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ON THE FANO SUBPLANES OF THE LEFT SEMIFIELD PLANE OF ORDER 9

Ziya Akça*, İbrahim Günaltılı* and Özgür Güney*

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Abstract

In this paper, we consider the projective plane of order 9, coordinatized by elements of a left semifield. It is shown that the number of Fano subplanes of this projective plane is at least 155760.

Keywords: Projective plane, Fano plane, Left semifield.

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1. Introduction

It is well known that every projective plane also has an algebraic structure obtained by coordinazation. Conversely, certain algebraic structures can be used to construct projective planes. Therefore, a general method of generating semifield has been given by Hall (1959).

A Fano plane is a projective plane of order 2. A Fano plane also occurs as a subplane of many larger projective planes. Therefore, the discovery of the Fano plane has played an important role in the improvement of the theory of finite geometries. Fano subplanes in some projective planes have been examined by many authors. For instance, Room-Kirpatrick [6], Çifçi-Kaya [2], Akça-Kaya [1], etc. A left semifield of order 9 is defined as follows:

1.1. Definition. A left semifield is a system $(S, \oplus, \odot)$, where $\oplus$ and $\odot$ are binary operations on the set S and:

(1) S is finite,
(2) $(S, \oplus)$ is a group, with identity 0,
(3) $(S \setminus \{0\}, \odot)$ is a semi-group with identity 1,
(4) $x \odot 0 = 0$ for all $x \in S$,
(5) $\odot$ is left distributive over $\oplus$, that is $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$ for all $x, y, z \in S$,

*Eskişehir Osmangazi University, Faculty of Science and Arts, Department of Mathematics, Eskişehir, Turkey

(6) Given $a, b, c \in S$ with $a \neq b$, there exists a unique $x \in S$ such that

$-a \odot x \oplus b \odot x = c$.

Let $(F_3, +, \cdot)$ be the field of integers modulo 3. Let

$$S = \{a + \lambda b : a, b \in F_3, \lambda \notin F_3\},$$

and consider the addition and multiplication on S given by

$$(a + \lambda b) \oplus (c + \lambda d) = (a + c) + \lambda(b + d) \tag{1.1}$$

and

$$(a + \lambda b) \odot (c + \lambda d) = \begin{cases} ac + \lambda(ad), & \text{if } b = 0, \\ ac - b^{-1}df(a) + \lambda(bc - (a-1)d), & \text{if } b \neq 0, \end{cases} \tag{1.2}$$

where $f(t) = t^2 - t - 1$ is an irreducible polynomial on F_3.

If, for the sake of brevity, we use ab instead of $a + \lambda b$ in equations (1.1) and (1.2), then the addition and multiplication tables are as follows:

Table 1

$\oplus$	00	01	02	10	11	12	20	21	22
00	00	01	02	10	11	12	20	21	22
01	01	02	00	11	12	10	21	22	20
02	02	00	01	12	10	11	22	20	21
10	10	11	12	20	21	22	00	01	02
11	11	12	10	21	22	20	01	02	00
12	12	10	11	22	20	21	02	00	01
20	20	21	22	00	01	02	10	11	12
21	21	22	20	01	02	00	11	12	10
22	22	20	21	02	00	01	12	10	11

Table 2

$\odot$	00	01	02	10	11	12	20	21	22
00	00	00	00	00	00	00	00	00	00
01	00	11	22	01	12	20	02	10	21
02	00	21	12	02	20	11	01	22	10
10	00	01	02	10	11	12	20	21	22
11	00	10	20	11	21	01	22	02	12
12	00	20	10	12	02	22	21	11	01
20	00	02	01	20	22	21	10	12	11
21	00	22	11	21	10	02	12	01	20
22	00	12	21	22	01	10	11	20	02

The system $(S, \oplus, \odot)$ is a left semifield of order 9.

Finally, we consider the projective plane of order 9 coordinatized by elements of the above left semifield, and investigate the Fano subplanes of this plane.

The Plane P_2S : The 91 points of P_2S are the elements of the set

$$\{(x,y)\ :\ x,y\in S\}\cup\{(m)\,:\,m\in S\}\cup\{(\infty)\}.$$

The points of the form (x, y) are called *proper points*, and the unique point (∞) and the points of the form (m) are called *ideal points*. The 91 lines of P_2S are defined to be sets of points satisfying one of the three conditions:

$$[m,k]=\{(x,y)\in S^2 : y=m\odot x\oplus k\}\cup\{(m)\}$$
$$[\lambda]=\{(x,y)\in S^2 : x=\lambda\}\cup\{(\infty)\}$$
$$[\infty]=\{(m)\in S\}\cup\{(\infty)\}$$

The 81 lines having the form $y = m \odot x \oplus k$ and the 9 lines having an equation of the form $x = \lambda$ are called the *proper lines* and the unique line $[\infty]$ is called the *ideal line.*

The system of points, lines and incidence relation given above defines a projective plane of order 9, which is the left semifield plane.

A regular quadrangle in a projective plane is a set of four points, no three of which are collinear. If $ABCD$ is a regular quadrangle, the six lines AB, AC, AD, BC, BD, CD are called the *sides* of the quadrangle, and the three points $V = AB \cap CD$, $W = AC \cap BD$, $U = AD \cap BC$ are called the *diagonal points* of the quadrangle. If the diagonal points of a regular quadrangle are collinear then the incidence structure $(\mathcal{P}, \mathcal{L})$ with

$$\mathcal{P}=\{A,B,C,D,U,V,W\}$$

and

$$\mathcal{L}=\{ABV,ACW,ADU,BCU,BDW,CDV,UVW\}$$

is a Fano plane. Such a Fano plane is called the completion of the regular quadrangle. If the diagonal points V, W, U are not collinear it is said that the quadrangle does not determine a Fano subplane.

2. Fano Subplanes of P_2S

Let $O = (0+\lambda 0, 0+\lambda 0) := (00, 00)$, $I = (1+\lambda 0, 1+\lambda 0) := (10, 10)$, $X = (0+\lambda 0) := (00)$ and $P_i = (a + \lambda b, c + \lambda d) := (ab, cd)$, $i \in \{1, 2, \ldots, 6\}$.

A regular quadrangle $OIXP_i$ can be completed to a Fano plane if and only if the diagonal points $OI \cap XP_i = V_i$, $OP_i \cap IX = U_i$, $OX \cap IP_i = W_i$, $i \in \{1, 2, \ldots, 6\}$, are collinear.

2.1. Proposition. *There are exactly eighteen Fano subplanes of* P_2S *which are completions of the regular quadrangles* $OIXP$ *with* $P = (a0, cd)$, $d \neq 0$, $a0 \neq cd$, $a, c, d \in \{0, 1, 2\}$,

Proof. Consider the quadrangles $OIXP$ with $O = (00, 00)$, $I = (10, 10)$, $X = (00)$ and $P = (a0, cd)$, $a, c, \in \mathbb{F}_3$, $d \in \mathbb{F}_3 \setminus \{0\}$. If $a = 0$ then $OIXP$ is a regular quadrangle with the diagonal points (cd, cd), $((c + 1)d, 00)$ and $(00, 01)$. Thus the completion of $OIXP$ is a Fano plane. If $a = 1$ then $OIXP$ is a regular quadrangle with the diagonal points (cd, cd), $((c + 2)d, 10)$ and $(10, 00)$. Thus, the completion of the regular quadrangle is also Fano plane. If $a = 2$ then the proof is similar to that of the above cases. □

It seems useful to find these Fano subplanes obtained in Proposition 2.1. For this, replace $P = (a0, cd)$ by P_i, R_i or S_i, $i \in \{1, 2, 3, 4, 5, 6\}$ according as P is on the line $x = 00$, $x = 10$, or $x = 20$, respectively. The list of these 18 Fano subplanes is given below by their diagonal points and the incidence tables:

Fano subplanes which are completions of $OIXP_i$:

1) $P_1 = (00, 01)$

$OI \cap XP_1 = (01, 01) = V_1$	U_1	P_1	W_1	O	I	X	V_1
$OP_1 \cap IX = (00, 01) = U_1$	P_1	W_1	O	I	X	V_1	U_1
$OX \cap IP_1 = (11, 00) = W_1$	O	I	X	V_1	U_1	P_1	W_1

2) $P_2 = (00, 02)$

$OI \cap XP_2 = (02, 02) = V_2$	U_1	P_2	W_2	O	I	X	V_2
$OP_2 \cap IX = (00, 01) = U_1$	P_2	W_2	O	I	X	V_2	U_1
$OX \cap IP_2 = (12, 00) = W_2$	O	I	X	V_2	U_1	P_2	W_2

3) $P_3 = (00, 11)$

$OI \cap XP_3 = (11, 11) = V_3$	U_1	P_3	W_3	O	I	X	V_3
$OP_3 \cap IX = (00, 01) = U_1$	P_3	W_3	O	I	X	V_3	U_1
$OX \cap IP_3 = (21, 00) = W_3$	O	I	X	V_3	U_1	P_3	W_3

4) $P_4 = (00, 12)$

$OI \cap XP_4 = (12, 12) = V_4$	U_1	P_4	W_4	O	I	X	V_4
$OP_4 \cap IX = (00, 01) = U_1$	P_4	W_4	O	I	X	V_4	U_1
$OX \cap IP_4 = (22, 00) = W_4$	O	I	X	V_4	U_1	P_4	W_4

5) $P_5 = (00, 21)$

$OI \cap XP_5 = (21, 21) = V_5$	U_1	P_5	W_5	O	I	X	V_5
$OP_5 \cap IX = (00, 01) = U_1$	P_5	W_5	O	I	X	V_5	U_1
$OX \cap IP_5 = (01, 00) = W_5$	O	I	X	V_5	U_1	P_5	W_5

6) $P_6 = (00, 22)$

$OI \cap XP_6 = (22, 22) = V_6$	U_1	P_6	W_6	O	I	X	V_6
$OP_6 \cap IX = (00, 01) = U_1$	P_6	W_6	O	I	X	V_6	U_1
$OX \cap IP_6 = (02, 00) = W_6$	O	I	X	V_6	U_1	P_6	W_6

Fano subplanes which are completions of $OIXR_i$:

1) $R_1 = (10, 01)$

$OI \cap XR_1 = (01, 01) = V_1$	Y_1	R_1	Z_1	O	I	X	V_1
$OR_1 \cap IX = (21, 10) = Y_1$	R_1	Z_1	O	I	X	V_1	Y_1
$OX \cap IR_1 = (10, 00) = Z_1$	O	I	X	V_1	Y_1	R_1	Z_1

2) $R_2 = (10, 02)$

$OI \cap XR_2 = (02, 02) = V_2$	Y_2	R_2	Z_1	O	I	X	V_2
$OR_2 \cap IX = (22, 10) = Y_2$	R_2	Z_1	O	I	X	V_2	Y_2
$OX \cap IR_2 = (10, 00) = Z_1$	O	I	X	V_2	Y_2	R_2	Z_1

3) $R_3 = (10, 11)$

$OI \cap XR_3 = (11, 11) = V_3$	Y_3	R_3	Z_1	O	I	X	V_3
$OR_3 \cap IX = (01, 10) = Y_3$	R_3	Z_1	O	I	X	V_3	Y_3
$OX \cap IR_3 = (10, 00) = Z_1$	O	I	X	V_3	Y_3	R_3	Z_1

4) $R_4 = (10, 12)$

$OI \cap XR_4 = (12, 12) = V_4$	Y_4	R_4	Z_1	O	I	X	V_4
$OR_4 \cap IX = (02, 10) = Y_4$	R_4	Z_1	O	I	X	V_4	Y_4
$OX \cap IR_4 = (10, 00) = Z_1$	O	I	X	V_4	Y_4	R_4	Z_1

5) $R_5 = (10, 21)$

$OI \cap XR_5 = (21, 21) = V_5$	Y_5	R_5	Z_1	O	I	X	V_5
$OR_5 \cap IX = (11, 10) = Y_5$	R_5	Z_1	O	I	X	V_5	Y_5
$OX \cap IR_5 = (10, 00) = Z_1$	O	I	X	V_5	Y_5	R_5	Z_1

6) $R_6 = (10, 22)$

$OI \cap XR_6 = (22, 22) = V_6$	Y_6	R_6	Z_1	O	I	X	V_6
$OR_6 \cap IX = (12, 10) = Y_6$	R_6	Z_1	O	I	X	V_6	Y_6
$OX \cap IR_6 = (10, 00) = Z_1$	O	I	X	V_6	Y_6	R_6	Z_1

Fano subplanes which are completions of $OIXS_i$:

1) $S_1 = (20, 01)$

$OI \cap XS_1 = (01, 01) = V_1$	Y_2	S_1	W_6	O	I	X	V_1
$OS_1 \cap IX = (22, 10) = Y_2$	S_1	W_6	O	I	X	V_1	Y_2
$OX \cap IS_1 = (02, 00) = W_6$	O	I	X	V_1	Y_2	S_1	W_6

2) $S_2 = (20, 02)$

$OI \cap XS_2 = (02, 02) = V_2$	Y_1	S_2	W_5	O	I	X	V_2
$OS_2 \cap IX = (21, 10) = Y_1$	S_2	W_5	O	I	X	V_2	Y_1
$OX \cap IS_2 = (01, 00) = W_5$	O	I	X	V_2	Y_1	S_2	W_5

3) $S_3 = (20, 11)$

$OI \cap XS_3 = (11, 11) = V_3$	Y_6	S_3	W_4	O	I	X	V_3
$OS_3 \cap IX = (12, 10) = Y_6$	S_3	W_4	O	I	X	V_3	Y_6
$OX \cap IS_3 = (22, 00) = W_4$	O	I	X	V_3	Y_6	S_3	W_4

4) $S_4 = (20, 12)$

$OI \cap XS_4 = (12, 12) = V_4$	Y_5	S_4	W_3	O	I	X	V_4
$OS_4 \cap IX = (11, 10) = Y_5$	S_4	W_3	O	I	X	V_4	Y_5
$OX \cap IS_4 = (21, 00) = W_3$	O	I	X	V_4	Y_5	S_4	W_3

5) $S_5 = (20, 21)$

$OI \cap XS_5 = (21, 21) = V_5$	Y_4	S_5	W_2	O	I	X	V_5
$OS_5 \cap IX = (02, 10) = Y_4$	S_5	W_2	O	I	X	V_5	Y_4
$OX \cap IS_5 = (12, 00) = W_2$	O	I	X	V_5	Y_4	S_5	W_2

6) $S_6 = (20, 22)$

$OI \cap XS_6 = (22, 22) = V_6$	Y_3	S_6	W_1	O	I	X	V_6
$OS_6 \cap IX = (01, 10) = Y_3$	S_6	W_1	O	I	X	V_6	Y_3
$OX \cap IS_6 = (11, 00) = W_1$	O	I	X	V_6	Y_3	S_6	W_1

Clearly, each of 18 Fano subplanes of P_2S containing O, I and X has a line passing through (∞). It is also known that every Fano subplane of P_2S has exactly one ideal point. Clearly, $X = (00)$ is the ideal point of the above 18 Fano subplanes which is paired with (∞).

In any Fano subplane let V be an ideal point with V', and let A and B be two proper points such that $V, V' \notin AB$. Then A, B, V can be mapped to O, I, X by a collination mapping the Fano subplane to a Fano subplane containing O, I, X.

2.2. Proposition. *The number of Fano subplanes which are completions of $AVBP$ is* 155520.

Proof. Let V be an ideal point, paired with V', in P_2S. Consider a Fano subplane which is completion of a regular quadrangle $ABVP$. As a proper point A can be chosen in 81 different ways, the second proper point B can be chosen in $8 \times 8 = 64$ different ways since it is not on the lines AV and AV'. There are 18 possibilities for the proper point P by Proposition 3 in [2]. It follows from the $3! = 6$ permutations of the proper points A, B and P that the total number of possibilities for A, B and P is $(81 \times 64 \times 18)/6 = 15552$. Finally, the ideal point V can be chosen in 10 different ways since P_2S is of order 9. Consequently the number of Fano planes which are completions of $ABVP$ in P_2S is 155520. □

2.3. Proposition. *If $P = (ab, cd)$ with $b \neq 0, d \neq 0$, then each the completion of $OIXP$ is a Fano subplane denoted by F_{abcd} (The total number of these subplanes is 30).*

Proof. If we check all configurations which are completions of $OIXP$, where $O = (00, 00)$, $I = (10, 10)$, $X = (00)$ and $P = (ab, cd)$, $b \neq 0, d \neq 0$, then it is easily seen that each completion of $OIXP$ determines a Fano subplane of P_2S as follows:

Consider the quadrangles $OIXP$ with $O = (00, 00)$, $I = (10, 10)$, $X = (00)$ and $P = (ab, cd)$, $d \neq 0$, $b \neq 0$. If $P = (01, 02)$ then $OIXP$ is a regular quadrangle with the diagonal points $OI \cap PX = (02, 02)$, $OX \cap IP = (21, 00)$, $OP \cap IX = (20, 10)$. Thus the completion of $OIXP$ is a Fano plane, denoted by F_{0102}.

$OI \cap PX = (02, 02) = D$	E	P	F	O	I	X	D
$OP \cap IX = (20, 10) = E$	P	F	O	I	X	D	E
$OX \cap IP = (21, 00) = F$	O	I	X	D	E	P	F.

Some Collinations of P_2S: For each $a \in S$ there exists a collination f_a of P_2S, as follows:

$(x, y) \to (x, y \oplus a)$, $a \in S$ and $[m, k] \to [m, k \oplus a]$, $a \in S$

$(m) \to (m)$ and $[\lambda] \to [\lambda]$

$(\infty) \to (\infty)$ and $[\infty] \to [\infty]$

□

2.4. Proposition. *Let $P = (ab, cd)$, $b \neq 0, d \neq 0$ and let F_{abcd} be the completion of the regular quadrangle $OIXP$. Then, there are exactly 8 different Fano planes $f_a(F_{abcd})$ which are isomorphic to F_{abcd}, for each F_{abcd}.*

Proof. To show $f_a(F_{abcd}) \neq f_b(F_{a_1b_1c_1d_1})$, we determine the image points of the diagonal points E and F in each plane $f_a(F_{abcd})$ and $f_b(F_{a_1b_1c_1d_1})$, respectively. Then checking the images of this pair of points, it can be easily seen that $f_a(F_{abcd})$ and $f_b(F_{a_1b_1c_1d_1})$ contain at least one distinct point. For every F_{abcd} there exist 8 such Fano planes $f_a(F_{abcd})$ with $a \in S$ since there are 8 collinations of P_2S distinct from the identity. □

2.5. Remark. From the above propositions the number of Fano subplanes of P_2S is at least $155520 + 8 \times 30 = 155760$.

References

[1] Akca, Z. and Kaya, R. *On the Subplanes of the Cartesian Group Plane of order 25*, Türk Matematik Derneği X. Ulusal Matematik Sempozyumu, 1–5 Eylül, 1–7, 1997.

[2] Çiftçi, S and Kaya, R. *On the Fano Subplanes in the Translation Plane of order 9*, Doğa-Tr. J. of Mathematics **14**, 1–7, 1990.

[3] Güney, Ö. *9. Mertebeden sol yarıcisim düzleminin altdüzlemleri üzerine* (OGÜ, Y. Lisans Tezi **68**, 2005).

[4] Hall, M. Jr. *Theory of Groups* (The Macmillan Company, New York 1959).

[5] Kaya, R. *Projektif Geometri*, (Osmangazi Üniversitesi **392**, 2005).

[6] Room, T. G and Kirkpatrick, P. B. *Miniquaternion Geometry* (Cambridge University Press, London **177**, 1971).

[7] Özcan, M. *Cebirsel Yapılardan Projektif Düzlem Elde Edilmesi Üzerine* (Anadolu Üniversitesi Y. Lisans Tezi **130**, 1998).