover, by [5, Lemma 3.9],

$$e^{\leftarrow}B=E^{\leftarrow}B=\epsilon^{-1}[B]=\begin{cases}B, & B\ne\mathbb{R}^+\\ \mathbb{R}, & B=\mathbb{R}^+\end{cases} \tag{3.1}$$

for all $B\in\mathcal{R}^+$. We have:

3.1. Lemma. $(e,E):(\mathbb{R},\mathcal{R})\to(\mathbb{R}^+,\mathcal{R}^+)$ *is a bijective difunction with inverse* $(e,E)^{\leftarrow}=(E^{\leftarrow},e^{\leftarrow}):(\mathbb{R}^+,\mathcal{R}^+)\to(\mathbb{R},\mathcal{R})$ *given by*

$$E^{\leftarrow}=\bigvee\{\overline{P}^{+,-}_{(\rho,r)}\mid \rho\in\mathbb{R}^+,\ r\in\mathbb{R},\ r\le\rho\},$$
$$e^{\leftarrow}=\bigcap\{\overline{Q}^{+,-}_{(\rho,r)}\mid \rho\in\mathbb{R}^+,\ r\in\mathbb{R},\ \rho\le r\}.$$

Here, $\overline{P}^{+,-}_{(,)}$*,* $\overline{Q}^{+,-}_{(,)}$ *denote the p-sets and q-sets for the texture* $(\mathbb{R}^+\times\mathbb{R},\mathcal{P}(\mathbb{R}^+)\otimes\mathcal{R})$*.*

Proof. The formulae for $E^{\leftarrow}$ and $e^{\leftarrow}$ follow at once from [5, Definition 2.3], while by [5, Theorem 2.31] bijectivity is equivalent to showing that $(e,E)^{\leftarrow}$ is a difunction. However for $\rho_1,\rho_2 \in \mathbb{R}^+$ with $P_{\rho_1} \not\subseteq Q_{\rho_2}$ we have $Q_{\rho_2} \neq \mathbb{R}^+$ and so $\rho_2 \in \mathbb{R}$, while clearly $E^{\leftarrow} \not\subseteq \overline{Q}^{+,-}_{(\rho_1,\rho_2)}$ and $\overline{P}^{+,-}_{(\rho_2,\rho_2)} \not\subseteq e^{\leftarrow}$ which verifies *DF1*. On the other hand, take $r_1, r_2 \in \mathbb{R}$ and $\rho \in \mathbb{R}^+$ satisfying $E^{\leftarrow} \not\subseteq \overline{Q}^{+,-}_{(\rho,r_1)}$ and $\overline{P}^{+,-}_{(\rho,r_2)} \not\subseteq e^{\leftarrow}$. Then $r_1 \le \rho \le r_2$ which gives $P_{r_2} \not\subseteq Q_{r_1}$ and so *DF2* is valid also.

This shows that (e,E) is bijective, and hence by [5, Proposition 3.14] an isomorphism in the category **dfTex** of textures and difunctions between them. Moreover, $(e,E)^{\leftarrow}$ is the morphism inverse to (e,E), that is $(e,E)^{\leftarrow}\circ(e,E) = (i,I)$, $(e,E)\circ(e,E)^{\leftarrow} = (i^+,I^+)$, (i,I) being the identity difunction on $(\mathbb{R},\mathcal{R})$ and (i^+,I^+) that on $(\mathbb{R}^+,\mathcal{R}^+)$. □

3.2. Lemma. *For a given texture $(S,\mathcal{S})$ the mapping*

$$(f,F) \mapsto (e,E)\circ(f,F)$$

is an bijection between the set $\mathrm{DF}(S)$ of real difunctions on $(S,\mathcal{S})$ and the set $\mathrm{DF}^+(S)$ of extended real difunctions on $(S,\mathcal{S})$.

Proof. Since $(S,\mathcal{S}) \xrightarrow{(f,F)} (\mathbb{R},\mathcal{R}) \xrightarrow{(e,E)} (\mathbb{R}^+,\mathcal{R}^+)$, certainly $(e,E)\circ(f,F) \in \mathrm{DF}^+(S)$ when $(f,F) \in \mathrm{DF}(S)$. The fact that this mapping is bijective is an immediate consequence of Lemma 3.1. □

We may obtain operations on $\mathrm{DF}^+(S)$ from operations on $(\mathbb{R}^+,\mathcal{R}^+)$ in the same way that the operations on $\mathrm{DF}(S)$ were obtained from those on $(\mathbb{R},\mathcal{R})$. Specifically, the mappings $\varphi : (\mathbb{R}^+\times\mathbb{R}^+, \mathcal{R}^+\otimes\mathcal{R}^+) \to (\mathbb{R}^+,\mathcal{R}^+)$ defined by

$$(\rho_1,\rho_2) \mapsto \min\{\rho_1,\rho_2\},\ (\rho_1,\rho_2) \mapsto \max\{\rho_1,\rho_2\} \text{ and } (\rho_1,\rho_2) \mapsto \rho_1+\rho_2$$

satisfy (a) (and indeed, (b) and (c)) and so induce operations of meet, join and sum on $(\mathbb{R}^+,\mathcal{R}^+)$. Indeed, for $\varphi(\rho_1,\rho_2) = \min\{\rho_1,\rho_2\}$ suppose that $P_{(\rho_1,\rho_2)} \not\subseteq Q_{(\mu_1,\mu_2)}$. Then $\mu_1,\mu_2 \in \mathbb{R}$ and $\mu_k \le \rho_k$ for $k = 1,2$. To establish (a) we must show that $P_{\varphi(\rho_1,\rho_2)} \not\subseteq Q_{\varphi(\mu_1,\mu_2)}$. However, certainly $\varphi(\mu_1,\mu_2) = \min\{\mu_1,\mu_2\} \in \mathbb{R}$, and clearly $\varphi(\mu_1,\mu_2) \le \varphi(\rho_1,\rho_2)$, so the result follows by the comment above.

The same proof holds for the other mappings, and (b), (c) may be established likewise. By [8] we obtain

$$f \wedge g = \bigvee\{\overline{P}^{+}_{(s,r_1\wedge r_2)} \mid \exists\, u\, \omega_S\, s \text{ with } f \not\subseteq \overline{Q}^{+}_{(u,r_1)} \text{ and } g \not\subseteq \overline{Q}^{+}_{(u,r_2)}\},$$
$$F \wedge G = \bigcap\{\overline{Q}^{+}_{(s,r_1\wedge r_2)} \mid \exists\, s\, \omega_S\, u \text{ with } \overline{P}^{+}_{(u,r_1)} \not\subseteq F \text{ and } \overline{P}^{+}_{(u,r_2)} \not\subseteq G\}.$$

Here, $\overline{P}^{+}_{(,)}$, $\overline{Q}^{+}_{(,)}$ denote the p-sets and q-sets for the texture $(S\times\mathbb{R}^+, \mathcal{P}(S)\otimes\mathcal{R}^+)$. Similar formulae hold for $(f,F)(\vee,\vee)(g,G)$ and $(f,F)(+,+)(g,G)$. As for $\mathrm{DF}(S)$, these operations make $\mathrm{DF}^+(S)$ an additive lattice.

3.3. Proposition. *The bijection $(f,F) \mapsto (e,E)\circ(f,F)$ from $\mathrm{DF}(S)$ to $\mathrm{DF}^+(S)$ preserves the lattice operations and the sum.*

Proof. The proof of Lemma 2.13 may easily be adapted to show that for (f,F), (g,G) in $\mathrm{DF}^+(S)$ we have $(f,F) \le (g,G) \iff f \subseteq g \iff F \subseteq G$. On the other hand, for $(f,F),(g,G) \in \mathrm{DF}(S)$ we have $e\circ f \subseteq e\circ g \iff (e\circ g)^{\leftarrow}B \subseteq (e\circ f)^{\leftarrow}B\ \forall B \in \mathcal{R}^+$ [5, Lemmas 2.4, 2.7]. It is clearly sufficient to consider only $B \in \mathcal{R}$, so

$$\begin{aligned} e \circ f \subseteq e \circ g &\iff (e \circ g)^{\leftarrow} B \subseteq (e \circ f)^{\leftarrow} B \ \forall\, B \in \mathcal{R} \\ &\iff g^{\leftarrow}(e^{\leftarrow} B) \subseteq f^{\leftarrow}(e^{\leftarrow} B) \ \forall\, B \in \mathcal{R} \\ &\iff g^{\leftarrow} B \subseteq f^{\leftarrow} B \ \forall\, B \in \mathcal{R} \\ &\iff f \subseteq g \end{aligned}$$

by (3.1). This shows that the bijection preserves the ordering, and hence the lattice structure.

To see that the sum is preserved we must show that $e \circ (f+g) = e \circ f + e \circ g$. Suppose that $e \circ (f+g) \not\subseteq e \circ f + e \circ g$, and take $s \in S$, $\rho \in \mathbb{R}^+$ with $e \circ (f+g) \not\subseteq \overline{Q}^{+}_{(s,\rho)}$ and $\overline{P}^{+}_{(s,\rho)} \not\subseteq e \circ f + e \circ g$. Now we have $\rho' \in \mathbb{R}^+$ with $\overline{P}^{+}_{(s,\rho')} \not\subseteq \overline{Q}_{(s,\rho)}$ and $r \in \mathbb{R}$ with $f+g \not\subseteq \overline{Q}_{(s,r)}$ and $e \not\subseteq \overline{Q}^{-,+}_{(r,\rho')}$. The latter gives $\rho' \leq r$ and from the former we have $r_1, r_2 \in \mathbb{R}$ with $\overline{P}_{(s,r_1+r_2)} \not\subseteq \overline{Q}_{(s,r)}$ and $u \in S$ satisfying $u \, \omega_S \, s$ for which $f \not\subseteq \overline{Q}_{(u,r_1)}$ and $g \not\subseteq \overline{Q}_{(u,r_2)}$. We deduce that $e \circ f \not\subseteq \overline{Q}^{+}_{(u,r_1)}$, $e \circ g \not\subseteq \overline{Q}^{+}_{(u,r_1)}$, whence $\overline{P}^{+}_{(s,r_1+r_2)} \subseteq e \circ f + e \circ g$, which gives a contradiction since $\rho \leq \rho' \leq r_1 + r_2$.

This establishes $e \circ (f+g) \subseteq e \circ f + e \circ g$, and the reverse inclusion may be proved similarly. □

In view of Proposition 3.3, a representation of the elements of $\mathrm{DF}^+(S)$ involving point functions from S to $\mathbb{R}^+$ will also provide a representation of the additive lattice of real difunctions on $(S, \mathcal{S})$. We may obtain such a representation by following much the same steps as in the last section. We begin by noting that for $(f,F) \in \mathrm{DF}^+(S)$ the functions $\lambda^{+}_{(f,F)}, \xi^{+}_{(f,F)} : S \to \mathbb{R}^+$ given by

$$\lambda^{+}_{(f,F)}(s) = \sup f^{\rightarrow} P_s, \ \xi^{+}_{(f,F)}(s) = \inf (F^{\rightarrow} Q_s)^c,$$

are well defined and satisfy condition (a). We denote by $\mathrm{A}^+(S)$ the set of all point functions from S to $\mathbb{R}^+$ satisfying (a). For $\varphi \in \mathrm{A}^+(S)$ the functions φ_*, φ^* may be defined as in Definition 2.5 but without the restriction of s to $S^\flat$, and taking the sup and inf in $\mathbb{R}^+$. It is easy to verify that these functions are well defined and belong to $\mathrm{A}^+(S)$. We may also define conditions $(b+)$, $(c+)$ corresponding to $(b*)$, $(c*)$ as follows:

(b+) $s \in S, \rho \in \mathbb{R}^+, \ \rho < \varphi(s) \implies \exists\, s' \in S$ with $s' \, \omega_S \, s, \ \rho < \varphi(s')$.
(c+) $s \in S, \rho \in \mathbb{R}^+, \ \varphi(s) < \rho \implies \exists\, s' \in S$ with $s \, \omega_S \, s', \ \varphi(s') < \rho$.

3.4. Theorem. *Let $(f,F) : (S, \mathcal{S}) \to (\mathbb{R}^+, \mathcal{R}^+)$ be a difunction. Then,*

(1) *For $\varphi \in \mathrm{A}^+(S)$, $(f,F) = (f_\varphi, F_\varphi)$ if and only if $\lambda^{+}_{(f,F)} \leq \varphi \leq \xi^{+}_{(f,F)}$.*
(2) *For $\varphi \in \mathrm{A}^+(S)$ with $\lambda^{+}_{(f,F)} \leq \varphi \leq \xi^{+}_{(f,F)}$ we have:*
  (i) $\varphi = \lambda^{+}_{(f,F)} \iff \varphi = \varphi_* \iff \varphi$ *satisfies* $(b+)$.
  (ii) $\varphi = \xi^{+}_{(f,F)} \iff \varphi = \varphi^* \iff \varphi$ *satisfies* $(c+)$.

Proof. It is straightforward to verify that $\varphi : S \to \mathbb{R}^+$ satisfies (a) if and only if the equalities

$$f_\varphi = \bigvee \{ \overline{P}^{+}_{(s,\varphi(u))} \mid s,u \in S, u \, \omega_S \, s \}, \ F_\varphi = \bigcap \{ \overline{Q}^{+}_{(s,\varphi(u))} \mid s,u \in S, s \, \omega_S \, u \},$$

define a difunction $(f_\varphi, F_\varphi) : (S, \mathcal{S}) \to (\mathbb{R}^+, \mathcal{R}^+)$, and we omit the details which are similar to the proof of Lemma 2.3 (1).

(1) Take $\varphi \in \mathrm{A}^+(S)$. Clearly for $s \in S$, $\lambda^{+}_{(f,F)}(s) = \sup_{\mathbb{R}^+} \{ r \in \mathbb{R} \mid f_\varphi^{\rightarrow} P_s \not\subseteq Q_r \}$, while $f_\varphi \not\subseteq \overline{Q}^{+}_{(s,r)} \implies \exists\, u \in S$ with $u \, \omega_S \, s$ and $\overline{P}^{+}_{(u,\varphi(u))} \not\subseteq \overline{Q}^{+}_{(s,r)}$. This gives $r \leq \varphi(u) \leq \varphi(s)$

by condition (a), so $\lambda^+_{(f,F)}(s) \le \varphi(s)$ as required. A dual argument shows that $\varphi(s) \le \xi^+_{(f,F)}$.

(2) Take $\varphi \in \mathrm{A}^+(S)$ with $\lambda^+_{(f,F)} \le \varphi \le \xi^+_{(f,F)}$.

(i) Let $\varphi = \lambda^+_{(f,F)}$. Since $\varphi_* \le \varphi$, suppose we have $s \in S$ with $\varphi_*(s) < \varphi(s)$. Since $\varphi(s) = \lambda^+_{(f,F)}(s) = \lambda^+_{(f_\varphi,F_\varphi)}(s)$ by (1) we have $r \in \mathbb{R}$ with $\varphi_*(s) < r$ and $f_\varphi^{\to} P_s \not\subseteq Q_r$. Now $f_\varphi \not\subseteq \overline{Q}^+_{(s,r)}$, and f_φ is a relation so applying the condition *R2* we have $u \in S$ with $u\,\omega_S\,s$ and $f_\varphi \not\subseteq \overline{Q}^+_{(u,r)}$. However this gives $r \le \lambda^+_{(f_\varphi,F_\varphi)}(u) = \varphi(u) \le \varphi_*(s)$, which is a contradiction.

Now suppose that $\varphi = \varphi_*$ and take $s \in S$, $\rho \in \mathbb{R}^+$ with $\rho < \varphi(s)$. Then $\rho < \varphi_*(s)$ so there exists $u \in S$ with $u\,\omega_S\,s$ and $\rho < \varphi(u)$, which establishes (b+).

Finally suppose that φ satisfies (b+). By hypothesis $\lambda^+_{(f,F)} \le \varphi$ so assume there exists $s \in S$ with $\lambda^+_{(f,F)}(s) < \varphi(s)$. By ($b$+) there exists $u \in S$ with $u\,\omega_S\,s$ for which $\lambda^+_{(f_\varphi,F_\varphi)}(s) = \lambda^+_{(f,F)}(s) < \varphi(u)$. Choose $u' \in S$ with $u\,\omega_S\,u'$ and $u'\,\omega_S\,s$. Now $\overline{P}^+_{(s,\varphi(u'))} \subseteq f_\varphi$, and $P_{\varphi(u')} \not\subseteq Q_{\varphi(u)}$ by (a), so $f = f_\varphi \not\subseteq \overline{Q}^+_{(s,\varphi(u))}$ and we obtain the contradiction $\varphi(u) \le \lambda^+_{(f,F)}(s)$.

(ii) The proof is dual to (i), and is omitted. □

If we define an equivalence relation $\sim$ on $\mathrm{A}^+(S)$ by setting $\varphi \sim \psi \iff (f_\varphi, F_\varphi) = (f_\psi, F_\psi)$ the above theorem states that $\widetilde{\varphi} = [\varphi_*,\ \varphi^*]$ for each $\varphi \in \mathrm{A}^+(S)$. This means that the elements of $\mathrm{A}^+(S)/\!\sim$ may be represented as interval valued functions

$$s \mapsto [\varphi_*(s),\ \varphi^*(s)]$$

on S, equivalently $(f,F) \in \mathrm{DF}^+(S)$ is represented by $s \mapsto [\lambda^+_{(f,G)}(s),\ \xi^+_{(f,F)}(s)]$. As mentioned above, $(f,F) \in \mathrm{DF}(S)$ may be represented by applying the above to $(e,E) \circ (f,F)$.

It will be seen that the representation in terms of extended real valued functions is formally very similar to that using real valued functions on $S^\flat$, but has the advantage of not involving the set $S^\flat$ which in general does not belong to $\mathcal{S}$. A similar relation also holds between the order and addition on $\mathbb{R}^+$ and that on $\mathrm{A}^+(S)/\!\sim$ as does between the order and addition on $\mathbb{R}$ and that on $\mathrm{A}(S^\flat)/\!\sim$, as the interested reader may easily verify. Finally, to consider bicontinuous difunctions we need only note that

$$\tau_{\mathbb{R}^+} = \{(-\infty, r) \mid r \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}^+\},\ \ \kappa_{\mathbb{R}^+} = \{(-\infty, r] \mid r \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}^+\}$$

is the natural ditopology on $(\mathbb{R}^+, \mathcal{R}^+)$, and that under this ditopology (e,E) and its inverse are bicontinuous. Hence the additive lattice isomorphism $(f,F) \mapsto (e,E) \circ (f,F)$ of $\mathrm{DF}(S)$ with $\mathrm{DF}^+(S)$ restricts to a isomorphism of $\mathrm{BDF}(S)$ with the additive lattice $\mathrm{BDF}^+(S)$ of bicontinuous extended real difunctions on $(S, \mathcal{S}, \tau, \kappa)$. Moreover, it is easy to verify that the elements of $\mathrm{BDF}^+(S)$ may be represented by restricting ones attention to the elements of $\mathrm{BA}^+(S)$, the bicontinuous extended real point functions on $(S, \mathcal{S}, \tau, \kappa)$ satisfying (a), and hence the same is true for the elements of $\mathrm{BDF}(S)$.

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