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CYCLES OF INDEFINITE QUADRATIC FORMS AND CYCLES OF IDEALS

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Abstract

Let δ denotes a real quadratic irrational integer with trace $t = \delta + \overline{\delta}$ and norm $n = \delta.\overline{\delta}$. Given a real quadratic irrational $\gamma \in \mathbb{Q}(\delta)$, there are rational integers P and Q such that $\gamma = \frac{P+\delta}{Q}$ with $Q|(\delta + P)(\overline{\delta} + P)$. Hence for each $\gamma = \frac{P+\delta}{Q}$, there is a corresponding ideal $I_\gamma = [Q, P + \delta]$, and an indefinite quadratic form $F_\gamma(x, y) = Q(x - \delta y)(x - \overline{\delta} y)$ of discriminant $t^2 - 4n$. In this paper, we consider the cycles of I_γ and cycles of F_γ for some specific values of $\delta = \sqrt{D}$, where $D \neq 1$ is a positive non-square integer.

Keywords: Quadratic forms, Ideals, Cycles of forms, Cycles of ideals.

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1. Introduction.

Binary quadratic forms play an important role in the theory of numbers and have been studied by many authors. A real binary quadratic form (or just a form) F is a polynomial in two variables x, y of the type

$$F = F(x, y) = ax^2 + bxy + cy^2, \tag{1.1}$$

with real coefficients a, b, c. We denote F briefly by $F = (a, b, c)$. The discriminant of F is defined by the formula $b^2 - 4ac$ and is denoted by Δ. A quadratic form F of discriminant Δ is called indefinite if $\Delta > 0$.

Gauss (1777-1855) defined the group action of the extended modular group $\overline{\Gamma}$ on the set of forms as follows:

$$\begin{aligned} gF(x, y) = &\left(ar^2 + brs + cs^2\right) x^2 + (2art + bru + bts + 2csu)\, xy \\ &+ \left(at^2 + btu + cu^2\right) y^2 \end{aligned} \tag{1.2}$$

for $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \overline{\Gamma}$ and $F = (a, b, c)$.

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Two forms F and G are called equivalent iff there exists a $g \in \overline{\Gamma}$ such that $gF = G$. If $\det g = 1$, then F and G are called properly equivalent. If $\det g = -1$, then F and G are called improperly equivalent. A quadratic form F is said to be ambiguous if it is improperly equivalent to itself. An indefinite quadratic form $F = (a, b, c)$ of discriminant Δ is said to be reduced if

$$\left|\sqrt{\Delta} - 2|a|\right| < b < \sqrt{\Delta}. \tag{1.3}$$

1.1. Theorem. [1, Sec:6.10, p.106] *Let $F = (a, b, c)$ be an indefinite quadratic form of discriminant Δ. Then the cycle of F is $F_0 \sim F_1 \sim F_2 \sim \cdots \sim F_{l-1}$ of length l, where $F_0 = F = (a_0, b_0, c_0)$,*

$$s_i = |s(F_i)| = \left[\frac{b_i + \sqrt{\Delta}}{2|c_i|}\right] \tag{1.4}$$

and

$$\begin{aligned} F_{i+1} &= (a_{i+1}, b_{i+1}, c_{i+1}) \\ &= \left(|c_i|,\ -b_i + 2s_i|c_i|,\ -(a_i + b_i s_i + c_i s_i^2)\right) \end{aligned} \tag{1.5}$$

for $1 \le i \le l-2$.

Mollin [4, Sec:1.1, p.4] considers the arithmetic of ideals in his book. Let $D \neq 1$ be a square free integer and let $\Delta = \frac{4D}{r^2}$, where $r = 2$ if $D \equiv 1 (\bmod 4)$, and $r = 1$, otherwise. If we set $\mathbb{K} = \mathbb{Q}(\sqrt{D})$, then $\mathbb{K}$ is called a quadratic number field of discriminant Δ, and O_Δ is the ring of integers of the quadratic field $\mathbb{K}$ of discriminant Δ.

Let $[\alpha, \beta]$ denote the $\mathbb{Z}-$ module $\alpha\mathbb{Z} \oplus \beta\mathbb{Z}$. Then every integer $w_\Delta \in O_\Delta$ can be uniquely expressed as $w_\Delta = x\alpha + y\beta$, where $x, y \in \mathbb{Z}$, and $\alpha, \beta \in O_\Delta$. We call α, β an integral basis for $\mathbb{K}$. The discriminant of $\mathbb{K}$ is D if $D \equiv 1 (\bmod 4)$, and is $4D$ otherwise. If $I = [a,\ b + cw_\Delta]$, then I is a non-zero ideal of O_Δ if and only if

$$c|b,\ c|a,\ ac|N(b + cw_\Delta). \tag{1.6}$$

Furthermore, for a given ideal I the integers a and c are unique and a is the least positive rational integer in I which we will denote as $L(I)$. The norm of an ideal I is defined as $N(I) = |ac|$. If I is an ideal of O_Δ with $L(I) = N(I)$, i.e. $c = 1$, then I is called primitive, which means that I has no rational integer factors other than ± 1. Every primitive ideal can be uniquely given by $I = [a,\ b + w_\Delta]$.

Mollin, Poorten and Cheng (see [3] and [5]) consider the ideals and their cycles extensively. The cycle of a primitive ideal $I = [a,\ b + w_\Delta]$ is defined as follows: Let $< m_0, \overline{m_1 m_2 \cdots m_{l-1}} >$ be the continued fraction expansion of $\frac{b+w_\Delta}{a}$ with period length $l = l(I)$, where

$$m_i = \left[\frac{P_i + \sqrt{D}}{Q_i}\right],\ \ P_{i+1} = m_i Q_i - P_i \text{ and } Q_{i+1} = \frac{D - P_{i+1}^2}{Q_i} \tag{1.7}$$

for $i \geq 0$. From the continued fraction factoring algorithm we get all reduced ideals equivalent to a given reduced ideal $I = [a,\ b+w_\Delta]$, i.e. in the continued fraction expansion of $\frac{b+w_\Delta}{a}$ we have $I = I_0 = [Q_0, P_0 + \sqrt{D}] \sim I_1 = [Q_1, P_1 + \sqrt{D}] \sim \cdots \sim I_{l-1} = [Q_{l-1}, P_{l-1} + \sqrt{D}]$. Finally, $I_l = I_0 = I$ for a complete cycle of reduced ideals of length $l(I) = l$. (see also [2] and [6]).

Let δ denotes a real quadratic irrational integer with trace $t = \delta + \overline{\delta}$ and norm $n = \delta\overline{\delta}$. Evidently, given a real quadratic irrational $\gamma \in \mathbb{Q}(\delta)$, there are rational integers P and Q such that $\gamma = \frac{P+\delta}{Q}$ with $Q|(\delta + P)(\overline{\delta} + P)$. Hence, for each $\gamma = \frac{P+\delta}{Q}$ there is a corresponding $\mathbb{Z}$–module $I_\gamma = [Q,\ P + \delta]$. In fact this module is an ideal by (1.6).

There is a connection between quadratic irrationals and quadratic forms. For any quadratic irrational $\gamma = \frac{P+\delta}{Q}$, we can generate a quadratic form

$$\begin{aligned} F_\gamma(x,y) &= Q(x-\delta y)(x-\overline{\delta}y) \\ &= Qx^2 - (t+2P)xy + \left(\frac{n+Pt+P^2}{Q}\right)y^2 \end{aligned}$$

of discriminant $\Delta = t^2 - 4n$ which corresponds to γ. Hence one associates with γ a quadratic form defined as above. If one takes $\delta = \sqrt{D}$, then $t = 0, n = -D$ and hence

$$F_\gamma = \left(Q, -2P, \frac{P^2-D}{Q}\right) \tag{1.8}$$

of discriminant $\Delta = 4D$.

2. Cycles of Indefinite Quadratic Forms and Cycles of Ideals.

Let $D \neq 1$ be a positive non-square integer, $\delta = \sqrt{D}$, and let $Q = k$, $P = -k$ for a positive integer k. Then $\gamma = \frac{-k+\sqrt{D}}{k}$ is a quadratic irrational, and hence $I_\gamma = [k, -k+\sqrt{D}]$ is an ideal and $F_\gamma = \left(k, 2k, \frac{k^2-D}{k}\right)$ is an indefinite quadratic form. We consider the cycles of I_γ and F_γ in four cases: $D = 4k^2 - k$, $D = k^2 + 2k$, $D = 3k^2$ and $D = 2k^2$. First we give the following theorem.

2.1. Theorem. *$F_\gamma = \left(k,\ 2k,\ \frac{k^2-D}{k}\right)$ is reduced if and only if $k^2 < D < 4k^2$.*

Proof. Let F_γ be reduced. Then by definition, we get

$$\left|\sqrt{\Delta} - 2|a|\right| < b < \sqrt{\Delta} \iff \left|\sqrt{4D} - 2k\right| < 2k < \sqrt{4D} \iff \sqrt{D} - k < k < \sqrt{D}.$$

Hence it is clear that $k^2 < D$ and $D < (2k)^2$. Therefore $k^2 < D < 4k^2$.

Conversely, let $k^2 < D < 4k^2$. Then

$$\begin{aligned} & k^2 < D < 4k^2 \\ & \implies \sqrt{k^2} < \sqrt{D} < \sqrt{4k^2} \\ & \implies k < \sqrt{D} < 2k \\ & \implies 0 < \left|\sqrt{D} - k\right| < k < \sqrt{D} \\ & \implies 2\left|\sqrt{D} - k\right| < 2k < 2\sqrt{D} \\ & \implies \left|\sqrt{4D} - 2k\right| < 2k < \sqrt{4D} \\ & \implies \left|\sqrt{\Delta} - 2|a|\right| < b < \sqrt{\Delta}. \end{aligned}$$

This is the definition of reduced forms. Therefore, F_γ is reduced. □

2.2. Theorem. *If $D = 4k^2 - k$ then:*

(1) *The continued fraction expansion of γ is $< 0, \overline{1, 4k-2, 1, 2} >$, and the cycle of I_γ is*

$$\begin{aligned} & I_\gamma^0 = [k, -k+\sqrt{4k^2-k}] \sim I_\gamma^1 = [3k-1, k+\sqrt{4k^2-k}] \\ & \sim I_\gamma^2 = [1, 2k-1+\sqrt{4k^2-k}] \sim I_\gamma^3 = [3k-1, 2k-1+\sqrt{4k^2-k}] \\ & \sim I_\gamma^4 = [k, k+\sqrt{4k^2-k}]; \end{aligned}$$

(2) *F_γ is ambiguous, and the cycle of F_γ is*

$$\begin{aligned} &F_\gamma^0 = (k, 2k, 1-3k) \\ \sim\ &F_\gamma^1 = (3k-1, 4k-2, -1) \\ \sim\ &F_\gamma^2 = (1, 4k-2, 1-3k) \\ \sim\ &F_\gamma^3 = (3k-1, 2k, -k). \end{aligned}$$

Proof. (1) For the quadratic irrational $\gamma = \frac{-k+\sqrt{4k^2-k}}{k}$ we have $m_0 = 0$ from (1.7). Hence

$$\begin{aligned} P_1 &= m_0Q_0 - P_0 = 0\cdot k - (-k) = k, \\ Q_1 &= \frac{D-P_1^2}{Q_0} = \frac{4k^2-k-k^2}{k} = 3k-1,\ m_1 = 1. \end{aligned}$$

For $i = 1$ we have

$$\begin{aligned} P_2 &= m_1Q_1 - P_1 = 1\cdot(3k-1) - k = 2k-1, \\ Q_2 &= \frac{D-P_2^2}{Q_1} = \frac{4k^2-k-(2k-1)^2}{3k-1} = \frac{3k-1}{3k-1} = 1,\ m_2 = 4k-2. \end{aligned}$$

For $i = 2$ we have

$$\begin{aligned} P_3 &= m_2Q_2 - P_2 = (4k-2).1 - (2k-1) = 2k-1, \\ Q_3 &= \frac{D-P_3^2}{Q_2} = \frac{4k^2-k-(2k-1)^2}{1} = 3k-1, m_3 = 1. \end{aligned}$$

For $i = 3$ we have

$$\begin{aligned} P_4 &= m_3Q_3 - P_3 = 1\cdot(3k-1) - (2k-1) = k, \\ Q_4 &= \frac{D-P_4^2}{Q_3} = \frac{4k^2-k-k^2}{3k-1} = k,\ m_4 = 2. \end{aligned}$$

For $i = 4$ we have

$$\begin{aligned} P_5 &= m_4Q_4 - P_4 = 2\cdot k - k = k = P_1, \\ Q_5 &= \frac{D-P_5^2}{Q_4} = \frac{4k^2-k-k^2}{k} = 3k-1 = Q_1,\ m_5 = 1 = m_1. \end{aligned}$$

Therefore, the continued fraction expansion of γ is $< 0, \overline{1, 4k-2, 1, 2} >$, and hence the cycle of I_γ is $I_\gamma^0 = [k, -k+\sqrt{4k^2-k}] \sim I_\gamma^1 = [3k-1, k+\sqrt{4k^2-k}] \sim I_\gamma^2 = [1, 2k-1+\sqrt{4k^2-k}] \sim I_\gamma^3 = [3k-1, 2k-1+\sqrt{4k^2-k}] \sim I_\gamma^4 = [k, k+\sqrt{4k^2-k}]$.

(2) Let $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \overline{\Gamma}$ and $F_\gamma = (k, 2k, 1-3k)$. Then by (1.2), the system of equations

$$\begin{aligned} kr^2 + 2krs + (1-3k)s^2 &= k \\ 2krt + 2kru + 2kts + 2(1-3k)su &= 2k \\ kt^2 + 2ktu + (1-3k)u^2 &= 1-3k \end{aligned}$$

has a solution for $g = \begin{pmatrix} 4k-1 & 4k \\ 2-4k & 1-4k \end{pmatrix} \in \overline{\Gamma}$. Hence F_γ is improperly equivalent to itself since $\det g = -1$. Therefore F_γ is ambiguous.

For the quadratic form $F_\gamma = (k, 2k, 1-3k)$ of discriminant $\Delta = 4(4k^2 - k)$ we get from (1.4), $s_0 = 1$ and from (1.5)

$$\begin{aligned} F_\gamma^1 &= (a_1, b_1, c_1) = \big(|c_0|,\ -b_0 + 2s_0|c_0|,\ -a_0 - b_0 s_0 - c_0 s_0^2\big) \\ &= (3k-1, -2k + 2(3k-1), -k - 2k.1 - (1-3k)\cdot 1) \\ &= (3k-1, 4k-2, -1)\,. \end{aligned}$$

For $i = 1$ we have $s_1 = 4k - 2$, and

$$\begin{aligned} F_\gamma^2 &= (a_2, b_2, c_2) = \big(|c_1|,\ -b_1 + 2s_1|c_1|,\ -a_1 - b_1 s_1 - c_1 s_1^2\big) \\ &= \big(1, 2 - 4k + 2(4k-2).1, 1 - 3k - (4k-2)(4k-2) + (4k-2)^2\big) \\ &= (1, 4k-2, 1-3k)\,. \end{aligned}$$

For $i = 2$ we have $s_2 = 1$, and

$$\begin{aligned} F_\gamma^3 &= (a_3, b_3, c_3) = \big(|c_2|,\ -b_2 + 2s_2|c_2|,\ -a_2 - b_2 s_2 - c_2 s_2^2\big) \\ &= (3k-1, 2 - 4k + 2(3k-1), -1 - (4k-2) - (1-3k)) \\ &= (3k-1, 2k, -k)\,. \end{aligned}$$

For $i = 3$ we have $s_3 = 2$, and

$$\begin{aligned} F_\gamma^4 &= (a_4, b_4, c_4) = \big(|c_3|,\ -b_3 + 2s_3|c_3|,\ -a_3 - b_3 s_3 - c_3 s_3^2\big) \\ &= (k, -2k + 2\cdot 2\cdot k, 1 - 3k - 2k\cdot 2 + k\cdot 4) \\ &= (k, 2k, 1-3k) \\ &= F_\gamma^0. \end{aligned}$$

Therefore the cycle of F_γ is completed and is $F_\gamma^0 = (k, 2k, 1-3k) \sim F_\gamma^1 = (3k-1, 4k-2, -1) \sim F_\gamma^2 = (1, 4k-2, 1-3k) \sim F_\gamma^3 = (3k-1, 2k, -k)$. □

2.3. Theorem. *If $D = k^2 + 2k$ then:*

(1) *The continued fraction expansion of γ is $< 0, \overline{k, 2} >$, and the cycle of I_γ is*

$$\begin{aligned} &I_\gamma^0 = [k, -k + \sqrt{k^2 + 2k}] \\ \sim\ &I_\gamma^1 = [2, k + \sqrt{k^2 + 2k}] \\ \sim\ &I_\gamma^2 = [k, k + \sqrt{k^2 + 2k}]; \end{aligned}$$

(2) *F_γ is ambiguous, and the cycle of F_γ is*

$$\begin{aligned} &F_\gamma^0 = (k, 2k, -2) \\ \sim\ &F_\gamma^1 = (2, 2k, -k). \end{aligned}$$

Proof. (1) For the quadratic irrational $\gamma = \frac{-k + \sqrt{k^2 + 2k}}{k}$ we have from (1.7), $m_0 = 0$. Hence

$$\begin{aligned} P_1 &= m_0 Q_0 - P_0 = 0.k - (-k) = k, \\ Q_1 &= \frac{D - P_1^2}{Q_0} = \frac{k^2 + 2k - k^2}{k} = 2,\ m_1 = k. \end{aligned}$$

For $i = 2$ we have

$$\begin{aligned} P_2 &= m_1 Q_1 - P_1 = k.2 - k = k, \\ Q_2 &= \frac{D - P_2^2}{Q_1} = \frac{k^2 + 2k - k^2}{2} = k,\ m_2 = 2. \end{aligned}$$

For $i=3$ we have

$$P_3 = m_2Q_2 - P_2 = 2\cdot k - k = k = P_1,$$

$$Q_3 = \frac{D-P_3^2}{Q_2} = \frac{k^2+2k-k^2}{k} = 2 = Q_1,\ m_3 = k = m_1.$$

Therefore, the continued fraction expansion of γ is $<0,\overline{k,2}>$, and hence the cycle of I_γ is $I_\gamma^0 = [k, -k+\sqrt{k^2+2k}] \sim I_\gamma^1 = [2, k+\sqrt{k^2+2k}] \sim I_\gamma^2 = [k, k+\sqrt{k^2+2k}]$.

(2) Let $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \overline{\Gamma}$ and $F_\gamma = (k, 2k, -2)$. Then by (1.2), the system of equations

$$\begin{aligned} kr^2 + 2krs - 2s^2 &= k \\ 2krt + 2kru + 2kts - 4su &= 2k \\ kt^2 + 2ktu - 2u^2 &= -2 \end{aligned}$$

has a solution for $g = \begin{pmatrix} 2k+1 & 2k^2+2k \\ -2 & -1-2k \end{pmatrix} \in \overline{\Gamma}$. Hence F_γ is improperly equivalent to itself since $\det g = -1$. Therefore F_γ is ambiguous.

For the quadratic form $F_\gamma = (k, 2k, -2)$ of discriminant $\Delta = 4(k^2+2k)$ we get from (1.4), $s_0 = k$ and from (1.5)

$$\begin{aligned} F_\gamma^1 = (a_1, b_1, c_1) &= \left(|c_0|,\ -b_0 + 2s_0|c_0|,\ -a_0 - b_0s_0 - c_0s_0^2\right) \\ &= \left(2, -2k + 2\cdot k\cdot 2, -k - 2k\cdot k + 2k^2\right) \\ &= (2, 2k, -k)\,. \end{aligned}$$

For $i=1$ we have $s_1 = 2$, and

$$\begin{aligned} F_\gamma^2 = (a_2, b_2, c_2) &= \left(|c_1|,\ -b_1 + 2s_1|c_1|,\ -a_1 - b_1s_1 - c_1s_1^2\right) \\ &= (k, -2k + 2\cdot 2\cdot k, -2 - 2k\cdot 2 + k\cdot 4) \\ &= (k, 2k, -2) \\ &= F_\gamma^0. \end{aligned}$$

Therefore the cycle of F_γ is completed and is $F_\gamma^0 = (k, 2k, -2) \sim F_\gamma^1 = (2, 2k, -k)$. □

2.4. Theorem. *If $D = 3k^2$ then:*

(1) *The continued fraction expansion of γ is $<0,\overline{1,2}>$, and the cycle of I_γ is*

$$\begin{aligned} &I_\gamma^0 = [k, -k+\sqrt{3k^2}] \\ \sim\ &I_\gamma^1 = [2k, k+\sqrt{3k^2}] \\ \sim\ &I_\gamma^2 = [k, k+\sqrt{3k^2}]; \end{aligned}$$

(2) *F_γ is ambiguous, and the cycle of F_γ is*

$$\begin{aligned} &F_\gamma^0 = (k, 2k, -2k) \\ \sim\ &F_\gamma^1 = (2k, 2k, -k). \end{aligned}$$

Proof. (1) For the quadratic irrational $\gamma = \frac{-k+\sqrt{3k^2}}{k}$ we have $m_0 = 0$ from (1.7). Hence

$$P_1 = m_0Q_0 - P_0 = 0\cdot k - (-k) = k,$$

$$Q_1 = \frac{D-P_1^2}{Q_0} = \frac{3k^2-k^2}{k} = 2k,\ m_1 = 1.$$

For $i = 1$ we have

$$P_2 = m_1Q_1 - P_1 = 1 \cdot 2k - k = k,$$
$$Q_2 = \frac{D - P_2^2}{Q_1} = \frac{3k^2 - k^2}{2k} = k,\ m_2 = 2.$$

For $i = 2$ we have

$$P_3 = m_2Q_2 - P_2 = 2 \cdot k - k = k = P_1,$$
$$Q_3 = \frac{D - P_3^2}{Q_2} = \frac{3k^2 - k^2}{k} = 2k = Q_1,\ m_3 = 1 = m_1.$$

Therefore, the continued fraction expansion of γ is $< 0, \overline{1, 2} >$, and hence the cycle of I_γ is $I_\gamma^0 = [k, -k + \sqrt{3k^2}] \sim I_\gamma^1 = [2k, k + \sqrt{3k^2}] \sim I_\gamma^2 = [k, k + \sqrt{3k^2}]$.

(2) Let $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \overline{\Gamma}$ and $F_\gamma = (k, 2k, -2k)$. Then by (1.2), the system of equations

$$\begin{aligned} kr^2 + 2krs - 2ks^2 &= k \\ 2krt + 2kru + 2kts - 4ksu &= 2k \\ kt^2 + 2ktu - 2ku^2 &= -2k \end{aligned}$$

has a solution for $g = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \in \overline{\Gamma}$. Hence F_γ is improperly equivalent to itself since $\det g = -1$. Therefore F_γ is ambiguous.

For the quadratic form $F_\gamma = (k, 2k, -2k)$ of discriminant $\Delta = 12k^2$ we get from (1.4), $s_0 = 1$ and from (1.5)

$$\begin{aligned} F_\gamma^1 = (a_1, b_1, c_1) &= \left(|c_0|,\ -b_0 + 2s_0|c_0|,\ -a_0 - b_0s_0 - c_0s_0^2\right) \\ &= (2k, -2k + 2 \cdot 2k, -k - 2k \cdot 1 + 2k) \\ &= (2k, 2k, -k)\,. \end{aligned}$$

For $i = 1$ we have $s_1 = 2$, and

$$\begin{aligned} F_\gamma^2 = (a_2, b_2, c_2) &= \left(|c_1|,\ -b_1 + 2s_1|c_1|,\ -a_1 - b_1s_1 - c_1s_1^2\right) \\ &= (k, -2k + 2 \cdot 2 \cdot k, -2k - 2k \cdot 2 + k \cdot 4) \\ &= (k, 2k, -2k) \\ &= F_\gamma^0. \end{aligned}$$

Therefore the cycle of F_γ is completed and is $F_\gamma^0 = (k, 2k, -2k) \sim F_\gamma^1 = (2k, 2k, -k)$. □

2.5. Theorem. *If $D = 2k^2$ then:*

(1) *The continued fraction expansion of γ is $< 0, \overline{2} >$, and the cycle of I_γ is*

$$\begin{aligned} &I_\gamma^0 = [k, -k + \sqrt{2k^2}] \\ \sim\ &I_\gamma^1 = [k, k + \sqrt{2k^2}]; \end{aligned}$$

(2) *F_γ is ambiguous, and the cycle of F_γ is $F_\gamma^0 = (k, 2k, -k)$.*

Proof. (1) For the quadratic irrational $\gamma = \frac{-k+\sqrt{2k^2}}{k}$ we have $m_0 = 0$ from (1.7). Hence

$$P_1 = m_0.Q_0 - P_0 = 0 \cdot k - (-k) = k,$$
$$Q_1 = \frac{D - P_1^2}{Q_0} = \frac{2k^2 - k^2}{k} = k,\ m_1 = 2.$$

For $i = 1$ we have

$$P_2 = m_1 Q_1 - P_1 = 2 \cdot k - k = k = P_1,$$

$$Q_2 = \frac{D - P_2^2}{Q_1} = \frac{2k^2 - k^2}{k} = k = Q_1, \ m_2 = 2 = m_1.$$

Therefore, the continued fraction expansion of γ is $< 0, \overline{2} >$, and hence the cycle of I_γ is $I_\gamma^0 = [k, -k + \sqrt{2k^2}] \sim I_\gamma^1 = [k, k + \sqrt{2k^2}]$.

(2) Let $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \overline{\Gamma}$ and $F_\gamma = (k, 2k, -k)$. Then by (1.2), the system of equations

$$\begin{aligned} kr^2 + 2krs - ks^2 &= k \\ 2krt + 2kru + 2kts - 2ksu &= 2k \\ kt^2 + 2ktu - ku^2 &= -k \end{aligned}$$

has a solution for $g = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \in \overline{\Gamma}$. Hence F_γ is improperly equivalent to itself since $\det g = -1$. Therefore F_γ is ambiguous.

For the quadratic form $F_\gamma = (k, 2k, -k)$ of discriminant $\Delta = 8k^2$ we get from (1.4), $s_0 = 2$ and from (1.5)

$$\begin{aligned} F_\gamma^1 = (a_1, b_1, c_1) &= \big(|c_0|, \ -b_0 + 2s_0|c_0|, \ -a_0 - b_0 s_0 - c_0 s_0^2\big) \\ &= (k, -2k + 2 \cdot 2k, -k - 2k \cdot 2 + 4k) \\ &= (k, 2k, -k) \\ &= F_\gamma^0. \end{aligned}$$

Therefore the cycle of F_γ is completed and is $F_\gamma^0 = (k, 2k, -k)$. □

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