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ON ALPHA-QUASI-UNIFORMLY CONVEX p-VALENT FUNCTIONS OF TYPE β IN TERMS OF RUSCHEWEYH DERIVATIVES

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Abstract

In the present paper we will establish some properties of the class of α - Quasi - Uniformly convex, p-valent functions of type β in the open unit disk, which we denote by $\text{QUCV}^{p,\lambda}_{\beta,\alpha}$, for $\lambda > -1;\ 0 \le \beta < p,\ \alpha \ge 0$ and $p \in \mathbb{N}$, by making use of the Ruscheweyh Derivatives.

Keywords: Ruscheweyh Derivative, Quasi-convex, Uniformly convex, Close-to-uniformly convex and alpha-quasi- uniformly convex p-valent functions.

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1. Introduction and Definitions

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=2}^{\infty} a_n z^n,\ p \in \mathbb{N}, \tag{1}$$

which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, normalized by conditions $f(0) = 0,\ f^{(p)}(0) = p!$. Also, let B denote the subclass of $\mathcal{A}_p$ that are p-valent in Δ. We also denote by UCV^p, $\text{C}^{*,p}$ and Q^p_α the subclasses of functions in $\mathcal{A}_p$ that are respectively uniformly convex p-valent, Quasi-convex p-valent and α-Quasi-convex p-valent in Δ. For $p = 1$ we obtain the classes UCV and C^* which were introduced and studied in [1], [2] respectively.

To prove our results, we need the following definitions.

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1.1. Definition. Let $f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p},\ p \in \mathbb{N}$. Then the Ruscheweyh Derivatives of $f(z)$ are defined by

$$D^\lambda f(z) = \frac{1}{\lambda!}\{z(z^{\lambda-1}f(z))^{(\lambda)}\} \ = S(\lambda,p)z^p + \sum_{n=1}^{\infty} S(\lambda,n)a_{n+p}z^{n+p},\ \lambda > -1,$$

where $S(\lambda,p) = \binom{\lambda+p-1}{\lambda}$.

1.2. Definition. The class $\mathrm{UQCV}_\beta^{p,\lambda}$ of *uniformly Quasi-convex p-valent functions of type* β consists of function f of the form (1) for which there exists a function $g \in \mathrm{UCV}^p$ of the form

$$g(z) = z^p + \sum_{n=2}^{\infty} b_n z^n \tag{2}$$

such that

$$Re\left\{\frac{[(z-\eta)(D^\lambda f(z))']'}{(D^\lambda g(z))'}\right\} > \beta,\ 0 \le \beta < p, \tag{3}$$

where $z \neq \eta \in \Delta$. When $\eta = 0$, the class of functions satisfying (3) is called the class of *Quasi-uniformly convex p-valent function of type* β, which we denote by $\mathrm{QUCV}_\beta^{p,\lambda}$, see [6]. For $p = 1,\ \lambda = 0$ and $\beta = 0$ we get the class UQCV introduced in [6].

1.3. Definition. The class $\mathrm{CUCV}_\beta^{p,\lambda}$ of *close-to-uniformly convex p-valent functions of order* β consists of functions f of the form (1) for which there exists a uniformly convex p-valent function $g \in \mathrm{UCV}^p$ of the form (2) such that

$$Re\left\{\frac{(D^\lambda f(z))'}{(D^\lambda g(z))'}\right\} > \beta,\ z \in \Delta,\ 0 \le \beta < p.$$

For $\lambda = 0,\ p = 1$ and $\beta = 0$ we get the class CUCV, which was introduced by K. S. Padmanabhan in [5].

We note that here

$$\mathrm{UCV}^p \subset \mathrm{UQCV}_\beta^{p,\lambda} \subset \mathrm{QUCV}_\beta^{p,\lambda} \subset \mathrm{CUCV}_\beta^{p,\lambda} \subset K \subset B,$$

where K is the class of close-to-convex functions of order β, and we have $D^\lambda f \in \mathrm{QUCV}_\beta^{p,\lambda}$ if and only if $z(D^\lambda f)' \in \mathrm{CUCV}_\beta^{p,\lambda}$. This is proved in [6] for $\lambda = 0,\ \beta = 0$ and $p = 1$.

1.4. Definition. Let $f(z) = z^p + \sum_{n=2}^{\infty} a_n z^n$, be analytic in Δ. Then $f(z)$ is said to $\alpha-$ *Quasi - uniformly convex p-valent of type* β *in* Δ if and only if, there exists a uniformly convex, p-valent function $g(z) = z^p + \sum_{n=2} b_n z^n$ in Δ such that

$$Re\left\{(1-\alpha)\frac{(D^\lambda f(z))'}{(D^\lambda g(z))'} + \alpha\frac{[z(D^\lambda f(z))']'}{(D^\lambda g(z))'}\right\} > \beta,\ 0 \le \alpha \le 1,\ 0 \le \beta < p.$$

This class of functions is denoted by $\mathrm{QUCV}_{\beta,\alpha}^{p,\lambda}$.

We note here that $\mathrm{QUCV}_{\beta,0}^{p,\lambda} = \mathrm{CUCV}_\beta^{p,\lambda}$, the class of close-to-uniformly convex functions; and $\mathrm{QUCV}_{\beta,1}^{p,\lambda} = \mathrm{QUCV}_\beta^{p,\lambda}$, the class of Quasi-uniformly functions. Thus, $\mathrm{QUCV}_{\beta,\alpha}^{p,\lambda}$ unifies the classes $\mathrm{CUCV}_\beta^{p,\lambda}$ and $\mathrm{QUCV}_\beta^{p,\lambda}$ in the same way as Q_α^p connects K and $\mathrm{C}^{*,p}$.

2. Main Results

2.1. Theorem. *Let*

$$D^{\lambda}d(z) = (1-\alpha)D^{\lambda}t(z) + \alpha z(D^{\lambda}t(z))', \ \lambda > -1, \ \alpha \geq 0, \ p \in \mathbb{N}, \tag{4}$$

and $|z| < 1$. *Then* $t(z) \in \mathrm{QUCV}^{p,\lambda}_{\beta,\alpha}$ *if and only if* $d(z) \in \mathrm{CUCV}^{p,\lambda}_{\beta}$.

Proof. Let $D^{\lambda}d(z) = (1-\alpha)D^{\lambda}t(z) + \alpha z(D^{\lambda}t(z))'$ and suppose $t(z) \in \mathrm{QUCV}^{p,\lambda}_{\beta,\alpha}$. Then there exists a function $\psi(z) \in \mathrm{UCV}^p$ such that

$$Re\left\{(1-\alpha)\frac{(D^{\lambda}(t(z))'}{(D^{\lambda}\psi(z))'} + \alpha\frac{[z(D^{\lambda}t(z))']'}{(D^{\lambda}\psi(z))'}\right\} > \beta. \tag{5}$$

Now by (4) we have

$$\frac{(D^{\lambda}d(z))'}{(D^{\lambda}\psi(z))'} = (1-\alpha)\frac{(D^{\lambda}t(z))'}{(D^{\lambda}\psi(z))'} + \alpha\frac{[z(D^{\lambda}t(z))']'}{(D^{\lambda}\psi(z))'},$$

so by (5) we have $Re\left\{\frac{(D^{\lambda}d(z))'}{(D^{\lambda}\psi(z))'}\right\} > \beta$, proving that $d(z)$ is in $\mathrm{CUCV}^{p,\lambda}_{\beta}$.

Conversely assume $d(z) \in \mathrm{CUCV}^{p,\lambda}_{\beta}$. Then there exists a function $\psi(z) \in \mathrm{UCV}^p$ such that

$$Re\left\{\frac{(D^{\lambda}d(z))'}{(D^{\lambda}\psi(z))'}\right\} > \beta. \tag{6}$$

From (4) we have

$$\frac{(D^{\lambda}d(z))'}{(D^{\lambda}\psi(z))'} = (1-\alpha)\frac{(D^{\lambda}t(z))'}{(D^{\lambda}\psi(z))'} + \frac{\alpha[z(D^{\lambda}t(z))']'}{(D^{\lambda}\psi(z))'}.$$

Then

$$Re\left\{(1-\alpha)\frac{(D^{\lambda}t(z))'}{(D^{\lambda}\psi(z))'} + \alpha\frac{[z(D^{\lambda}t(z))']'}{(D^{\lambda}\psi(z))'}\right\} > \beta \quad \text{(by 6)}$$

which implies $t(z) \in \mathrm{QUCV}^{p,\lambda}_{\beta,\alpha}$. This completes the proof. □

2.2. Corollary. *For* $p = 1$, $\lambda = 0$ *and* $\beta = 0$ *we get the result due to C. Selvaraj [8] involving* CUCV *and* QUCV_{α}.

2.3. Theorem. *A function* f *of the form (1) is in* $\mathrm{QUCV}^{p,\lambda}_{\beta,\alpha}$ *if and only if there exists a function* $S(z) \in \mathrm{CUCV}^{p,\lambda}_{\beta}$ *such that*

$$D^{\lambda}f(z) = \frac{1}{\alpha z^{(\frac{1}{\alpha})-1}}\int_0^z t^{(1/\alpha)-2}D^{\lambda}S(t)dt, \quad 0 < \alpha \leq 1, \lambda > -1. \tag{7}$$

Proof. From the representation (7) we have

$$\alpha z^{(1/\alpha)-1}D^{\lambda}f(z) = \int_0^z t^{(1/\alpha)-2}D^{\lambda}S(t)dt. \tag{8}$$

After differentiating both sides of (8) we obtain

$$(1-\alpha)z^{(\frac{1}{\alpha})-2}D^{\lambda}f(z) + \alpha z^{(\frac{1}{\alpha})-1}(D^{\lambda}f(z))' = z^{(\frac{1}{\alpha})-2}D^{\lambda}S(z),$$

or equivalently,

$$(1-\alpha)D^{\lambda}f(z) + \alpha z(D^{\lambda}f(z))' = D^{\lambda}S(z).$$

So we obtain (4), and hence by Theorem 2.1 the proof is complete. □

The same result was obtained for $p=1,\ \lambda=0$ and $\beta=0$ by C. Selvaraj [8].

Goodman [1] showed that the classical Alexander result $f\in K \iff zf'\in S^*$ does not hold between the classes UCV^p and UST^p (uniformly starlike functions). Rønning [7] introduced the class $S_{\mathbb{P}}(\gamma)$ consisting of functions $zf',\ f\in \mathrm{UCV}^p$. To prove the next theorem we require the definition of the class $S_{\mathbb{P}}(\gamma)$.

2.4. Definition. Let $f(z)=z^p+\sum\limits_{k=2}^{\infty}a_kz^k\in\mathcal{A}_p$. Then $f\in S_{\mathbb{P}}(\gamma)$ if and only if

$$\left|\frac{zf'(z)}{f(z)}-p\right|\le Re\left\{\frac{zf'(z)}{f(z)}-\gamma\right\}, z\in\Delta, -1\le\gamma<1.$$

He also defined the class $\mathrm{UCV}^p(\gamma)$ of functions f for which $zf'\in S_{\mathbb{P}}(\gamma)$. In this paper we obtain this class by letting $\gamma=0$, that is $S_{\mathbb{P}}(0)=S_{\mathbb{P}}$.

2.5. Theorem. *Let $D^\lambda S(z)\in S_{\mathbb{P}}$,*

$$D^\lambda t(z)=\frac{p}{\alpha z^{(1/\alpha)-1}}\int_0^z t^{(1/\alpha)-2}D^\lambda S(t)dt,\ \alpha>0,\ \lambda>-1,\ p\in\mathbb{N}, \tag{9}$$

and let $1/\alpha$ be a positive integer. Then $D^\lambda d(z)\in S^*(\frac{1}{2}p)$.

Proof. By using logarithmic differentiation, from (9) we have

$$\begin{aligned}\frac{(D^\lambda d(z))'}{D^\lambda d(z)}&=\frac{pz^{(1/\alpha)-2}D^\lambda S(z)}{p\int_0^z t^{(1/\alpha)-2}D^\lambda S(t)dt}-\frac{(1-\alpha)z^{(1/\alpha)-2}}{\alpha z^{(1/\alpha)-1}}\\&=\frac{z^{(1/\alpha)-2}D^\lambda S(z)-(\frac{1}{\alpha}-1)z^{-1}\int_0^z t^{(1/\alpha)-2}D^\lambda S(t)dt}{\int_0^z t^{(1/\alpha)-2}D^\lambda S(t)dt}.\end{aligned}$$

So

$$\begin{aligned}\frac{z(D^\lambda d(z))'}{D^\lambda d(z)}&=\frac{z^{(1/\alpha)-1}D^\lambda S(z)+(1-1/\alpha)\int_0^z z^{(1/\alpha)-2}D^\lambda S(z)dz}{\int_0^z t^{(1/\alpha)-2}D^\lambda S(t)dt}\\&=\frac{z\mu'(z)+(1-1/\alpha)\mu(z)}{\mu(z)},\end{aligned}\tag{10}$$

where $\mu(z)=\int_0^z t^{(1/\alpha)-2}D^\lambda S(t)dt$. Now we have

$$\begin{aligned}&Re\left\{\frac{[z\mu'(z)+(1-1/\alpha)\mu(z)]'}{\mu'(z)}\right\}=\\&=Re\left\{\frac{z\mu''(z)}{\mu'(z)}+2-1/\alpha\right\}\\&=Re\left\{\frac{z[(1/\alpha-2)z^{(1/\alpha)-3}D^\lambda S(z)+z^{(1/\alpha)-2}(D^\lambda S(z))']}{Z^{(1/\alpha)-2}D^\lambda S(z)}+2-1/\alpha\right\}\\&=Re\left\{\frac{z(D^\lambda S(z))'}{D^\lambda S(z)}\right\}\\&>\frac{1}{2}p\end{aligned}$$

as $D^\lambda S(z)\in S_{\mathbb{P}}$. Hence, $Re\left\{\frac{[z\mu'(z)+(1-1/\alpha)\mu(z)]'}{\mu'(z)}\right\}>\frac{1}{2}p$. By a Lemma of Libera [3],

$$Re\left\{\frac{z\mu'(z)+(1-1/\alpha)\mu(z)}{\mu(z)}\right\}>\frac{1}{2}p,\quad z\in\Delta.$$

Therefore, by (10) we have $Re\left\{\frac{z(D^\lambda d(z))'}{D^\lambda d(z)}\right\}>\frac{1}{2}p\ \ z\in\Delta$. Then $D^\lambda d(z)\in S^*(\frac{1}{2}p)$ (classes of starlike function of order $\frac{1}{2}p$) and the proof is complete. □

2.6. Corollary. *For $\lambda = 0$ and $p = 1$ we get the result due to C. Selvaraj [8] .*

2.7. Theorem. *If $\eta(z) \in \mathrm{CUCV}_{\beta}^{p,\lambda}$ and $F_\lambda(z) = \frac{1}{2}[zD^\lambda \eta(z)]'$, then $F_\lambda(z)$ is close-to-convex of order β for $|z| < 1/2$.*

Proof. Because $\eta(z) \in \mathrm{CUCV}_{\beta}^{p,\lambda}$ there exists a function $h(z) \in \mathrm{UCV}^p$ such that

$$Re\left\{\frac{(D^\lambda \eta(z))'}{(D^\lambda h(z))'}\right\} > \beta, z \in \Delta,$$

and since $h(z) \in \mathrm{UCV}^p$ we have $D^\lambda h(z) \in \mathrm{UCV}^p$ which implies $z(D^\lambda h(z))' \in S_{\mathbb{P}}$. Therefore we have $Re\left\{\frac{z(D^\lambda \eta(z))'}{z(D^\lambda h(z))'}\right\} > \beta$, so $Re\left\{\frac{z(D^\lambda \eta(z)'}{H_\lambda(z)}\right\} > \beta$ where $H_\lambda(z) = z(D^\lambda h(z))' \in S_{\mathbb{P}}$. If $G_\lambda(z) = \frac{1}{2}[zH_\lambda(z)]'$, from Livingston [4] we have $G_\lambda(z) \in S^*, |z| < \frac{1}{2}$.

To prove F_λ is close-to-convex of order β, it is sufficient to show that $Re\left\{\frac{zF_\lambda'(z)}{G_\lambda(z)}\right\} > \beta$ for $|z| < \frac{1}{2}$. Now proceeding as in Theorem 3 of [5], we get F_λ is close-to-convex of order β for $|z| < \frac{1}{2}$. This completes the proof. □

2.8. Corollary. *For $p = 1, \lambda = 0$ and $\beta = 0$ we get the result due to C. Selvaraj [8].*

2.9. Theorem. *Let $\mu(z) \in \mathrm{QUCV}_{\beta,\alpha}^{p,\lambda}$ and let $F_\lambda(z) = \frac{1}{2}[zD^\lambda \mu(z)]', z \in \Delta$. Then F_λ is close-to-convex of order β for $|z| < \frac{1}{2}$.*

Proof. Since $\mu(z) \in \mathrm{QUCV}_{\beta,\alpha}^{p,\lambda}$, by Theorem 2.1 we have

$$D^\lambda \phi(z) = (1-\alpha)D^\lambda \mu(z) + \alpha z(D^\lambda \mu(z))', \quad z \in \Delta \tag{11}$$

such that $\phi(z) \in \mathrm{CUCV}_{\beta}^{p,\lambda}$. Suppose $H_\lambda(z) = \frac{1}{2}[zD^\lambda \phi(z)]'$. By Theorem 2.7, $H_\lambda(z)$ is close-to-convex of order β for $|z| < \frac{1}{2}$. From (11) we have

$$(D^\lambda \phi(z))' = (1-\alpha)(D^\lambda \mu(z))' + \alpha(D^\lambda \mu(z))' + \alpha z[D^\lambda \mu(z)]''.$$

Therefore

$$\begin{aligned} \frac{1}{2}[zD^\lambda \phi(z)]' &= \frac{1}{2}[D^\lambda \phi(z) + z(D^\lambda \phi(z))'] \\ &= \frac{1}{2}[(1-\alpha)(zD^\lambda \mu(z))' + \alpha z[z(D^\lambda \mu(z))'' + 2(D^\lambda \mu(z))']. \end{aligned} \tag{12}$$

To prove F_λ is close-to-convex, we have to prove $G(z) = (1-\alpha)F_\lambda(z) + \alpha z(F_\lambda(z))'$ is close-to-convex of order β for $|z| < \frac{1}{2}$ and for $\alpha \geq 0$ (by Theorem 2.1), where $F_\lambda(z) = \frac{1}{2}[zD^\lambda \mu(z)]',\ z \in \Delta$, so we have

$$\begin{aligned} G(z) &= (1-\alpha)\frac{1}{2}[zD^\lambda \mu(z)]' + \alpha z\frac{1}{2}[D^\lambda \mu(z) + z(D^\lambda \mu(z))'] \\ &= \frac{1}{2}[(1-\alpha)(zD^\lambda \mu(z))' + \alpha z(D^\lambda \mu(z))' + \alpha z^2(D^\lambda \mu(z))''] \\ &= \frac{1}{2}[zD^\lambda \phi(z)]' \ \ \text{(by (12))}. \end{aligned}$$

So $G(z)$ is close-to-convex of order β in $|z| < \frac{1}{2}$ and consequently $F_\lambda(z)$ is close-to-convex of order β in $|z| < \frac{1}{2}$, and the proof is complete. □

In the final theorem we obtain a necessary condition for a function belonging to $\mathrm{QUCV}_{\beta,\alpha}^{p,\lambda}$.

2.10. Theorem. *Let* $f(z) = z^p + \sum_{n=2}^{\infty} a_n z^n$. *If* $f(z) \in \mathrm{QUCV}_{\beta,\alpha}^{p,\lambda}$ *then*

$$\sum_{n=2}^{\infty} n\delta(\lambda,n)[\beta b_n - a_n(1+\alpha(n-1))] < p\delta(\lambda,p)[(1-\beta)+\alpha(p-1)],\ p \in \mathbb{N},\ 0 \le \beta \le p \tag{13}$$

where $g(z) = z^p + \sum_{n=2}^{\infty} b_n z^n$ *is in* UCV^p.

Proof. Let $f(z) \in \mathrm{QUCV}_{\beta,\alpha}^{p,\lambda}$. By Definition 1.4 we have

$$Re\left\{(1-\alpha)\frac{(D^\lambda f(z))'}{(D^\lambda g(z))'} + \alpha\frac{[z(D^\lambda f(z))']'}{[D^\lambda g(z)]'}\right\} > \beta, \tag{14}$$

or

$$\left|\frac{(D^\lambda f(z))' + \alpha z(D^\lambda f(z))''}{(D^\lambda g(z))'}\right| > \beta.$$

So we have

$$\left|\frac{p\delta(\lambda,p)z^{p-1}(1+\alpha(p-1)) + \sum_{n=2}^{\infty} n\delta(\lambda,n)a_n(1+\alpha(n-1))z^{n-1}}{p\delta(\lambda,p)z^{p-1} + \sum_{n=2}^{\infty} n\delta(\lambda,n)b_n z^{n-1}}\right| > \beta$$

$$\Longrightarrow p\delta(z,p)|z|^{p-1}(1+\alpha(p-1)) + \sum_{n=2}^{\infty} n\delta(\lambda,n)a_n(1+\alpha(n-1))|z|^n$$

$$-\beta p\delta(\lambda,p)|z|^{p-1} - \beta\sum_{n=2}^{\infty} n\delta(\lambda,n)b_n|z|^{n-1} > 0.$$

Now letting $z \to 1^-$ we obtain

$$\sum_{n=2}^{\infty} n\delta(\lambda,n)[\beta b_n - a_n(1+\alpha(n-1))] < p\delta(\lambda,p)[(1-\beta)+\alpha(p-1)].$$

Hence the proof is complete. □

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