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ESTIMATION OF PARAMETERS OF THE LOGLOGISTIC DISTRIBUTION BASED ON PROGRESSIVE CENSORING USING THE EM ALGORITHM

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Abstract

Although the maximum likelihood estimation method based on progressively censored data has been studied extensively, traditionally the Newton–Raphson method has been used to obtain the estimates (Ng *et al.*, 2002). As pointed out by Little and Rubin in 1983, the EM algorithm will converge reliably but rather slowly (as compared to the Newton–Raphson method) when the amount of information in the missing data is relatively large. Therefore, in this study, maximum likelihood estimates for the parameters of the Loglogistic distribution are obtained using the EM algorithm based on a progressive Type-II right censored sample. An illustrative example is also given.

Keywords: EM algorithm, Loglogistic distribution, Maximum likelihood estimate.

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1. Introduction

The loglogistic distribution is a commonly used lifetime distribution in lifetime data analysis since the logarithm of the lifetime variables are logistically distributed. Because of the well-known properties of the logistic distribution and because it belongs to the location-scale family, the log-lifetimes will be used in this paper. If X has a logistic distribution with location parameter μ and scale parameter σ, then the probability density function (p.d.f.) and distribution function (d.f.) of X are given respectively by

$$f(x;\boldsymbol{\theta}) = \frac{\exp\left(\frac{x-\mu}{\sigma}\right)}{\sigma\left\{1+\exp\left(\frac{x-\mu}{\sigma}\right)\right\}^2},\ x\in\mathbb{R},\ \mu\in\mathbb{R},\ \sigma\in\mathbb{R}_+$$

$$F(x;\boldsymbol{\theta}) = \frac{\exp\left(\frac{x-\mu}{\sigma}\right)}{1+\exp\left(\frac{x-\mu}{\sigma}\right)}$$

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where $\boldsymbol{\theta} = (\mu, \sigma)$.

In this study, the censoring scheme known as progressive Type-II right censoring is considered. The progressive Type-II right censoring model is of importance in the field of reliability and life testing. Suppose n identical units are placed on a lifetime test. At the time of the ith failure, R_i surviving units are randomly withdrawn from the experiment, $1 \leq i \leq m$. Thus, if m failures are observed then $R_1 + \cdots + R_m$ units are progressively censored; hence $n = m + R_1 + \cdots + R_m$, $X^{\mathbf{R}}_{1:m:n} \leq X^{\mathbf{R}}_{2:m:n} \leq \cdots \leq X^{\mathbf{R}}_{m:m:n}$ describe the progressively censored failure times, where $\mathbf{R} = (R_1, \ldots, R_m)$ denotes the censoring scheme (Bairamov and Eryılmaz, [4]). If the failure times of the n items originally on test are from a continuous population with d.f. $F(x)$ and p.d.f. $f(x)$, the joint probability density function for $X^{\mathbf{R}}_{1:m:n}, X^{\mathbf{R}}_{2:m:n}, \ldots, X^{\mathbf{R}}_{m:m:n}$ is given by

$$f_{1,2,\ldots,m}(x^{\mathbf{R}}_{1:m:n}, \ldots, x^{\mathbf{R}}_{m:m:n}) = c \prod_{j=1}^{m} f(x^{\mathbf{R}}_{j:m:n})(1 - F(x^{\mathbf{R}}_{j:m:n}))^{R_j}$$

$$(-\infty < x^{\mathbf{R}}_{1:m:n} < x^{\mathbf{R}}_{2:m:n} < \cdots < x^{\mathbf{R}}_{m:m:n} < \infty),$$

where

$$c = n\,(n - R_1 - 1)\,(n - R_1 - R_2 - 2) \cdots (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1)$$

(Balakrishnan and Aggarwala, [6]). As a special case, if $\mathbf{R} = (0, \ldots, 0)$, where no withdrawals are made, the ordinary order statistics are obtained (Bairamov and Eryılmaz, [4]). For more details see Balakrishnan and Aggarwala [6].

Statistical inference on the parameters of some distributions under progressive Type-II censoring has been investigated by several authors such as Cohen [10], Mann [16], Thomas and Wilson [20], Wong [24], Viveros and Balakrishnan [21], Balakrishnan and Sandhu [5], Balasooriya *et al.* [9], Wu [22], Ng *et al.* [19], Balakrishnan and Lin [7], Balakrishnan *et al.* [8], Wu *et al.* [23] and Fernandez [12].

Ng *et al.* [19] determined the maximum likelihood estimates when the data are progressively Type-II right censored, and they considered the lognormal and Weibull lifetime distributions to illustrate their methodology. In this study, the results in their paper are used for the loglogistic distribution.

This paper is organized as follows: In Section 2, how the EM algorithm is used to determine the maximum likelihood estimates in a general setting . In Section 3, it is explained how the maximum likelihood estimates (MLEs) for the two parameters μ and σ of the loglogistic distribution are derived based on progressive Type-II right censored samples using the EM algorithm. In Section 4, a numerical example is presented to close the paper.

2. The EM algorithm

The MLEs of μ and σ must be derived numerically. The Newton–Raphson algorithm is one of the standard methods to determine the MLEs of the parameters. To employ this algorithm, the second derivatives of the log-likelihood are required for all iterations. The EM algorithm is a very powerful tool in handling the incomplete data problem (Dempster *et al.*, [11], McLachlan and Krishnan, [17]). It is an iterative method which repeatedly replaces the missing data with estimated values and updates the parameter estimates. It is especially useful if the complete data set is easy to analyze. As pointed out by Little and Rubin [15], the EM algorithm will converge reliably but rather slowly (as compared to the Newton–Raphson method) when the amount of information in the missing data is relatively large (Ng *et al.,* [19]). Recently, the EM algorithm has been used by several authors, including Adamidis and Loukas [1], Adamidis [2], Ng *et al.* [19], Karlis [13], Adamidis *et al.* [3] and Kuş [14].

First of all, we denote the observed and censored data by

$$\mathbf{Y} = (Y^{\mathbf{R}}_{1:m:n}, Y^{\mathbf{R}}_{2:m:n}, \ldots, Y^{\mathbf{R}}_{m:m:n})$$

and

$$\mathbf{Z} = (\mathbf{Z}_1,\ \mathbf{Z}_2, \ldots, \mathbf{Z}_m),$$

respectively. Here, $\mathbf{Z}_j$ is a $1 \times R_j$ vector with $\mathbf{Z}_j = (Z_{j1}, Z_{j2}, \ldots, Z_{jR_j})$, for $j = 1, 2, \ldots, m$. The censored data vector $\mathbf{Z}$ can be thought of as the missing data. Therefore, the progressive right censoring model problem can be viewed as an incomplete data problem and then the EM algorithm may be applied to the problem of obtaining the MLEs of the parameters. We combine $\mathbf{Y}$ and $\mathbf{Z}$ to form $\mathbf{X}$, which is the complete data set. The corresponding log-likelihood function is denoted by $\ell(\mathbf{X}, \boldsymbol{\theta})$. Then the E-step of the algorithm requires the computation of the conditional expectation

$$E[\ell(\mathbf{X},\ \boldsymbol{\theta}_{(h+1)}) \,|\, \mathbf{Y} = \mathbf{y},\ \boldsymbol{\theta}_{(h)}],$$

which mainly involves the computation of the conditional expectation of functions of $\mathbf{Z}$ conditional on the observed values $\mathbf{Y}$ and the current value of the parameters. Therefore, in order to facilitate the EM algorithm, the conditional distribution of $\mathbf{Z}$, conditional on $\mathbf{Y}$ and the current value of the parameters, needs to be determined (Ng *et al.,* [19]). The following theorem given by Ng *et al.* [19] gives a basic result on which the EM algorithm for progressively right censored data can be developed.

2.1. Theorem. *Given that* $Y^{\mathbf{R}}_{1:m:n} = y^{\mathbf{R}}_{1:m:n}, Y^{\mathbf{R}}_{2:m:n} = y^{\mathbf{R}}_{2:m:n}, \ldots, Y^{\mathbf{R}}_{j:m:n} = y^{\mathbf{R}}_{j:m:n}$, *the conditional distribution of* Z_{jk}, $k = 1, 2, \ldots, R_j$ *is*

$$\begin{aligned} f_{Z|Y}&\left(z_j | Y^{\mathbf{R}}_{1:m:n} = y^{\mathbf{R}}_{1:m:n}, Y^{\mathbf{R}}_{2:m:n} = y^{\mathbf{R}}_{2:m:n}, \ldots, Y^{\mathbf{R}}_{j:m:n} = y^{\mathbf{R}}_{j:m:n}\right) \\ &= f_{Z|Y}(z_j | Y^{\mathbf{R}}_{j:m:n} = y^{\mathbf{R}}_{j:m:n}) = \frac{f_X(z_j)}{[1 - F_X(y^{\mathbf{R}}_{j:m:n})]}, \quad z_j > y^{\mathbf{R}}_{j:m:n}, \end{aligned}$$

and Z_{jk}, Z_{jl}, $k \neq l$, *are conditionally independent given* $Y^{\mathbf{R}}_{j:m:n} = y^{\mathbf{R}}_{j:m:n}$.

Proof. Ng *et al.* [19]. □

The theorem states that given $Y^{\mathbf{R}}_{j:m:n} = y^{\mathbf{R}}_{j:m:n}$, the $\mathbf{Z}_j$ form a random sample from the truncated population and hence the expectations of functions of $\mathbf{Z}_j$ can be obtained.

In the M-step on the $(h+1)th$ iteration of the EM algorithm, the value of $\boldsymbol{\theta}$ which maximizes $E[\ell(\mathbf{X}, \boldsymbol{\theta}_{(h+1)}) \,|\, \mathbf{Y} = \mathbf{y}, \theta_{(h)}]$ will be used as the next estimate of $\boldsymbol{\theta}_{(h+1)}$. The MLE of $\boldsymbol{\theta}$ can be obtained by repeating the E-step and M-step until convergence occurs.

3. Estimation based on progressive Type-II right censored samples

In this section, the MLEs for the parameters of the Loglogistic distribution are obtained using the EM algorithm based on a progressive Type-II right censored sample. The log-likelihood function based on the complete log-lifetimes $\mathbf{X}$ is

$$\begin{aligned} \ell(\mathbf{X}\ ;\ \boldsymbol{\theta}) &= -n\log(\sigma) + \sum_{i=1}^{n} \frac{x_i - \mu}{\sigma} - 2\sum_{i=1}^{n} \log\left\{1 + \exp\left(\frac{x_i - \mu}{\sigma}\right)\right\} \\ &= -n\log(\sigma) + \sum_{j=1}^{m} \frac{y^{\mathbf{R}}_{j:m:n} - \mu}{\sigma} - 2\sum_{j=1}^{m} \log\left\{1 + \exp\left(\frac{y^{\mathbf{R}}_{j:m:n} - \mu}{\sigma}\right)\right\} \\ &\quad + \sum_{j=1}^{m}\sum_{k=1}^{R_j} \frac{Z_{jk} - \mu}{\sigma} - 2\sum_{j=1}^{m}\sum_{k=1}^{R_j} \log\left\{1 + \exp\left(\frac{Z_{jk} - \mu}{\sigma}\right)\right\}. \end{aligned}$$

Here, logarithms are to base e.

Using Theorem 2.1, the conditional distribution of Z_j given $Y^{\mathbf{R}}_{j:m:n} = y^{\mathbf{R}}_{j:m:n}$ is a truncated logistic distribution which is left truncated at $y^{\mathbf{R}}_{j:m:n}$ and has p.d.f:

$$
(1) \qquad f_{Z_j}\left(z_j|Z_j > y^{\mathbf{R}}_{j:m:n},\ \mu,\ \sigma\right) = \frac{\exp\left(\dfrac{z_j-\mu}{\sigma}\right)\left\{1+\exp\left(\dfrac{y^{\mathbf{R}}_{j:m:n}-\mu}{\sigma}\right)\right\}}{\sigma\left\{1+\exp\left(\dfrac{z_j-\mu}{\sigma}\right)\right\}^2},\ z_j > y^{\mathbf{R}}_{j:m:n}.
$$

Using (1), the required conditional expectations in the E-step are obtained in the form

$$
(2) \qquad \begin{aligned} E\left(Z_j|y^{\mathbf{R}}_{j:m:n},\mu,\sigma\right) &= \sigma\left\{1+\exp\left(\frac{y^{\mathbf{R}}_{j:m:n}-\mu}{\sigma}\right)\right\} \\ &\quad\times\left\{\log\left[1+\exp\left(\frac{y^{\mathbf{R}}_{j:m:n}-\mu}{\sigma}\right)\right]-\log\left[\exp\left(-\frac{\mu}{\sigma}\right)\right]\right\} \\ &\quad - y^{\mathbf{R}}_{j:m:n}\exp\left(\frac{y^{\mathbf{R}}_{j:m:n}-\mu}{\sigma}\right), \end{aligned}
$$

$$
(3) \qquad E\left\{\frac{\exp\left(\dfrac{Z_j-\mu}{\sigma}\right)}{1+\exp\left(\dfrac{Z_j-\mu}{\sigma}\right)}\ \middle|\ y^{\mathbf{R}}_{j:m:n},\ \mu,\ \sigma\right\} = \frac{1}{2}\,\frac{\exp\left(\dfrac{\mu}{\sigma}\right)+2\exp\left(\dfrac{y^{\mathbf{R}}_{j:m:n}}{\sigma}\right)}{\exp\left(\dfrac{\mu}{\sigma}\right)+\exp\left(\dfrac{y^{\mathbf{R}}_{j:m:n}}{\sigma}\right)}
$$

and

$$
(4) \qquad \begin{aligned} &E\left\{\frac{(Z_j-\mu)\exp\left(\dfrac{Z_j-\mu}{\sigma}\right)}{1+\exp\left(\frac{Z_j-\mu}{\sigma}\right)}\ \middle|\ y^{\mathbf{R}}_{j:m:n},\ \mu,\ \sigma\right\} \\ &= \left\{\left[-\exp\left(\frac{2y^{\mathbf{R}}_{j:m:n}}{\sigma}\right)-2\exp\left(\frac{\mu+y^{\mathbf{R}}_{j:m:n}}{\sigma}\right)-\exp\left(\frac{2\mu}{\sigma}\right)\right]\right. \\ &\quad\times\sigma\log\left[1+\exp\left(\frac{y^{\mathbf{R}}_{j:m:n}-\mu}{\sigma}\right)\right]+(y-\mu)\exp\left(\frac{2y_{j:m:n}}{\sigma}\right) \\ &\quad\left.-\sigma\left[\exp\left(\frac{2\mu}{\sigma}\right)-\exp\left(\frac{\mu+y^{\mathbf{R}}_{j:m:n}}{\sigma}\right)\right]\right\}\left[\exp\left(\frac{y^{\mathbf{R}}_{j:m:n}}{\sigma}\right)+\exp\left(\frac{\mu}{\sigma}\right)\right]^{-1}. \end{aligned}
$$

The expectations (2), (3) and (4) can be evaluated quickly using Maple software.

The MLEs of the parameters based on the complete data cannot be solved explicitly. The MLEs of the parameters can be obtained by solving the equations

$$
\frac{n}{2}-\sum_{j=1}^{n}\frac{\exp\left(\dfrac{x_j-\mu}{\sigma}\right)}{1+\exp\left(\dfrac{x_j-\mu}{\sigma}\right)} = 0,
$$

$$
n\sigma - n\mu + \sum_{j=1}^{n}x_j - 2\sum_{j=1}^{n}\frac{(x_j-\mu)\exp\left(\dfrac{x_j-\mu}{\sigma}\right)}{\left[1+\exp\left(\dfrac{x_j-\mu}{\sigma}\right)\right]} = 0,
$$

simultaneously. The EM cycle is completed with an M-step, which obtains a complete-data maximum likelihood over $\boldsymbol{\theta}$, the missing values being replaced by their conditional expectations.

Thus, in the M-step of the $(h+1)th$ iteration of the EM algorithm, the value of $\mu_{(h+1)}$ is first obtained by solving the equation

$$\frac{n}{2}-\sum_{j=1}^{m}\frac{\exp\left(\dfrac{y^{\mathbf{R}}_{j:m:n}-\mu_{(h+1)}}{\sigma_{(h)}}\right)}{1+\exp\left(\dfrac{y^{\mathbf{R}}_{j:m:n}-\mu_{(h+1)}}{\sigma_{(h)}}\right)}=\sum_{j=1}^{m}R_j\mathrm{E}\left\{\frac{\exp\left(\dfrac{Z_j-\mu}{\sigma}\right)}{1+\exp\left(\dfrac{Z_j-\mu}{\sigma}\right)}\,\Bigg|\,y^{\mathbf{R}}_{j:m:n},mu_{(h)},\ \sigma_{(h)}\right\}$$

and then $\sigma_{(h+1)}$ is obtained by solving the equation

$$\begin{aligned}n\sigma_{(h+1)}&=n\mu_{(h+1)}-\sum_{j=1}^{m}y^{\mathbf{R}}_{j:m:n}-\sum_{j=1}^{m}R_j\mathrm{E}\big(Z_j\mid y^{\mathbf{R}}_{j:m:n},\ \mu_{(h+1)},\ \sigma_{(h)}\big)\\&+2\sum_{j=1}^{m}\frac{\big(y^{\mathbf{R}}_{j:m:n}-\mu_{(h+1)}\big)\exp\left(\dfrac{y^{\mathbf{R}}_{j:m:n}-\mu_{(h+1)}}{\sigma_{(h+1)}}\right)}{1+\exp\left(\dfrac{y^{\mathbf{R}}_{j:m:n}-\mu}{\sigma_{(h+1)}}\right)}\\&+2\sum_{j=1}^{m}R_j\mathrm{E}\left\{\frac{(Z_j-\mu)\exp\left(\dfrac{Z_j-\mu}{\sigma}\right)}{1+\exp\left(\dfrac{Z_j-\mu}{\sigma}\right)}\,\Bigg|\,y^{\mathbf{R}}_{j:m:n},\ \mu_{(h+1)},\ \sigma_{(h)}\right\}\end{aligned}$$

4. An illustrative example

Nelson [18, p.228] presented data on the time to breakdown of an insulating fluid in an accelerated test at 34 kilovolts. This data is given in Table 1.

Table 1. Nelson's Data

0.19	0.78	0.96	1.31	2.78	3.16	4.15	4.67	4.85	6.50	7.35	8.01
8.27	12.06	31.75	32.52	33.91	36.71	72.89					

Figure 1. Loglogistic probability plot for the Nelson data.

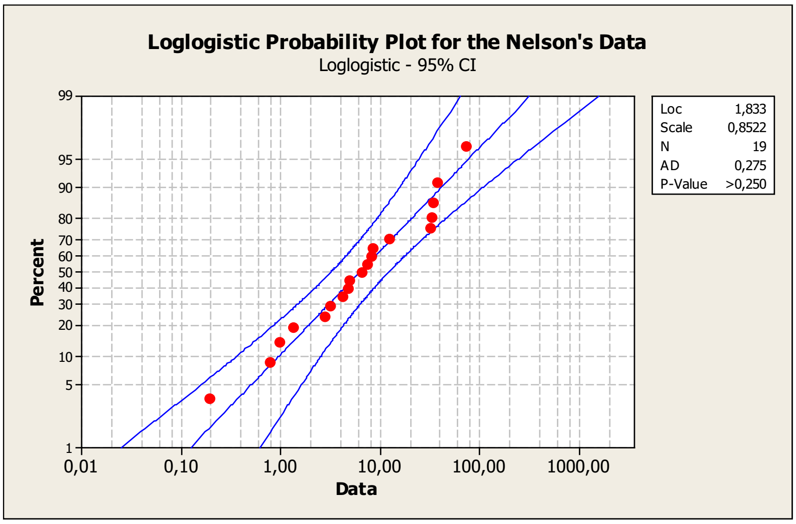
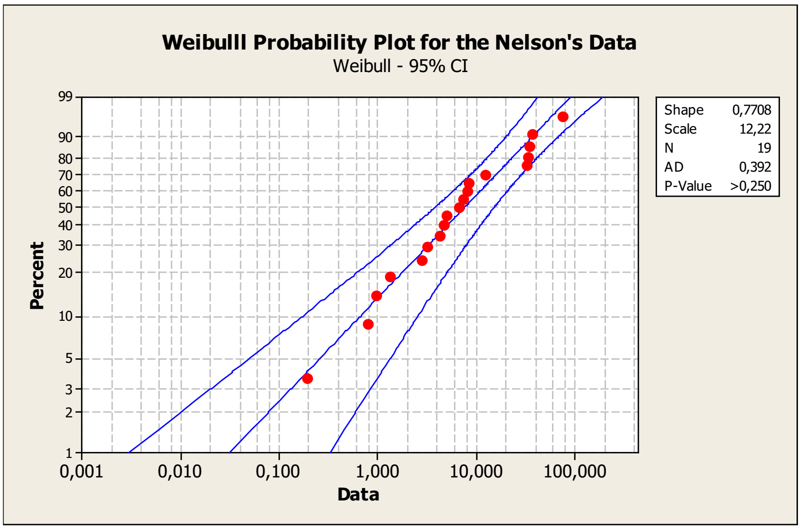


Figure 2. Weibull probability plot for the Nelson data.



In analyzing this data set, Nelson [18] considered a Weibull Distribution which is further discussed in Viveros and Balakrishnan [21] and Ng *et al.* [19]. Figs. 1 and 2 show that the Loglogistic distribution provides better fits to Nelson's data than does the Weibull distribution (see AD statistics).

A progressively censored sample generated from the log-lifetimes to breakdown data on an insulating fluid tested at 34 KV, presented by Viveros and Balakrishnan [21], is used to demonstrate the above estimation procedure. This data is presented in Table 2.

Table 2. Progressively censored sample presented by Viveros and Balakrishnan

i	1	2	3	4	5	6	7	8
R_i	0	0	3	0	3	0	0	5
$x^{\mathbf{R}}_{i:m:n}$	-1.6608	-0.2485	-0.0409	0.2700	1.0224	1.5789	1.8718	1.9947

For this data, the EM algorithm was used with starting values $\mu_{(0)} = 1$ and $\sigma_{(0)} = 1$. The EM algorithm converged to the values $\mu_{(\infty)} = 1.8757$ and $\mu_{(\infty)} = 0.9027$. The values of $\mu_{(h)}$ and $\sigma_{(h)}$ were plotted against h, and the results presented in Figures 3 and 4.

The level of accuracy was fixed at 10^{-5}. The EM algorithm took 41 iterations to converge.

Figure 3. Trace plot for $\mu_{(h)}$ under EM-iterations.

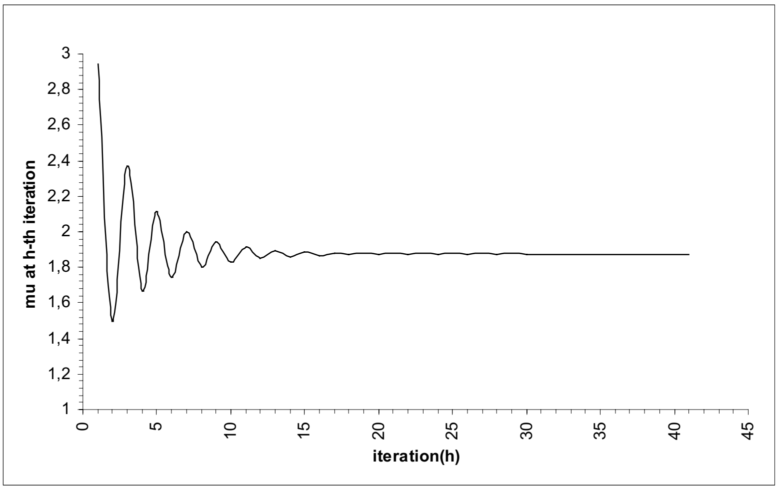
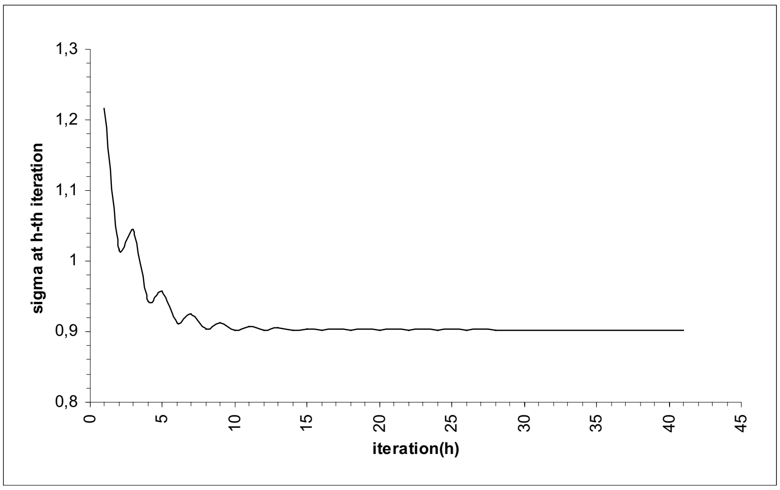


Figure 3. Trace plot for $\sigma_{(h)}$ under EM-iterations.



5. Conclusion

In the current literature, the maximum likelihood estimation method based on progressively censored data has been studied extensively. However, the vast majority of the existing studies traditionally use the Newton–Raphson method to obtain the estimates.

As pointed out by Little and Rubin [15], the EM algorithm will converge reliably but rather slowly as compared to the Newton–Raphson method when the amount of information in the missing data is relatively large. Ng *et al.* [19] used this algorithm to determine the maximum likelihood estimates when the data are progressively Type-II right censored, and they considered lognormal and Weibull lifetime distributions to illustrate their methodology.

In this paper, we have exploited the EM algorithm in order to obtain the maximum likelihood estimates for the parameters of a Loglogistic distribution based on a progressive Type-II right censored sample using the methodology of Ng *et al.* [19]. We have also provided a real example to illustrate the use of the EM algorithm to obtain the MLEs for the parameters of a Loglogistic distribution.

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