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MIXED ESTIMATORS FOR ORDERED SCALE PARAMETERS OF TWO WEIBULL DISTRIBUTIONS

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Abstract

The problem of the estimation of ordered scale parameters in two Weibull populations is considered when their shape parameters are assumed to be known and unequal. Mixed estimators for ordered scale parameters are obtained and their risks and biases are compared with the maximum likelihood estimators with the help of a simulation, and shown to improve upon them.

Keywords: Mixed Estimators, Maximum Likelihood Estimators, Ordered Scale Parameters, Weibull Distribution.

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1. Introduction

The Weibull distribution has been named after the Swedish scientist, Weibull, who proposed it for the first time in 1939 in connection with his studies on the strength of materials. Weibull [7] showed that the distribution was also useful in describing "wear-out" or fatigue failures. Kao [2] used it as a model for ball bearing failures. Mann [4] gave a variety of situations in which the distribution can be used for other types of failure data.

In this study we are interested in the estimation of the scale parameters when it is assumed *a priori* that the scale parameter of one population is smaller than that of the other.

Vijayasree and Singh [6] carried out some work on a negative exponential distribution. Srivastava [5] has obtained maximum likelihood estimators of ordered scale parameter in two Weibull populations.

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Let X_{ij}, $j = 1, 2, \dots, n$, be independent random samples from the Weibull population Pi_i, $i = 1, 2$. The probability density function of Pi_i is given by

$$f(x, \alpha_i, \beta_i) = \frac{\alpha_i}{\beta_i}\left(\frac{x}{\beta_i}\right)^{\alpha_i - 1} \exp\left(-\frac{x}{\beta_i}\right)^{\alpha_i},\ x \geq 0,\ \beta_i,\ \alpha_i > 0,$$

where the α_i's are shape parameters and the β_i's scale parameters (Johnson and Kotz [1]).

The loss functions used here are as follows:

Squared Error Loss Function: $L_1(d, \beta) = (d - \beta)^2$,

Scale Invariant Squared Error Loss Function: $L_2(d, \beta) = (\frac{d}{\beta} - 1)^2$.

Thus the corresponding risk functions are

$$R_1(d, \beta) = \mathrm{E}(d - \beta)^2 \text{ and } R_2(d, \beta) = \mathrm{E}(\frac{d}{\beta} - 1)^2,$$

respectively. The biases will be the absolute bias, $B_1(d, \beta) = |\mathrm{E}(d - \beta)|$ and the standardized bias, $B_2(d, \beta) = |E(\frac{d}{\beta} - 1)|$.

When the parameters α_1 and α_2 are known the maximum likelihood estimators of β_1 and β_2 are $\left(\sum\limits_{j=1}^{n} \frac{X_{1j}^{\alpha_1}}{n_1}\right)^{1/\alpha_1}$ and $\left(\sum\limits_{j=1}^{n} \frac{X_{2j}^{\alpha_2}}{n_2}\right)^{1/\alpha_2}$, respectively. Let $Z_1 = \sum\limits_{j=1}^{n_1} X_{1j}^{\alpha_1}$ and $Z_2 = \sum\limits_{j=1}^{n_2} X_{2j}^{\alpha_2}$. Then Z_1 and Z_2 are independent random variables which have gamma distributions with parameters $(n_1, \beta_1^{\alpha_1})$ and $(n_2, \beta_2^{\alpha_2})$, respectively. Hence the joint density density function of Z_1 and Z_2 is given by

$$f(z_1, z_2) = \frac{1}{\Gamma(n_1)\Gamma(n_2)\beta_1^{n_1\alpha_1}\beta_2^{n_2\alpha_2}} \mathrm{e}^{-z_1/\beta_1^{\alpha_1}} \mathrm{e}^{-z_2/\beta_2^{\alpha_2}} z_1^{n_1 - 1} z_2^{n_2 - 1},$$

for $z_1 > 0$, $z_2 > 0$. In terms of Z_1 and Z_2 the maximum likelihood estimators of β_1 and β_2 are $d_1 = \left(\frac{z_1}{n_1}\right)^{1/\alpha_1}$ and $d_2 = \left(\frac{Z_2}{n_2}\right)^{1/\alpha_2}$, respectively.

In Section 2 the risks and biases of the maximum likelihood estimators are given. Section 3 discusses the mixed estimators and their biases and risks. In Section 4 the simulation results are given.

2. The risks and biases of the maximum likelihood estimators (MLE)

The following results are reproduced from the paper of Srivastava [5].

2.1. Lemma. *With the notation defined above we have:*

$$\mathrm{E}\left(\sum_{j=1}^{n_i} X_{ij}\right)^{2/\alpha_i} = \beta_i^2 \frac{\Gamma(n_i + 2/\alpha_i)}{\Gamma(n_i)},\ i = 1, 2$$

and

$$E\left(\sum_{j=1}^{n_i} X_{ij}\right)^{1/\alpha_i} = \beta_i \frac{\Gamma(n_i + 1/\alpha_i)}{\Gamma(n_i)},\ i = 1, 2$$

The risks and biases for the maximum likelihood estimators are given in the following theorem.

2.2. Theorem. *For the maximum likelihood estimator d_1 the risks using the squared error loss function and the scale invariant error loss function, respectively, are as follows:*

$$R_1(d_1,\beta_1) = \beta_1^2\left[\frac{\Gamma(n_1+2/\alpha_1)}{n_1^{2/\alpha_1}\Gamma(n_1)} + 1 - \frac{2\Gamma(n_1+1/\alpha_1)}{n_1^{1/\alpha_1}\Gamma(n_1)}\right],$$
$$R_2(d_1,\beta_1) = \frac{\Gamma(n_1+2/\alpha_1)}{n_1^{2/\alpha_1}\Gamma(n_1)} + 1 - \frac{2\Gamma(n_1+1/\alpha_1)}{n_1^{1/\alpha_1}\Gamma(n_1)}.$$

The corresponding biases are:

$$B_1(d_1,\beta_1) = \beta_1\left|\frac{2\Gamma(n_1+1/\alpha_1)}{n_1^{1/\alpha_1}\Gamma(n_1)} - 1\right|,$$
$$B_2(d_1,\beta_1) = \left|\frac{2\Gamma(n_1+1/\alpha_1)}{n_1^{1/\alpha_1}\Gamma(n_1)} - 1\right|.$$

For the maximum likelihood estimator d_2 the risks are

$$R_1(d_2,\beta_2) = \beta_2^2\left[\frac{\Gamma(n_2+2/\alpha_2)}{n_2^{2/\alpha_2}\Gamma(n_2)} + 1 - \frac{2\Gamma(n_2+1/\alpha_2)}{n_2^{1/\alpha_2}\Gamma(n_2)}\right],$$
$$R_2(d_2,\beta_2) = \frac{\Gamma(n_2+2/\alpha_2)}{n_2^{2/\alpha_2}\Gamma(n_2)} + 1 - \frac{2\Gamma(n_2+1/\alpha_2)}{n_2^{1/\alpha_2}\Gamma(n_2)},$$

and the corresponding biases are

$$B_1(d_2,\beta_2) = \beta_2\left|\frac{2\Gamma(n_2+1/\alpha_2)}{n_2^{1/\alpha_2}\Gamma(n_2)} - 1\right|,$$
$$B_2(d_2,\beta_2) = \left|\frac{2\Gamma(n_2+1/\alpha_2)}{n_2^{1/\alpha_2}\Gamma(n_2)} - 1\right|.$$

3. The mixed estimator of ordered scale parameters

We now consider the mixed estimator of β_1 and β_2 with $\beta_1 < \beta_2$ on the lines of Kumar and Sharma [3]. The mixed estimators of β_1 and β_2 are given by,

$$d_{1\theta} = \min\left\{\left(\frac{\sum X_{1j}^{\alpha_1}}{n_1}\right)^{1/\alpha_1},\ \theta\left(\frac{\sum X_{1j}^{\alpha_1}}{n_1}\right)^{1/\alpha_1} + (1-\theta)\left(\frac{\sum X_{2j}^{\alpha_2}}{n_2}\right)^{1/\alpha_2}\right\},$$
$$d_{2\theta} = \max\left\{\left(\frac{\sum X_{2j}^{\alpha_2}}{n_2}\right)^{1/\alpha_2},\ \theta\left(\frac{\sum X_{1j}^{\alpha_1}}{n_1}\right)^{1/\alpha_1} + (1-\theta)\left(\frac{\sum X_{2j}^{\alpha_2}}{n_2}\right)^{1/\alpha_2}\right\},$$

where $0 < \theta < 1$.

The mixed estimators for $0 < \theta < 1$ can be rewritten in terms of Z_1 and Z_2 as follows:

$$d_{1\theta} = \min\left\{\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1},\ \theta\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1} + (1-\theta)\left(\frac{Z_2}{n_2}\right)^{1/\alpha_2}\right\},$$
$$d_{2\theta} = \min\left\{\left(\frac{Z_2}{n_2}\right)^{1/\alpha_2},\ \theta\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1} + (1-\theta)\left(\frac{Z_2}{n_2}\right)^{1/\alpha_2}\right\}.$$

The risk functions $R_1(d_{1\theta},\beta_1)$ and $R_1(d_{2,\theta},\beta_2)$ corresponding to the squared error loss function are given below.

$$\begin{aligned}
R_1(d_{1\theta},\beta_1) &= \mathrm{E}(d_{1\theta}-\beta_1)^2 \\
&= \mathrm{E}\left(\min\left\{\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1}, \theta\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1} + (1-\theta)\left(\frac{Z_2}{n_2}\right)^{1/\alpha_2}\right\} - \beta_1\right)^2 \\
&= \int_0^\infty \int_0^{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}} \left(\left(\frac{z_1}{n_1}\right)^{1/\alpha_1} - \beta_1\right)^2 f(z_1,z_2)dz_1dz_2 \\
&\quad + \int_0^\infty \int_{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}}^\infty \left(\theta\left(\frac{z_1}{n_1}\right)^{1/\alpha_1} + (1-\theta)\left(\frac{z_2}{n_2}\right)^{1/\alpha_2} - \beta_1\right)^2 \\
&\qquad \times f(z_1,z_2)dz_1dz_2 \\
&= \int_0^\infty \int_0^{n_1(\frac{z_2}{n_2})^{\alpha_1/\alpha_2}} \left(\left(\frac{z_1}{n_1}\right)^{1/\alpha_1} - \beta_1\right)^2 \frac{\mathrm{e}^{-z_1/\beta_1^{\alpha_1}}\mathrm{e}^{-z_2/\beta_2^{\alpha_2}} z_1^{n_1-1} z_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\beta_1^{n_1\alpha_1}\beta_2^{n_2\alpha_2}} dz_1dz_2 \\
&\quad + \int_0^\infty \int_{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}}^\infty \left(\theta\left(\frac{z_1}{n_1}\right)^{1/\alpha_1} + (1-\theta)\left(\frac{z_2}{n_2}\right)^{1/\alpha_2} - \beta_1\right)^2 \\
&\qquad \times \frac{\mathrm{e}^{-z_1/\beta_1^{\alpha_1}}\mathrm{e}^{-z_2/\beta_2^{\alpha_2}} z_1^{n_1-1} z_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\beta_1^{n_1\alpha_1}\beta_2^{n_2\alpha_2}} dz_1dz_2, \\
R_1(d_{2\theta},\beta_2) &= \mathrm{E}(d_{2\theta}-\beta_2)^2 \\
&= \mathrm{E}\left(\max\left\{\left(\frac{Z_2}{n_2}\right)^{1/\alpha_2}, \theta\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1} + (1-\theta)\left(\frac{Z_2}{n_2}\right)^{1/\alpha_2}\right\} - \beta_2\right)^2 \\
&= \int_0^\infty \int_0^{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}} \left(\left(\frac{z_2}{n_2}\right)^{1/\alpha_2} - \beta_2\right)^2 f(z_1,z_2)dz_1dz_2 \\
&\quad + \int_0^\infty \int_{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}}^\infty \left(\theta\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1} + (1-\theta)\left(\frac{z_2}{n_2}\right)^{1/\alpha_2} - \beta_2\right)^2 \\
&\qquad \times f(z_1,z_2)dz_1dz_2 \\
&= \int_0^\infty \int_0^{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}} \left(\left(\frac{z_2}{n_2}\right)^{1/\alpha_2} - \beta_2\right)^2 \frac{\mathrm{e}^{-z_1/\beta_1^{\alpha_1}}\mathrm{e}^{-z_2/\beta_2^{\alpha_2}} z_1^{n_1-1} z_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\beta_1^{n_1\alpha_1}\beta_2^{n_2\alpha_2}} dz_1dz_2 \\
&\quad + \int_0^\infty \int_{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}}^\infty \left(\theta\left(\frac{z_1}{n_1}\right)^{1/\alpha_1} + (1-\theta)\left(\frac{z_2}{n_2}\right)^{1/\alpha_2} - \beta_2\right)^2 \\
&\qquad \times \frac{\mathrm{e}^{-z_1/\beta_1^{\alpha_1}}\mathrm{e}^{-z_2/\beta_2^{\alpha_2}} z_1^{n_1-1} z_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\beta_1^{n_1\alpha_1}\beta_2^{n_2\alpha_2}} dz_1dz_2.
\end{aligned}$$

The risk functions corresponding to the scale invariant loss function are given below.

$$
\begin{aligned}
R_2(d_{1\theta},\beta_1) &= \mathrm{E}\left(\frac{d_{1\theta}}{\beta_1}-1\right)^2 \\
&= \mathrm{E}\left\{\frac{1}{\beta_1}(min\left\{\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1},\theta\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1}+(1-\theta)\left(\frac{Z_2}{n_2}\right)^{1/\alpha_2}\right\}-1\right)^2 \\
&= \int_0^\infty \int_0^{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}} \left(\frac{1}{\beta_1}\left(\frac{z_1}{n_1}\right)^{1/\alpha_1}-1\right)^2 f(z_1,z_2)dz_1dz_2 \\
&\quad + \int_0^\infty \int_{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}}^\infty \left(\frac{1}{\beta_1}\theta\left(\frac{z_1}{n_1}\right)^{1/\alpha_1}+\frac{1}{\beta_1}(1-\theta)\left(\frac{z_2}{n_2}\right)^{1/\alpha_2}-1\right)^2 \\
&\qquad \times f(z_1,z_2)dz_1dz_2 \\
&= \int_0^\infty \int_0^{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}} \left(\frac{1}{\beta_1}(z_1/n_1)^{1/\alpha_1}-1\right)^2 \frac{\mathrm{e}^{-z_1/\beta_1^{\alpha_1}}\mathrm{e}^{-z_2/\beta_2^{\alpha_2}}z_1^{n_1-1}z_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\beta_1^{n_1\alpha_1}\beta_2^{n_2\alpha_2}}dz_1dz_2 \\
&\quad + \int_0^\infty \int_{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}}^\infty \left(\frac{1}{\beta_1}\theta(z_1/n_1)^{1/\alpha_1}+\frac{1}{\beta_1}(1-\theta)\left(\frac{z_2}{n_2}\right)^{1/\alpha_2}-1\right)^2 \\
&\qquad \times \frac{\mathrm{e}^{-z_1/\beta_1^{\alpha_1}}\mathrm{e}^{-z_2/\beta_2^{\alpha_2}}z_1^{n_1-1}z_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\beta_1^{n_1\alpha_1}\beta_2^{n_2\alpha_2}}dz_1dz_2,
\end{aligned}
$$

$$
\begin{aligned}
R_2(d_{2\theta},\beta_2) &= \mathrm{E}\left(\frac{d_{2\theta}}{\beta_2}-1\right)2 \\
&= \mathrm{E}\left(\frac{1}{\beta_2}\max\left\{\left(\frac{Z_2}{n_2}\right)^{1/\alpha_2},\theta\left(\frac{z_1}{n_1}\right)^{1/\alpha_1}+(1-\theta)\left(\frac{Z_2}{n_2}\right)^{1/\alpha_2}\right\}-1\right)^2 \\
&= \int_0^\infty \int_0^{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}} \left(\frac{1}{\beta_2}\left(\frac{z_2}{n_2}\right)^{1/\alpha_2}-1\right)^2 f(z_1,z_2)dz_1dz_2 \\
&\quad + \int_0^\infty \int_{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}}^\infty \left(\frac{1}{\beta_2}\theta\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1}+\frac{1}{\beta_2}(1-\theta)\left(\frac{z_2}{n_2}\right)^{1/\alpha_2}-1\right)^2 \\
&\qquad \times f(z_1,z_2)dz_1dz_2 \\
&= \int_0^\infty \int_0^{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}} \left(\frac{1}{\beta_2}\left(\frac{z_2}{n_2}\right)^{1/\alpha_2}-1\right)^2 \frac{\mathrm{e}^{-z_1/\beta_1^{\alpha_1}}\mathrm{e}^{-z_2/\beta_2^{\alpha_2}}z_1^{n_1-1}z_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\beta_1^{n_1\alpha_1}\beta_2^{n_2\alpha_2}}dz_1dz_2 \\
&\quad + \int_0^\infty \int_{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}}^\infty \left(\frac{1}{\beta_2}\theta\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1}+\frac{1}{\beta_2}(1-\theta)\left(\frac{z_2}{n_2}\right)^{1/\alpha_2}-1\right)^2 \\
&\qquad \times \frac{\mathrm{e}^{-z_1/\beta_1^{\alpha_1}}\mathrm{e}^{-z_2/\beta_2^{\alpha_2}}z_1^{n_1-1}z_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\beta_1^{n_1\alpha_1}\beta_2^{n_2\alpha_2}}dz_1dz_2
\end{aligned}
$$

The absolute biases of $d_{1\theta}$ and $d_{2\theta}$ are

$$\begin{aligned}
B_1(d_{1\theta},\beta_1) &= |\mathrm{E}(d_{1\theta}-\beta_1)| \\
&= \left|\mathrm{E}\left(\min\left\{\left(\frac{z_1}{n_1}\right)^{1/\alpha_1}, \theta\left(\frac{z_1}{n_1}\right)^{1/\alpha_1} + (1-\theta)\left(\frac{Z_2}{n_2}\right)^{1/\alpha_2}\right\} - \beta_1\right)\right| \\
&= \Bigg| \int_0^{\infty} \int_0^{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}} \left(\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1} - \beta_1\right) \frac{\mathrm{e}^{-z_1/\beta_1^{\alpha_1}}\mathrm{e}^{-z_2/\beta_2^{\alpha_2}} z_1^{n_1-1} z_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\beta_1^{n_1\alpha_1}\beta_2^{n_2\alpha_2}} dz_1 dz_2 \\
&\quad + \int_0^{\infty} \int_{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}}^{\infty} \left(\theta\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1} + (1-\theta)\left(\frac{z_2}{n_2}\right)^{1/\alpha_2} - \beta_1\right) \\
&\qquad \times \frac{\mathrm{e}^{-z_1/\beta_1^{\alpha_1}}\mathrm{e}^{-z_2/\beta_2^{\alpha_2}} z_1^{n_1-1} z_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\beta_1^{n_1\alpha_1}\beta_2^{n_2\alpha_2}} dz_1 dz_2 \Bigg|
\end{aligned}$$

and

$$\begin{aligned}
B_1(d_{2\theta},\beta_2) &= |\mathrm{E}(d_{2\theta}-\beta_2)| \\
&= \left|\mathrm{E}\left(\max\left\{\left(\frac{Z_2}{n_2}\right)^{1/\alpha_2}, \theta\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1} + (1-\theta)\left(\frac{Z_2}{n_2}\right)^{1/\alpha_2}\right\} - \beta_2\right)\right| \\
&= \Bigg| \int_0^{\infty} \int_0^{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}} \left(\left(\frac{z_2}{n_2}\right)^{1/\alpha_2} - \beta_2\right) \frac{\mathrm{e}^{-z_1/\beta_1^{\alpha_1}}\mathrm{e}^{-z_2/\beta_2^{\alpha_2}} z_1^{n_1-1} z_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\beta_1^{n_1\alpha_1}\beta_2^{n_2\alpha_2}} dz_1 dz_2 \\
&\quad + \int_0^{\infty} \int_{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}}^{\infty} \left(\theta\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1} + (1-\theta)\left(\frac{z_2}{n_2}\right)^{1/\alpha_2} - \beta_2\right) \\
&\qquad \times \frac{\mathrm{e}^{-z_1/\beta_1^{\alpha_1}}\mathrm{e}^{-z_2/\beta_2^{\alpha_2}} z_1^{n_1-1} z_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\beta_1^{n_1\alpha_1}\beta_2^{n_2\alpha_2}} dz_1 dz_2 \Bigg|
\end{aligned}$$

The standardized biases of $d_{1\theta}$ and $d_{2\theta}$ are

$$\begin{aligned}
B_2(d_{1\theta},\beta_1) &= |\mathrm{E}(d_{1\theta}/\beta_1 - 1)| \\
&= \left|\mathrm{E}\left(\frac{1}{\beta_1}\min\left\{\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1}, \theta\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1} + (1-\theta)\left(\frac{Z_2}{n_2}\right)^{1/\alpha_2}\right\} - 1\right)\right| \\
&= \Bigg| \int_0^{\infty} \int_0^{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}} \left(\frac{1}{\beta_1}\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1} - 1\right) \frac{\mathrm{e}^{-z_1/\beta_1^{\alpha_1}}\mathrm{e}^{-z_2/\beta_2^{\alpha_2}} z_1^{n_1-1} z_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\beta_1^{n_1\alpha_1}\beta_2^{n_2\alpha_2}} dz_1 dz_2 \\
&\quad + \int_0^{\infty} \int_{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}}^{\infty} \left(\frac{1}{\beta_1}\left\{\theta\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1} + (1-\theta)\left(\frac{z_2}{n_2}\right)^{1/\alpha_2}\right\} - 1\right) \\
&\qquad \times \frac{\mathrm{e}^{-z_1/\beta_1^{\alpha_1}}\mathrm{e}^{-z_2/\beta_2^{\alpha_2}} z_1^{n_1-1} z_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\beta_1^{n_1\alpha_1}\beta_2^{n_2\alpha_2}} dz_1 dz_2 \Bigg|
\end{aligned}$$

and

$$
\begin{aligned}
B_2(d_{2\theta},\beta_2) &= |\mathrm{E}(d_{2\theta}/\beta_2-1)| \\
&= \left|\mathrm{E}\left(\frac{1}{\beta_2}max\{\left(\frac{Z_2}{n_2}\right)^{1/\alpha_2},\theta\left(\frac{Z_1}{n_1}\right)^{1/\alpha_1}+(1-\theta)\left(\frac{Z_2}{n_2}\right)^{1/\alpha_2}\}-1\right)\right| \\
&= \left|\int_0^\infty \int_0^{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}} \left(\frac{1}{\beta_2}\left(\frac{z_2}{n_2}\right)^{1/\alpha_2}-1\right)\frac{\mathrm{e}^{-z_1/\beta_1^{\alpha_1}}\mathrm{e}^{-z_2/\beta_2^{\alpha_2}}z_1^{n_1-1}z_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\beta_1^{n_1\alpha_1}\beta_2^{n_2\alpha_2}}dz_1dz_2\right. \\
&\quad +\int_0^\infty\int_{n_1\left(\frac{z_2}{n_2}\right)^{\alpha_1/\alpha_2}}^\infty\left(\frac{1}{\beta_2}\left\{\theta(z_1/n_1)^{1/\alpha_1}+(1-\theta)\left(\frac{z_2}{n_2}\right)^{1/\alpha_2}\right\}-1\right) \\
&\quad \left.\times\frac{\mathrm{e}^{-z_1/\beta_1^{\alpha_1}}\mathrm{e}^{-z_2/\beta_2^{\alpha_2}}z_1^{n_1-1}z_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\beta_1^{n_1\alpha_1}\beta_2^{n_2\alpha_2}}dz_1dz_2\right|
\end{aligned}
$$

Since all these integrals are incomplete gamma integrals, the risks and biases cannot be obtained in closed form. Hence these will be obtained using simulation in section 4.

4. Simulation results

In this section the risks and biases of the mixed estimators which are obtained in section 3 are used. The maximum likelihood estimators of unordered β_1 and β_2 are compared with the MLE of ordered parameters in Srivastava's study [5] for the values $\theta = 0.365,\ 0.5$ and 0.791.

For the simulation, various sets of scale and shape parameters with different sample sizes have been considered. The samples were generated using "Statistical Tables" by Bob Wheeler Version 1. (Copyright 1996 by ECHIP Inc., http://www.echip.com).

For the parameters sets $\alpha_1 = 3.0, 5.0$; $\alpha_2 = 3.5, 5.5$; $\beta_1 = 1.0, 3.0, 5.0$ and $\beta_2 = 1.5, 3.2, 5.2$, samples of size 5, 10 and 15 were generated. These were replicated 20 times each and then the complete set repeated 5 times. The biases B_1 , B_2, and the risks R_1 ,R_2 were simulated for d_i, d_{1i} and $d_{i\theta}$,$i = 1, 2$.

These are tabulated in Tables 1-4. From these tables it is clear that the mixed estimators have smaller bias and risk compared to the usual maximum likelihood estimator and the ordered maximum likelihood estimator. Thus they improve upon the previous estimators.

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Table 1 Comparison of B_1 for d_i with d_{1i} and $d_{i\theta}$

α_1	α_2	β_1	β_2	n_1	n_2	$B_1(d_1)$	$B_1(d_{11})$	$B_1(d_{1\theta})$	$B_1(d_2)$	$B_1(d_{12})$	$B_1(d_{2\theta})$
$\Theta = 0.365$											
3.0	3.5	1.0	1.2	5	10	0.1306	0.1142	0.1149	0.0791	0.0708	0.0701
3.0	3.5	1.0	1.2	10	5	0.0825	0.0763	0.0707	0.1344	0.1221	0.1276
3.0	3.5	1.0	1.2	15	15	0.0664	0.0634	0.0626	0.0744	0.0714	0.0722
5.0	5.5	3.0	3.2	5	10	0.2535	0.2017	0.2007	0.2207	0.1840	0.1819
5.0	5.5	3.0	3.2	10	5	0.1173	0.0917	0.0808	0.2305	0.1742	0.1976
5.0	5.5	3.0	3.2	15	15	0.1170	0.1122	0.1109	0.1117	0.1069	0.1082
3.0	3.5	5.0	5.2	5	10	0.5124	0.4668	0.4689	0.3410	0.3388	0.3410
3.0	3.5	5.0	5.2	10	5	0.3406	0.3381	0.3666	0.5831	0.4358	0.4877
3.0	3.5	5.0	5.2	15	15	0.2191	0.1943	0.1908	0.2211	0.2195	0.2161
$\Theta = 0.5$											
3.0	3.5	1.0	1.2	5	10	0.1187	0.1071	0.1100	0.0734	0.0676	0.0647
3.0	3.5	1.0	1.2	10	5	0.0858	0.0751	0.0765	0.1683	0.1422	0.1488
3.0	3.5	1.0	1.2	15	15	0.0740	0.0724	0.0724	0.0726	0.0710	0.0710
5.0	5.5	3.0	3.2	5	10	0.2123	0.1742	0.1808	0.1292	0.1120	0.1052
5.0	5.5	3.0	3.2	10	5	0.1746	0.1655	0.1686	0.2325	0.1885	0.1995
5.0	5.5	3.0	3.2	15	15	0.1117	0.0998	0.0998	0.1104	0.0985	0.0985
3.0	3.5	5.0	5.2	5	10	0.4601	0.3332	0.3488	0.4172	0.3055	0.2641
3.0	3.5	5.0	5.2	10	5	0.3242	0.2837	0.2832	0.6588	0.5017	0.5247
3.0	3.5	5.0	5.2	15	15	0.2914	0.2518	0.2518	0.3129	0.2660	0.2660
$\Theta = 0.791$											
3.0	3.5	1.0	1.2	5	10	0.1765	0.1445	0.1665	0.1031	0.0999	0.1101
3.0	3.5	1.0	1.2	10	5	0.0805	0.0772	0.0772	0.1059	0.0955	0.0935
3.0	3.5	1.0	1.2	15	15	0.0662	0.0643	0.0654	0.0761	0.0741	0.0730
5.0	5.5	3.0	3.2	5	10	0.1628	0.1415	0.1561	0.1588	0.1513	0.1500
5.0	5.5	3.0	3.2	10	5	0.1217	0.1062	0.1108	0.2754	0.1857	0.1726
5.0	5.5	3.0	3.2	15	15	0.1016	0.0915	0.0974	0.1335	0.1233	0.1174
3.0	3.5	5.0	5.2	5	10	0.5198	0.3663	0.4694	0.2900	0.2796	0.3723
3.0	3.5	5.0	5.2	10	5	0.3418	0.3104	0.3173	0.4573	0.3288	0.3108
3.0	3.5	5.0	5.2	15	15	0.2708	0.2717	0.2626	0.3218	0.2782	0.2529

Table 2. Comparison of B_2 for d_i with d_{1i} and $d_{i\theta}$

α_1	α_2	β_1	β_2	n_1	n_2	$B_2(d_1)$	$B_2(d_{11})$	$B_2(d_{1\theta})$	$B_2(d_2)$	$B_2(d_{12})$	$B_2(d_{2\theta})$
$\Theta = 0.365$											
3	3.5	1	1.2	5	10	0.1306	0.1142	0.1149	0.0659	0.0590	0.0584
3	3.5	1	1.2	10	5	0.0777	0.0736	0.0699	0.0760	0.0691	0.0722
3	3.5	1	1.2	15	15	0.0879	0.0874	0.0873	0.0436	0.0432	0.0433
5	5.5	3	3.2	5	10	0.0617	0.0566	0.0568	0.0442	0.0459	0.0461
5	5.5	3	3.2	10	5	0.0391	0.0306	0.0269	0.0720	0.0544	0.0618
5	5.5	3	3.2	15	15	0.0339	0.0319	0.0325	0.0395	0.0374	0.0379
3	3.5	5	5.2	5	10	0.1025	0.0934	0.0938	0.0656	0.0652	0.0656
3	3.5	5	5.2	10	5	0.0681	0.0676	0.0733	0.1121	0.0838	0.0938
3	3.5	5	5.2	15	15	0.0616	0.0513	0.0495	0.0508	0.0350	0.0388
$\Theta = 0.5$											
3	3.5	1	1.2	5	10	0.1187	0.1071	0.1100	0.0734	0.0676	0.0647
3	3.5	1	1.2	10	5	0.1082	0.1051	0.1035	0.0906	0.0855	0.0868
3	3.5	1	1.2	15	15	0.0674	0.0664	0.0664	0.0641	0.0632	0.0632
5	5.5	3	3.2	5	10	0.0702	0.0658	0.0669	0.0489	0.0460	0.0446
5	5.5	3	3.2	10	5	0.0547	0.0545	0.0549	0.0644	0.0578	0.0595
5	5.5	3	3.2	15	15	0.0336	0.0292	0.0292	0.0403	0.0361	0.0361
3	3.5	5	5.2	5	10	0.0920	0.0666	0.0698	0.0802	0.0587	0.0508
3	3.5	5	5.2	10	5	0.0648	0.0567	0.0566	0.1267	0.0965	0.1009
3	3.5	5	5.2	15	15	0.0583	0.0504	0.0504	0.0602	0.0512	0.0512
$\Theta = 0.791$											
3	3.5	1	1.2	5	10	0.1765	0.1445	0.1665	0.1031	0.0999	0.1101
3	3.5	1	1.2	10	5	0.0805	0.0772	0.0772	0.0882	0.0796	0.0780
3	3.5	1	1.2	15	15	0.0792	0.0783	0.0788	0.0648	0.0640	0.0635
5	5.5	3	3.2	5	10	0.0543	0.0472	0.0520	0.0496	0.0473	0.0469
5	5.5	3	3.2	10	5	0.0406	0.0354	0.0369	0.0861	0.0580	0.0540
5	5.5	3	3.2	15	15	0.0306	0.0264	0.0288	0.0385	0.0346	0.0326
3	3.5	5	5.2	5	10	0.1040	0.0733	0.0939	0.0558	0.0538	0.0716
3	3.5	5	5.2	10	5	0.0593	0.0515	0.0521	0.1141	0.0624	0.0564
3	3.5	5	5.2	15	15	0.0542	0.0543	0.0525	0.0619	0.0535	0.0486

Table 3. Comparison of R_1 for d_i with d_{1i} and $d_{i\theta}$

α_1	α_2	β_1	β_2	n_1	n_2	$R_1(d_1)$	$R_1(d_{11})$	$R_1(d_{1\theta})$	$R_1(d_2)$	$R_1(d_{12})$	$R_1(d_{2\theta})$
$\Theta = 0.365$											
3	3.5	1	1.2	5	10	0.0171	0.0144	0.0145	0.0119	0.0117	0.0117
3	3.5	1	1.2	10	5	0.0122	0.011	0.0107	0.0176	0.0126	0.0144
3	3.5	1	1.2	15	15	0.0069	0.0063	0.0062	0.0095	0.0087	0.0089
5	5.5	3	3.2	5	10	0.0531	0.0454	0.0456	0.037	0.0322	0.0319
5	5.5	3	3.2	10	5	0.0225	0.0142	0.0118	0.0665	0.0412	0.0497
5	5.5	3	3.2	15	15	0.017	0.0161	0.0162	0.0287	0.0251	0.0259
3	3.5	5	5.2	5	10	0.426	0.3377	0.3395	0.1689	0.1534	0.1532
3	3.5	5	5.2	10	5	0.1947	0.1863	0.2382	0.4954	0.2177	0.3092
3	3.5	5	5.2	15	15	0.134	0.0962	0.0966	0.125	0.0581	0.07
$\Theta = 0.5$											
3	3.5	1	1.2	5	10	0.0196	0.0156	0.0164	0.0088	0.0077	0.0073
3	3.5	1	1.2	10	5	0.0111	0.0111	0.0112	0.0165	0.0153	0.0156
3	3.5	1	1.2	15	15	0.0076	0.0072	0.0072	0.0079	0.0063	0.0063
5	5.5	3	3.2	5	10	0.0529	0.0441	0.0453	0.036	0.033	0.0324
5	5.5	3	3.2	10	5	0.0304	0.0189	0.0154	0.0516	0.0286	0.0321
5	5.5	3	3.2	15	15	0.0149	0.0114	0.0114	0.024	0.019	0.019
3	3.5	5	5.2	5	10	0.4644	0.4017	0.4119	0.2962	0.2904	0.2929
3	3.5	5	5.2	10	5	0.155	0.1223	0.1236	0.5647	0.4168	0.4361
3	3.5	5	5.2	15	15	0.232	0.164	0.164	0.1184	0.0616	0.0616
$\Theta = 0.791$											
3	3.5	1	1.2	5	10	0.0316	0.0291	0.0306	0.0152	0.0138	0.0128
3	3.5	1	1.2	10	5	0.0082	0.0075	0.0077	0.0188	0.0179	0.0178
3	3.5	1	1.2	15	15	0.0062	0.0061	0.0062	0.0085	0.0078	0.0074
5	5.5	3	3.2	5	10	0.0412	0.0307	0.0371	0.0311	0.0294	0.03
5	5.5	3	3.2	10	5	0.0233	0.0244	0.0233	0.0636	0.0269	0.0225
5	5.5	3	3.2	15	15	0.0146	0.0123	0.0134	0.0255	0.0218	0.0206
3	3.5	5	5.2	5	10	0.3997	0.2632	0.329	0.1614	0.0991	0.1193
3	3.5	5	5.2	10	5	0.4309	0.3179	0.3458	0.384	0.1397	0.1484
3	3.5	5	5.2	15	15	0.1152	0.0945	0.1023	0.1153	0.0716	0.0621

Table 4. Comparison of R_2 for d_i with d_{1i} and $d_{i\theta}$

α_1	α_2	β_1	β_2	n_1	n_2	$R_2(d_1)$	$R_2(d_{11})$	$R_2(d_{1\theta})$	$R_2(d_2)$	$R_2(d_{12})$	$R_2(d_{2\theta})$
$\Theta = 0.365$											
3	3.5	1	1.2	5	10	0.0171	0.0144	0.0145	0.0083	0.0081	0.0081
3	3.5	1	1.2	10	5	0.0122	0.011	0.0107	0.0122	0.0088	0.01
3	3.5	1	1.2	15	15	0.0087	0.0087	0.0087	0.0058	0.0058	0.0058
5	5.5	3	3.2	5	10	0.0078	0.0063	0.0064	0.0029	0.003	0.003
5	5.5	3	3.2	10	5	0.0025	0.0016	0.0013	0.0065	0.004	0.0049
5	5.5	3	3.2	15	15	0.0019	0.0018	0.0018	0.0028	0.0025	0.0025
3	3.5	5	5.2	5	10	0.017	0.0135	0.0136	0.0062	0.0057	0.0057
3	3.5	5	5.2	10	5	0.0085	0.0071	0.0082	0.0165	0.0087	0.011
3	3.5	5	5.2	15	15	0.0054	0.0038	0.0039	0.0046	0.0021	0.0026
$\Theta = 0.5$											
3	3.5	1	1.2	5	10	0.0221	0.0198	0.0203	0.0093	0.0091	0.0091
3	3.5	1	1.2	10	5	0.0118	0.0099	0.0097	0.0276	0.0188	0.0205
3	3.5	1	1.2	15	15	0.0068	0.0066	0.0066	0.0054	0.0052	0.0052
5	5.5	3	3.2	5	10	0.0059	0.0049	0.005	0.0035	0.0032	0.0032
5	5.5	3	3.2	10	5	0.0034	0.0021	0.0017	0.005	0.0028	0.0031
5	5.5	3	3.2	15	15	0.0017	0.0013	0.0013	0.0023	0.0019	0.0019
3	3.5	5	5.2	5	10	0.013	0.0071	0.0069	0.011	0.0057	0.0046
3	3.5	5	5.2	10	5	0.0062	0.0049	0.0049	0.0209	0.0154	0.0161
3	3.5	5	5.2	15	15	0.0058	0.005	0.005	0.0047	0.004	0.004
$\Theta = 0.791$											
3	3.5	1	1.2	5	10	0.0316	0.0291	0.0306	0.0105	0.0096	0.0089
3	3.5	1	1.2	10	5	0.0096	0.0092	0.0093	0.0125	0.0091	0.0087
3	3.5	1	1.2	15	15	0.0035	0.0034	0.0035	0.0058	0.0056	0.0056
5	5.5	3	3.2	5	10	0.0046	0.0034	0.0041	0.003	0.0029	0.0029
5	5.5	3	3.2	10	5	0.0026	0.0028	0.0027	0.0114	0.0062	0.0056
5	5.5	3	3.2	15	15	0.0017	0.0013	0.0015	0.0021	0.0017	0.0015
3	3.5	5	5.2	5	10	0.016	0.0105	0.0132	0.006	0.0037	0.0044
3	3.5	5	5.2	10	5	0.005	0.0042	0.0041	0.018	0.0061	0.0051
3	3.5	5	5.2	15	15	0.0042	0.0026	0.0032	0.0058	0.0028	0.0023

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