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COMPARISON OF THE LEAST SQUARE ESTIMATORS WITH THEIR ROBUST VERSIONS IN LINEAR CALIBRATION PROBLEMS

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Abstract

The calibration problem is concerned with the prediction of x_0, the true but unknown value, by using y_0 given by a quick method. There are some approaches to solve this problem. In this study, these approaches are compared by using a simulation method with and without an outlier. It is observed that the Inverse estimator, generally yielded better results for interpolation, but that the Nazsodi estimators were good for extrapolation and the preference of small bias.

Keywords: Linear calibration, Classical estimator, Inverse estimator, Nazsodi estimator, Ali and Singh estimator, Srivastava and Singh estimator, Robust calibration, Outliers, Huber versions, Monte Carlo simulation, Mean squared errors, Design matrix.

1. The Calibration Problem and its Solutions

The simple linear calibration problem can be explained as follows. Let $x_1, x_2, \dots, x_n$ represent the unknown true values which can obtained by a slow and expensive method, and $y_1, y_2, \dots, y_n$ the corresponding observed values using a quick and cheap method. Thus, y_i is measured with error but x_i without error. The simple linear relationship between Y and X is given by

(1) $$y_i = \alpha + \beta x_i + \epsilon_i,\ i = 1, 2, \dots, n,$$

where α and β are unknown parameters; $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ are identically and independently distributed random variables with the same mean zero and variance σ^2. In general, ϵ is distributed as $N(0, \sigma^2)$. Further, suppose that m additional observations on y_0 corresponding to an unknown x_0 are available. The calibration problem is concerned with the prediction of x_0 the true but unknown value, by using y_0, the mean of $y_{01}, y_{02}, \dots, y_{0m}$ on a calibrated scale. There are two main approaches to solve this problem. First, the

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parameters (α, β) are estimated by least squares as (a, b) and then the prediction of x_0 is obtained by

$$\widehat{x}_{oc} = (y_0 - a)/b = \overline{x} + (y_0 - \overline{y})/b, \ b \neq 0, \tag{2}$$

where $\overline{x} = \sum x_i/n$, $\overline{y} = \sum y_j/n$, $y_0 = \sum y_{0i}/m$. The estimator of $\widehat{x}_{0c}$ in (2) is known as the "Classical Estimator". The regression of X on Y, without regard to which variables are measured with error, can be used to estimate x_0 as

$$\widehat{x}_{0I} = \overline{x} + \frac{S_{xy}}{S_{yy}}(y_0 - \overline{y}), \tag{3}$$

where $S_{xy} = \sum(x_i - \overline{x})(y_i - \overline{y})$ and $S_{yy} = \sum(y_i - \overline{y})^2$ while $b = \frac{S_{xy}}{S_{xx}}$ is used in (2) with $S_{xx} = \sum(x_i - \overline{x})^2$. The equation (3) can be used to estimate the equation $x_i = \gamma + \delta y_i + u_i$, $i = 1, 2, \ldots, n$. The parameters (γ,δ) can be estimated by least square as (c, d) where u_i has the same assumptions with ϵ_i. Here, $\widehat{x}_{0I}$ is known as the "Inverse Estimator". The main problem with the classical estimator is that it has an undefined mean and infinite mean squared error (MSE) for fixed x_i and finite n (Willams, 1969).

Eisenhart [4] rejected the inverse method because the regression obtained by minimizing $\sum(x_i - \widehat{x}_i)^2$ is meaningless when the values of x have been fixed and the corresponding y values observed. Krutchkoff [8,9] compared the two methods using a simulation study based on the MSE for the case $m = 1$, in which the values of $|b| < 0.001$ were replaced by ± 0.001. He found the inverse method to be better than the classical method for interpolation. According to Hoadley [6], the inverse estimator is a Bayesian approach if a prior distribution on x is selected carefully. Lwin and Maritz [10] also support the inverse estimator of x_0 when using a compound estimation approach.

Berkson [3] showed that the estimators a and b were consistent, but that the inverse estimators c and d were inconsistent. However, he pointed out that $\widehat{x}_{0\mathrm{I}}$ has a smaller MSE than $\widehat{x}_{0\mathrm{C}}$ over a large region around $\overline{x}$.

Halperin [5] revised the equation (3) as $\widehat{x}_0 = (1 - R)\overline{x} + R\widehat{x}_{0c}$, where

$$R = nb^2 S_{xx}/[\mathrm{RSS} + nb^2_{xx}].$$

He also suggested an alternative estimator for x_0, by using the ratio RSS$/m$ instead of RS $= \sum(y_i - \widehat{y}_i)^2$:

$$\widehat{x}_{0H} = \overline{x} + \frac{mS_{xy}}{mb^2 S_{xx} + RSS}(y_0 - \overline{y}), \tag{4}$$

(Halperin [5]). Halperin regarded this expression as a generalized Krutchkoff estimator.

Shukla [12] studied the two estimators for the truncated normal case $|\beta| > 0$. He obtained results supporting Berkson's conclusion, and showed that the moments can be generated when the ratio β/σ is large.

Aitchison and Dunsmore [1] used the following equation, which is similar to Halperin's estimator, to estimate x_0. Here, $V = \sum_{j=1}^{m}(y_{0j} - y_0)^2$.

$$\widehat{x}_{0AD} = \overline{x} + \frac{S_{xy}}{S_{yy} + V}(y_0 - \overline{y})^2. \tag{5}$$

For $m = 1$, $\widehat{x}_{0AD}$ is equal to x_{0I}. But if $m > 1$, the variation in $y_{01}, y_2, \ldots, y_{0m}$ is taken into account in the estimate.

Aitchison and Dunimore [1] evaluated this approach from a Bayesian viewpoint. Naszodi (1978) suggested a correction $1/(b + c)$ instead of $1/b$ in the classical regression line before estimating x_0, where $c = m_1/b =^2 /b_{xx}$, m_1 being the second order central moment of b. If the values y and the estimators a and b are distributed independently and normally, Naszodi determined that $\widehat{x}_0$ has a Cauchy distribution, which has no expected value. Then he suggested an approximation to the expected value of $\widehat{x}_0$, and after rearranging the estimator of x_0 is given by

$$\widehat{x}_{0N} = \overline{x} + [\frac{b}{b^2 + S^2/S_{xx}}](y_0 - \overline{y}) \tag{6}$$

where S^2 is an unbiased estimator of σ^2. In his simulation work, he obtained the unbiased values by using $\hat{x}_{0\mathrm{N}}$, while $\hat{x}_{0C}$ and $\hat{x}_{0I}$ are biased. These estimators are derived from the experimental designs of $x_1, x_2, \ldots, x_n$.

Ali and Singh [2] also suggested an alternative estimator in linear calibration given by

$$\widehat{x}_{0AS} = \lambda\widehat{x}_{0C} + (1-\lambda)\overline{x},\ \lambda \in [0,1]. \tag{7}$$

Ali and Singh described the asymptotic MSE properties of $\hat{x}_{0S}$. They showed the existence of a value of λ for which $\hat{x}_{0S}$ is better than $\hat{x}_{0C}$. An optimum value λ^* of λ was obtained as $\lambda = \beta^2 S^2(\beta^2\gamma^2+\sigma^2)^(-1)$ by Ali and Singh, while $\widehat{\delta} = \widehat{x}_{0C} - \overline{x}$ or $\widehat{\delta} = \widehat{x}_{0I} - \overline{x}$.

Srivastava and Singh [13] considered a weighted average of the classical and inverse estimators for x_0 based on a suggestion of Ali and Singh [2]. They expressed this estimator in the form

$$\begin{aligned} \widehat{x}_{0SS} &= \lambda\widehat{x}_{0C} + (1-\lambda)\widehat{x}_{OI} \\ &= [\widehat{x}_{0C} + (n-3)\widehat{x}_{0I}]/(n-2), \end{aligned} \tag{8}$$

where λ is a constant between 0 and 1.

On the other hand, in this paper we investigate techniques for the robust estimation of x_0 in calibration problems based on the regression model.

2. Robust Calibration

Robust calibration procedures are procedures that work well even if there is some contamination in the data, or if the model assumptions are slightly violated. Several approaches to robustifying calibration are given in the literature. Kitsos and Müller [7] introduced calibration estimators based on robust one-step M-estimators. One step M-estimators have a bounded asymptotic bias. There are two basic approaches to robust calibration. Firstly, in the standard regression model (1), let us write the classical regression M-estimates in the forms $\widehat{\alpha} = S(F)$ and $\widehat{\beta} = T(F)$, where $S(F)$ and $T(F)$ are solutions of

$$\int \rho(y - S(F) - T(F)x)dF(x,y), \tag{9}$$

where F_n is the empirical distribution function and ρ is the loss function with derivative Ψ. To obtain S and T, the equations are written as

$$\begin{aligned} &\int \psi(y - S(F) - T(F)x)\, dF(x,y) = 0, \\ &\int \psi(y - S(F) - T(F)x)x\, dF(x,y) = 0. \end{aligned} \tag{10}$$

The estimates S and T obtained from (10) are not so sensitive to large deviations. If we center the data about the means $\dot{x}$ and $\dot{y}$, equations (10) reduce to

$$\int \psi(\dot{y} - T(G)\dot{x})\dot{x}dG(\dot{x},\dot{y}) = 0,$$

where G is the distribution function of $\dot{x}$, $\dot{y}$, and $T(F) = T(G)$. Therefore it can be used to focus on the estimation of β, and estimate $\alpha = \mu_y - \beta\mu_x$ without taking into account μ_x and μ_y. That is, it substitutes robust estimates $\widehat{\mu_x}$ and $\widehat{\mu_y}$ in place of μ_x and μ_y while estimating β using the estimated data. Now we redefine S(F) to conform with our use of centered data:

$$S(F) = \mu_y(F) - T(F)\mu_x, \tag{11}$$

where $\mu_x(F)$ and $\mu_y(F)$ are robust estimators of the respective means. Further, we define the estimate for x_0 as

$$V(F) = \frac{y_0 - S(F)}{T(F)}. \tag{12}$$

In summary, for x_0 the procedure is as follows:

1. Obtain $\widehat{\mu_x} = \mu_x(F)$ and $\widehat{\mu_y} = \mu_y(F)$.
2. Center the data on these robust estimates.
3. From the centered data, find $T(F_n)$ of β.
4. Define x_0 by

$$V(F) = \frac{y_0 - S(F_n)}{T(F_n)}, \tag{13}$$

where $S(F_n)$ is given by equation (12).

To obtain the robust inverse regression model, we interchange the roles of X and Y in the above and make the following changes:

$$Q(F) = \mu_x(F) - R(F)\mu_y(F) \text{ and } W(F) = Q(F) + R(F)y_0.$$

Here, $R(F)$ is the robust estimator based on the centered data. $Q(F)$ estimates γ and $W(F)$ estimates x_0. We estimate x_0 by $W(F_n) = Q(F_n) + R(F_n)y_0$.

3. The Monte Carlo Simulation Study

The studies given in Section 1 are related to the classic, the inverse and some alternative estimators. In general, the Bayesian approximations support the inverse estimators. The alternative estimators are the weighted averages of the classical and inverse estimators. In many studies, these estimators had been compared only with the classical and/or inverse estimators. Besides, in data about calibration (especially in analytical and pharmacology), the number of observation is small. So a little deviation in the data has a large influence as an outlier. In this study, we aimed to compare all the methods given above, and to study the influences of an outlier and the robust versions of the previous estimators by using a Monte Carlo Simulation.

The simulation experiments were used to estimate the mean squared errors (MSE) for all estimators. The steps are described below. At the beginning, to compare the results with other studies, we took the model (1) and set σ=0.1 and α=1. To see the effects of the various factors, we considered combinations of β, the design matrix of X, the true values of x_0, and the sample size n for $m = 1$. These factors are as follows:

(1) $\beta :$ 0.2, 0.5, 1.0, 2.0
(2) $x_0 :$ 0.1, 0.4, 0.7, 1.0, 3.0
(3) $n :$ 6, 10, 20
(4) The Design matrix of X(D): The end point design, the equidistant design, the unconstrained design

The range of the controlled variable X is taken to be $[0, 1]$ and the mean of X to be 0.5 in all designs. Here, in the end point design, half of the observations are taken to be zero (or min), and the other half to be one (or max). In 1968, Ott and Myers determined this as an optimal design in a linear calibration. In the equidistant design given by Nazsodi [11], the range of X is divided equally into n parts. In this study, for n=6, this gives $X = [0.0, 0.2, 0.4, 0.6, 0.8, 1.0]$. For the unconstrained design, the values of x_i are selected not to be constrained as above in the interval $(0, 1)$. So the values of $\sum(x_i - \overline{x})^2$, which is considered to be a factor over MSE, are the highest in the end point design and the lowest in the mixed design.

On the other hand, the values 0.1 and 1.0 of x_0 above were selected as the end points of X, the points 0.4 and 0.7 near to the mean of x and the extrapolation point as 3.0. Another factor on MSE is the ratio β/σ, according to the formulations of Shukla [12]. In this study, β/σ is taken as $2, 5, 10$ and 20 because β was changed while σ was fixed.

To constitute data for each combinations of the factors, ϵ_i in model (1) was generated from $\epsilon_i \sim N(0, \sigma^2)$. The future value of y corresponding to x_0, the means of y_0, was also obtained from the same model but x_0 was taken instead of x_i, and $\epsilon_{n+1},\ldots,\epsilon_{m+n}$ were generated instead of ϵ_i. After obtaining the n values, the estimators were evaluated. However, since $b \neq 0$ for the classical estimator, we considered a truncated version of this estimator by assigning the value 0.001 to b in equation (2) when $b < 0.001$.

The other estimators $\widehat{x}_0$ were the Inverse estimator from (3), the Nazsodi from (6), the Ali and Singh from (7) and the Srivastava and Singh from (8). Note that (4) and (5) were not calculated because the estimators (3),(4) and (5) are the same for $m = 1$. On the other hand, the third observation of the dependent variable Y was taken as $y_3 = 3.0$ in all designs to give an outlier. In this case, the robust Huber versions of the classical and inverse methods were evaluated as given in Section 2. Some versions of the alternative estimators were also considered. Because these estimators are the weighted averages of the Classical and Inverse estimators, Huber versions of classical estimators were used for b and S^2 in the Nazsodi and the Ali and Singh methods. Huber versions of the Srivastava and Sing method were also taken as weighted averages of the classical Huber and inverse-Huber estimators.

In all simulations, 2000 repetitions were used. To calculate the estimate of MSE for each $\widehat{x}_0$, the average of 2000 sample values of $(\widehat{x}_0 - x_0)^2$ were obtained. These results are given in the tables. All calculations and the generation of the random normal variates were performed on a PC using the MATLAB programme.

4. Comparisons

The effects of the factors in the case of no outlier are given in Table 1 and Table 2. The results for $n = 10$ are not given in the tables as they add nothing new.

Due to the effects of the slope of β, that is β/σ, the MSE's for the classic estimator and for Srivastava and Sing (SS) at $n < 20$ are very large for $\beta < 0.5$.

The inverse method yielded generally better results then the others for interpolation.

The Nazsodi estimator(NE) can be preferred outside the interval $(0, 1)$ in all cases.

The MSE's for the inverse and Ali and Singh (AS) estimators were smaller than the MSE of the others, and SS was also as good as the inverse estimator at the ends of the interval.

All estimators gave very similar results for $\beta \geq 1.0$.

We deduce that the location of x_0 and the values of β all affect the MSE's of the estimators $\widehat{x}_0$.

We note that AS is clearly better than the others for $n \geq 10$.

The values of the MSE generally increased when the design matrix was changed from Design 1 to Design 3, that is, the variability of X was increased.

The end point design was the most efficient for the classical estimator. However, the design has no effect on calibration methods. The greater the sample size, the clearer were the differences between the estimators, except for a few cases which may have been due to sampling fluctuations. In this case, AS could be preferred when x_0 was near the center of x. The Inverse and SS estimators could be preferred at the ends of the interval of x.

For extrapolation, Nazsodi was better than the others. Besides this the classical estimator could also be used as β is very large regardless of the other factors.

However when we look at the bias of the estimators, the best performance is related to Nazsodi. These results are given in Table 3. When β is not very large, the Inverse and AS estimators have the highest bias. In design 1, the different methods have a small bias, while β is large. But the bias of Nazsodi is small in Design 2 and Design 3. So the design of X is important in terms of bias. This is also valid for $n > 6$, but the classical estimator also has a small bias at the center of the data, according to the simulation results. Our results are consistent with those obtained by Krutchkoff [8] and Nazsodi [11].

In the end point design and/or case $\beta > 1$, the classical estimator can be preferred on practical grounds.

The effects of factors in the case of an outlier are given in Table 4 and Table 5. The results of Design 2 are not given in these tables because the results are the same as for Design 3.

The MSE's for the least squares and Huber classical estimators were very large, especially for small β, as a result of which β was near zero. Least Squares and the Huber version of the Nazsodi and SS estimators were also unstable for $\beta \leq 0.5$. But the inverse and Ali-Singh estimators were not affected very much by the outlier when x_0 was near to $\hat{x}$. Naturally, in the Inverse method and that of Ali-Sing, which is derived from the Inverse Estimator, the outlier takes place in the dependent variable. However, the MSE of the Huber-inverse estimator was smaller than that of $\widehat{x}_I$ for $\beta \geq 1$. This difference is most clear in Design 2 and Design 3 for $\beta \geq 2$.

For $n \geq 10$ and $\beta \geq 0.5$, the MSE's for both least squares and Huber Nazsodi estimators decreased. But the decrease for Huber Nazsodi was larger than that for the least squares.

For extrapolation, the best estimator was Huber-Nazsodi in all cases.

When the design matrix was changed from 1 to 3, or the values of $\sum(x_i - \overline{x})^2$ in relation to x, the MSE's for all estimators increased. This can be considered as an important result for the classical estimators.

For a large sample size, the differences between the estimators are clear in case of an outlier. But the results for the Huber inverse or the Huber SS estimators are fairly good regarding MSE. On the other hand, if we look at the bias as above, all Huber estimators gave a smaller bias than the original forms unless β is very small. The Nazsodi estimators have the smallest bias for interpolation and extrapolation with no outlier.

Table 1. MSE's for estimators in case of no outliers ($n = 6$).

Design	Beta	Estimator	$X_0 = 0.10$	$X_0 = 0.40$	$X_0 = 0.70$	$X_0 = 1.00$	$X_0 = 3.00$
D1	$\beta = 0.20$	Classic	17.314	18.744	16.9917	13.4792	209.51
		Inverse	**0.1607**	**0.1275**	**0.1269**	**0.1775**	1.5187
		Nazsodi	0.3000	0.2772	0.2764	0.3205	**1.2218**
		AS	0.2242	0.1372	0.1383	0.2805	2.1252
		SS	1.2848	1.3328	1.2173	1.0516	14.1949
	$\beta = 0.50$	Classic	0.0556	0.0498	0.0517	0.0588	0.2744
		Inverse	**0.0447**	**0.0303**	**0.0408**	**0.0472**	0.2133
		Nazsodi	0.0516	0.0469	0.0482	0.0542	**0.2080**
		AS	0.0595	**0.0298**	**0.0399**	0.0671	0.2658
		SS	0.0467	0.0425	0.0432	0.0490	**0.2045**
	$\beta = 1.00$	Classic	0.0128	0.0122	0.0123	0.0141	0.0580
		Inverse	**0.0120**	**0.0115**	**0.0117**	**0.0133**	**0.0529**
		Nazsodi	0.0126	0.0120	0.0121	0.0139	0.0553
		AS	0.0145	0.0124	0.0137	0.0152	0.0576
		SS	**0.0122**	**0.0117**	**0.0118**	**0.0134**	**0.0529**
	$\beta = 2.00$	Classic	0.0033	0.0030	0.0030	0.0033	0.0140
		Inverse	0.0033	0.0029	0.0030	0.0032	0.0140
		Nazsodi	0.0033	0.0029	0.0030	0.0033	0.0139
		AS	0.0035	0.0033	0.0035	0.0034	0.0140
		SS	0.0033	0.0029	0.0030	0.0033	0.0139
D2	$\beta = 0.20$	Classic	86.708	27.978	58.643	156.71	1142.8
		Inverse	**0.1300**	**0.0721**	**0.0796**	**0.1612**	2.7433
		Nazsodi	0.2226	0.1941	0.1874	0.2503	**2.1039**
		AS	0.1865	0.1119	0.1049	0.2703	3.8664
		SS	5.5071	1.8624	3.7702	9.8229	71.099
	$\beta = 0.50$	Classic	0.0798	0.0650	0.0597	0.0929	0.9339
		Inverse	**0.0436**	**0.0366**	**0.0361**	**0.0501**	0.4353
		Nazsodi	0.0581	0.0518	0.0464	0.0626	**0.3889**
		AS	0.0685	**0.0346**	**0.0371**	0.0792	0.7674
		SS	0.0473	0.0420	0.0382	0.0525	**0.3885**
	$\beta = 1.00$	Classic	0.0149	0.0129	0.0128	0.0161	0.1182
		Inverse	**0.0132**	**0.0115**	**0.0113**	**0.0144**	0.1039
		Nazsodi	0.0143	0.0125	0.0123	0.0154	0.1072
		AS	0.0166	**0.0109**	0.0137	0.0172	0.1173
		SS	**0.0134**	**0.0118**	**0.0116**	**0.0146**	**0.1010**
	$\beta = 2.00$	Classic	0.0035	0.0031	0.0031	0.0040	0.0265
		Inverse	0.0034	0.0030	0.0031	0.0039	0.0258
		Nazsodi	0.0035	0.0031	0.0031	0.0039	0.0260
		AS	0.0036	0.0035	0.0037	0.0040	0.0265
		SS	0.0034	0.0030	0.0031	0.0039	0.0256

Table 1 (Continued)

Design	Beta	Estimator	$X_0 = 0.10$	$X_0 = 0.40$	$X_0 = 0.70$	$X_0 = 1.00$	$X_0 = 3.00$
D3	$\beta = 0.20$	Classic	23.5296	38.1301	44.2903	49.4643	654.5195
		Inverse	**0.1361**	**0.0307**	**0.0485**	**0.1969**	4.3676
		Nazsodi	0.1727	0.0864	0.0956	0.2204	**3.5947**
		AS	0.1661	0.0450	0.0631	0.2351	4.1475
		SS	1.5806	2.4351	2.8245	3.2891	43.3392
	$\beta = 0.50$	Classic	47.0640	4.5824	1.2148	90.3120	877.2394
		Inverse	**0.0507**	**0.0238**	**0.0278**	**0.0633**	1.1306
		Nazsodi	0.0617	0.0443	0.0469	0.0655	**0.6734**
		AS	0.0851	**0.0259**	**0.0300**	0.1062	1.7037
		SS	2.9473	0.3146	0.1027	5.6360	54.4469
	$\beta = 1.00$	Classic	0.0217	0.0145	0.0170	0.0290	0.3801
		Inverse	**0.0162**	**0.0104**	**0.0116**	**0.0196**	0.2524
		Nazsodi	0.0185	0.0129	0.0147	0.0229	0.2698
		AS	0.0226	0.0112	0.0167	0.0265	0.3725
		SS	**0.0163**	**0.0112**	**0.0125**	**0.0197**	**0.2325**
	$\beta = 2.00$	Classic	0.0045	0.0031	0.0034	0.0054	0.0640
		Inverse	0.0042	0.0029	0.0031	0.0051	0.0604
		Nazsodi	0.0044	0.0031	0.0033	0.0052	0.0606
		AS	0.0046	0.0035	0.0038	0.0054	0.0639
		SS	0.0042	0.0030	0.0032	0.0051	0.0590

Table 2. MSE's for estimators in case of no outliers (n=20).

Design	Beta	Estimator	$X_0 = 0.10$	$X_0 = 0.40$	$X_0 = 0.70$	$X_0 = 1.00$	$X_0 = 3.00$
D1	$\beta = 0.20$	Classic	0.3362	0.2950	0.3145	0.3233	1.0412
		Inverse	**0.1126**	0.0745	0.0846	**0.1316**	1.4971
		Nazsodi	0.2813	0.2532	0.2612	0.2737	**0.5838**
		AS	0.1400	**0.0579**	**0.0807**	0.1816	0.6551
		SS	**0.1165**	0.0820	0.0912	**0.1328**	1.334
	$\beta = 0.50$	Classic	0.0440	0.0418	0.0446	0.0443	0.0976
		Inverse	**0.0350**	0.0318	0.0346	**0.0360**	0.1518
		Nazsodi	0.0431	0.0411	0.0438	0.0434	**0.0932**
		AS	0.0514	**0.0211**	**0.0328**	0.0545	0.0960
		SS	**0.0353**	0.0323	0.0351	**0.0361**	0.1422
	$\beta = 1.00$	Classic	0.0110	0.0103	0.0107	0.0116	0.0230
		Inverse	**0.0103**	0.0096	**0.0100**	**0.0111**	0.0278
		Nazsodi	0.0109	0.0102	0.0106	0.0116	**0.0228**
		AS	0.0128	**0.0080**	**0.0104**	0.0131	**0.0229**
		SS	**0.0104**	0.0096	**0.0100**	**0.0111**	0.0271
	$\beta = 2.00$	Classic	0.0028	0.0026	0.0025	0.0026	0.0057
		Inverse	0.0027	0.0026	0.0025	0.0026	0.0059
		Nazsodi	0.0028	0.0026	0.0025	0.0026	0.0057
		AS	0.0029	0.0031	0.0029	0.0027	0.0057
		SS	0.0027	0.0026	0.0025	0.0026	0.0059

Table 2 (Continued).

Design	Beta	Estimator	$X_0 = 0.10$	$X_0 = 0.40$	$X_0 = 0.70$	$X_0 = 1.00$	$X_0 = 3.00$
D2	$\beta = 0.20$	Classic	98.0855	14.29	8.031	23.8753	378.9561
		Inverse	**0.1045**	0.0296	**0.0454**	**0.1518**	3.2952
		Nazsodi	0.2324	0.2502	0.2441	0.2435	**0.7313**
		AS	0.1281	**0.0263**	**0.0495**	0.1877	1.5861
		SS	0.4042	0.0809	0.0746	0.2091	4.0402
	$\beta = 0.50$	Classic	0.0475	0.045	0.0446	0.0511	0.2004
		Inverse	**0.0343**	0.0229	**0.0254**	0.0435	0.5293
		Nazsodi	0.0446	0.0428	0.0423	0.0479	**0.1709**
		AS	0.0551	**0.0193**	0.0309	0.0643	0.1872
		SS	**0.0339**	0.0239	**0.026**	**0.0422**	0.4747
	$\beta = 1.00$	Classic	0.0117	0.0108	0.0110	0.0128	0.0462
		Inverse	**0.0107**	0.0090	**0.0091**	**0.0123**	0.0808
		Nazsodi	0.0115	0.0107	0.0108	0.0126	**0.0449**
		AS	0.0138	**0.0080**	0.0126	0.0143	0.0460
		SS	**0.0106**	0.0091	**0.0092**	**0.0122**	0.0758
	$\beta = 2.00$	Classic	0.0029	0.0027	0.0027	0.0031	0.0117
		Inverse	0.0028	0.0026	0.0026	0.003	0.0137
		Nazsodi	0.0029	0.0027	0.0027	0.0031	0.0116
		AS	0.003	0.0031	0.0031	0.0032	0.0117
		SS	0.0028	0.0026	0.0026	0.003	0.0134
D3	$\beta = 0.20$	Classic	29.7166	12.5081	10.0201	70.8879	761.2932
		Inverse	**0.0858**	0.0199	**0.0526**	**0.1880**	3.7417
		Nazsodi	0.2097	0.2181	0.2203	0.2384	**0.8579**
		AS	0.1049	**0.0172**	**0.0617**	0.2227	1.9146
		SS	0.1687	0.0636	0.0809	0.3623	5.1874
	$\beta = 0.50$	Classic	0.0516	0.0468	0.0481	0.0571	0.2665
		Inverse	**0.0326**	0.0205	**0.0253**	0.0532	0.7220
		Nazsodi	0.0474	0.0439	0.0446	0.0521	**0.2181**
		AS	0.0520	**0.0159**	0.0369	0.0699	0.2438
		SS	**0.0322**	0.0216	**0.0257**	**0.0507**	0.6451
	$\beta = 1.00$	Classic	0.0110	0.0106	0.0113	0.0123	0.0549
		Inverse	**0.0097**	0.0085	**0.0094**	0.0131	0.1078
		Nazsodi	0.0108	0.0105	0.0111	**0.0121**	**0.0528**
		AS	0.0132	**0.0057**	0.0137	0.0137	0.0546
		SS	**0.0097**	0.0086	**0.0094**	0.0128	0.1002
	$\beta = 2.00$	Classic	0.0029	0.0026	0.0028	0.0032	0.0131
		Inverse	0.0028	0.0024	0.0026	0.0033	0.0170
		Nazsodi	0.0029	0.0026	0.0027	0.0032	0.0130
		AS	0.0031	0.0021	0.0031	0.0033	0.0131
		SS	0.0028	0.0024	0.0026	0.0033	0.0165

Table 3. The bias values for estimators in case of no outlier ($n = 6$).

Design	Beta	Estimator	$X_0 = 0.10$	$X_0 = 0.40$	$X_0 = 0.70$	$X_0 = 1.00$	$X_0 = 3.00$
D1	$\beta = 0.20$	Classic	0.0464	0.0976	0.1060	0.0469	0.3385
		Inverse	0.1579	0.0234	0.0882	0.1797	1.0076
		Nazsodi	**0.0426**	**0.0064**	**0.0325**	**0.0310**	**0.2968**
		AS	0.1807	0.0327	0.1083	0.2009	0.2542
		SS	0.1300	0.0068	0.0396	0.1230	0.6711
	$\beta = 0.50$	Classic	0.0134	0.0048	0.0126	0.0154	0.0740
		Inverse	0.0297	0.0156	0.0098	0.0382	0.1966
		Nazsodi	**0.0011**	**0.0079**	**0.0062**	**0.0001**	**0.0031**
		AS	0.0699	0.0381	0.0484	0.0669	0.0505
		SS	0.0189	0.0129	**0.0042**	0.0248	0.1290
	$\beta = 1.00$	Classic	0.0040	0.0051	**0.0013**	0.0065	0.0242
		Inverse	0.0066	**0.0023**	0.0065	0.0069	0.0422
		Nazsodi	**0.0013**	0.0044	0.0026	**0.0030**	**0.0071**
		AS	0.0213	0.0211	0.0354	0.0144	0.0199
		SS	0.0040	0.0030	0.0052	0.0035	0.0256
	$\beta = 2.00$	Classic	0.0011	0.0015	**0.0007**	0.0011	0.0020
		Inverse	0.0016	**0.0008**	**0.0006**	0.0022	0.0146
		Nazsodi	0.**0004**	0.0013	**0.0004**	**0.0002**	**0.0012**
		AS	0.0052	0.0149	0.0114	0.0040	0.0010
		SS	0.0009	0.0010	**0.0003**	0.0014	0.0104
D3	$\beta = 0.20$	Classic	**0.0277**	**0.0339**	**0.0330**	**0.2458**	**0.2832**
		Inverse	0.3227	0.0807	0.1602	0.4058	2.0202
		Nazsodi	0.2583	0.0654	0.1252	0.3279	1.6243
		AS	0.3252	0.0841	0.1629	0.4097	1.6062
		SS	0.2351	0.0690	0.1119	0.3658	1.4443
	$\beta = 0.50$	Classic	0.3144	**0.0038**	0.0728	0.3892	1.4882
		Inverse	0.1489	0.0400	0.0676	0.1821	0.9087
		Nazsodi	**0.0418**	0.0148	**0.0138**	**0.0477**	**0.1720**
		AS	0.1324	0.0467	0.0781	0.1353	**0.1890**
		SS	**0.0331**	0.0290	0.0325	**0.0463**	0.3094
	$\beta = 1.00$	Classic	0.0167	0.0034	0.0089	0.0244	0.1176
		Inverse	0.0417	0.0114	0.0214	0.0505	0.2546
		Nazsodi	**0.0009**	**0.0011**	**0.0004**	**0.0016**	**0.0046**
		AS	0.0209	0.0273	0.0361	0.0076	0.1104
		SS	0.0271	0.0077	0.0139	0.0318	0.1615
	$\beta = 2.00$	Classic	0.0028	**0.0004**	0.0033	0.0041	0.0194
		Inverse	0.0115	0.0033	0.0039	0.0137	0.0697
		Nazsodi	**0.0009**	**0.0004**	**0.0014**	**0.0005**	**0.0038**
		AS	0.0041	0.0172	0.0097	0.0014	0.0183
		SS	0.0079	0.0024	0.0021	0.0092	0.0474

Table 3. (Continued) The bias values for estimators in case of no outlier ($n = 20$).

Design	Beta	Estimator	$X_0 = 0.10$	$X_0 = 0.40$	$X_0 = 0.70$	$X_0 = 1.00$	$X_0 = 3.00$
D1	$\beta = 0.20$	Classic	0.0206	**0.0012**	**0.0067**	0.0247	0.1562
		Inverse	0.1896	0.0465	0.1009	0.2369	1.1718
		Nazsodi	**0.0066**	**0.0043**	0.0094	**0.0082**	**0.0162**
		AS	0.2213	0.0584	0.1244	0.2658	0.2999
		SS	0.1779	0.0439	0.0949	0.2224	1.0980
	$\beta = 0.50$	Classic	0.0083	**0.0045**	**0.0083**	0.0134	0.0311
		Inverse	0.0449	0.0170	0.0335	0.0532	0.2976
		Nazsodi	**0.0048**	**0.0053**	**0.0099**	**0.0091**	**0.0099**
		AS	0.0816	0.0404	0.0727	0.0730	0.0089
		SS	0.0420	0.0163	0.0321	0.0495	0.2793
	$\beta = 1.00$	Classic	0.0031	**0.0018**	**0.0025**	**0.0005**	0.0035
		Inverse	0.0110	0.0052	0.0094	0.0180	0.0840
		Nazsodi	**0.0022**	**0.0020**	**0.0029**	0.0016	**0.0016**
		AS	0.0232	0.0305	0.0398	0.0221	0.0009
		SS	0.0103	0.0050	0.0091	0.0170	0.0792
	$\beta = 2.00$	Classic	**0.0003**	**0.0007**	**0.0007**	**0.0003**	0.0021
		Inverse	0.0039	0.0016	0.0012	0.0048	0.0204
		Nazsodi	**0.0005**	**0.0008**	**0.0006**	**0.0005**	**0.0008**
		AS	0.0068	0.0188	0.0118	0.0054	0.0011
		SS	0.0037	0.0015	0.0011	0.0045	0.0192
D3	$\beta = 0.20$	Classic	**0.1905**	0.0384	0.1827	0.5623	1.9508
		Inverse	0.2598	0.0395	0.1819	0.4102	1.9173
		Nazsodi	0.0282	**0.0026**	**0.0271**	**0.0740**	**0.3696**
		AS	0.2826	0.0454	0.1990	0.4315	1.1568
		SS	0.2348	0.0352	0.1616	0.3562	1.7024
	$\beta = 0.50$	Classic	0.0176	0.0064	0.0156	0.0141	0.0875
		Inverse	0.1098	0.0140	0.0739	0.1771	0.8078
		Nazsodi	**0.0064**	**0.0046**	**0.0077**	**0.0027**	**0.0087**
		AS	0.1106	0.0225	0.0930	0.1191	0.0442
		SS	0.1027	0.0128	0.0689	0.1664	0.7580
	$\beta = 1.00$	Classic	0.0047	**0.0027**	0.0046	**0.0002**	0.0138
		Inverse	0.0346	0.0082	0.0226	0.0593	0.2654
		Nazsodi	**0.0022**	**0.0030**	**0.0029**	**0.0036**	**0.0037**
		AS	0.0292	0.0212	0.0363	0.0235	0.0087
		SS	0.0324	0.0079	0.0211	0.0560	0.2498
	$\beta = 2.00$	Classic	**0.0003**	**0.0022**	0.0010	**0.0006**	0.0028
		Inverse	0.0106	0.0037	0.0062	0.0152	0.0708
		Nazsodi	0.0009	**0.0023**	**0.0006**	**0.0003**	**0.0014**
		AS	0.0078	0.0167	0.0096	0.0042	0.0018
		SS	0.0100	0.0036	0.0058	0.0143	0.0667

Table 4. MSE's for estimators in case of outliers ($n = 6$). Design of X: 1 (The End-Point Design)

Beta	Estimator	$X_0 = 0.10$	$X_0 = 0.40$	$X_0 = 0.70$	$X_0 = 1.00$	$X_0 = 3.00$
$\beta = 0.20$	Classic	1.7911	0.8597	0.2689	**0.0673**	8.2870
	Inverse	0.2467	0.0341	0.0181	0.1997	6.4487
	Nazsodi	0.4755	0.1254	**0.0059**	0.1211	6.8524
	AS	**0.1629**	**0.0105**	0.0394	0.2491	6.2521
	SS	0.4947	0.1348	0.0066	0.1165	6.8793
	H-Classic	363.7498	122.7539	194.3761	101.0208	1685.3431
	H-Inverse	0.2537	0.0377	**0.0184**	**0.1970**	6.4309
	H-Nazsodi	0.3299	0.2147	0.2250	0.3528	**3.7379**
	H-AS	**0.1585**	**0.0114**	0.0413	0.2494	6.1911
	H-SS	22.5933	7.6363	12.1770	6.3932	106.2586
$\beta = 0.50$	Classic	248.2793	154.0131	32.4787	32.6348	886.1075
	Inverse	0.2031	0.0188	0.0322	0.2425	6.6892
	Nazsodi	0.3507	0.0588	**0.0174**	0.2240	7.9721
	AS	0.1605	**0.0101**	0.0400	0.2500	6.2652
	SS	16.3193	9.7773	1.9522	2.1819	67.6085
	H-Classic	0.8347	0.5075	0.2223	**0.1323**	5.3961
	H-Inverse	0.1935	0.0183	0.0304	0.2299	6.3524
	H-Nazsodi	0.1836	0.1510	0.1253	**0.1310**	**0.3956**
	H-AS	0.1454	**0.0097**	0.0388	0.2380	4.7562
	H-SS	**0.1032**	0.0300	0.0345	**0.1295**	3.0581
$\beta = 1.00$	Classic	4.2283	1.8509	0.5245	0.1591	23.7741
	Inverse	0.0752	**0.0012**	0.0505	0.2230	4.5200
	Nazsodi	**0.0089**	0.0284	0.0826	0.1677	1.5523
	AS	0.1545	0.0097	0.0401	0.2499	5.4921
	SS	0.1086	0.1053	0.1156	0.1320	0.4999
	H-Classic	0.0367	0.0233	0.0172	**0.0169**	0.1879
	H-Inverse	0.0220	0.0067	**0.0149**	0.0472	0.8685
	H-Nazsodi	0.1204	0.0987	0.0877	0.0758	**0.0897**
	H-AS	0.0335	0.0088	0.0240	0.0566	0.1211
	H-SS	0.0135	0.0084	**0.0140**	**0.0303**	0.4099
$\beta = 2.00$	Classic	0.2094	0.0971	0.0289	0.0078	1.0703
	Inverse	**0.0025**	0.0116	0.0325	0.0646	0.5809
	Nazsodi	0.0815	0.0526	0.0298	0.0144	0.1073
	AS	0.0208	0.0039	0.0398	0.1825	0.4176
	SS	0.0194	0.0250	0.0312	0.0378	0.1092
	H-Classic	0.0067	0.0047	**0.0036**	**0.0039**	0.0333
	H-Inverse	0.0032	**0.0030**	**0.0034**	**0.0044**	0.0267
	H-Nazsodi	0.0792	0.0722	0.0649	0.0578	**0.0266**
	H-AS	0.0047	0.0036	0.0078	0.0053	0.0314
	H-SS	0.0035	**0.0033**	**0.0034**	**0.0040**	**0.0271**

Table 4. (Continued) MSE's for estimators in case of outliers ($n = 6$). Design of X: 3 (The Unconstrained Design)

Beta	Estimator	$X_0 = 0.10$	$X_0 = 0.40$	$X_0 = 0.70$	$X_0 = 1.00$	$X_0 = 3.00$
$\beta = 0.20$	Classic	1086.1534	797.3395	582.1076	471.6873	253.7337
	Inverse	0.1645	**0.0110**	**0.0385**	**0.2471**	6.2630
	Nazsodi	0.1781	0.0144	**0.0346**	**0.2390**	6.3004
	AS	**0.1600**	**0.0100**	0.0400	**0.2500**	6.2500
	SS	68.1789	49.8592	36.3715	29.6170	22.1928
	H-Classic	94.3619	35.0822	66.6361	79.4451	1235.0048
	H-Inverse	0.1623	**0.0110**	**0.0386**	**0.2463**	6.2158
	H-Nazsodi	0.**1510**	0.0438	0.0813	**0.2470**	**4.8495**
	H-AS	**0.1588**	**0.0108**	**0.0398**	**0.2481**	6.1196
	H-SS	5.9599	2.1962	4.1885	5.0871	80.3935
$\beta = 0.50$	Classic	1565.4605	734.5616	240.5462	76.7243	5805.9292
	Inverse	0.1527	0.0088	0.0415	0.2511	6.1633
	Nazsodi	0.1341	**0.0064**	0.0463	0.2542	5.9233
	AS	0.1600	0.0100	0.0400	0.2500	6.2483
	SS	97.5658	45.8435	15.1763	5.0508	363.3927
	H-Classic	5.0306	2.4441	0.7861	2.4448	85.9360
	H-Inverse	0.1387	0.0087	**0.0383**	0.2281	5.5652
	H-Nazsodi	**0.0586**	0.0366	0.0509	**0.1037**	**1.4558**
	H-AS	0.1430	0.0101	**0.0385**	0.2285	4.8247
	H-SS	0.3675	0.1564	0.0752	0.2766	8.0523
$\beta = 1.00$	Classic	4.9312	3.6183	0.4315	0.3956	61.7473
	Inverse	0.1156	0.0046	0.0423	0.2288	5.2688
	Nazsodi	0.0431	**0.0015**	0.0484	0.1847	3.3552
	AS	0.1565	0.0098	0.0400	0.2497	5.5566
	SS	0.3037	0.2193	0.0724	0.1805	6.2024
	H-Classic	0.0399	0.0210	**0.0172**	**0.0285**	0.4995
	H-Inverse	0.0711	0.0045	0.0267	0.1379	3.1403
	H-Nazsodi	**0.0223**	0.0173	0.0207	0.0323	**0.2981**
	H-AS	0.0703	0.0081	0.0305	0.1253	0.8321
	H-SS	0.0350	0.0041	0.0191	0.0802	1.6449
$\beta = 2.00$	Classic	0.0299	0.0175	0.0101	0.0081	0.1257
	Inverse	0.0222	**0.0008**	0.0217	0.0843	1.5815
	Nazsodi	**0.0037**	0.0067	0.0136	0.0240	0.1918
	AS	0.0276	0.0056	0.0372	0.1142	0.0860
	SS	0.0068	0.0022	0.0177	0.0529	0.8022
	H-Classic	0.0064	0.0042	**0.0039**	**0.0060**	0.0704
	H-Inverse	0.0093	0.0017	0.0087	0.0301	0.5457
	H-Nazsodi	0.0113	0.0102	0.0106	0.0125	**0.0660**
	H-AS	0.0055	0.0034	0.0115	0.0107	0.0678
	H-SS	0.0056	0.0021	0.0065	0.0187	0.3018

Table 5. MSE's for estimators in case of outliers ($n = 20$).
Design of X: 1 (The End-Point Design)

Beta	Estimator	$X_0 = 0.10$	$X_0 = 0.40$	$X_0 = 0.70$	$X_0 = 1.00$	$X_0 = 3.00$
$\beta = 0.20$	Classic	1783.5839	1249.2645	609.3808	420.9247	6092.2458
	Inverse	0.1598	**0.0101**	**0.0401**	0.2502	6.2486
	Nazsodi	0.1918	0.0307	0.0530	0.2607	6.3614
	AS	0.1600	**0.0100**	**0.0400**	0.2500	6.2500
	SS	5.5365	3.8349	1.9343	1.5813	23.1651
	H-Classic	0.4843	0.4251	0.4330	0.4266	1.3919
	H-Inverse	**0.1161**	0.0340	0.0449	**0.1469**	3.2361
	H-Nazsodi	0.3484	0.3295	0.3321	0.3195	**0.6741**
	H-AS	0.1384	0.0289	0.0476	**0.1967**	1.7852
	H-SS	**0.1101**	0.0402	0.0503	0.1382	2.8111
$\beta = 0.50$	Classic	0.5203	0.3101	0.1650	0.1362	2.5138
	Inverse	0.0887	**0.0035**	0.0410	0.2008	4.3895
	Nazsodi	0.1302	0.1168	0.0934	0.0792	0.6834
	AS	0.1549	0.0097	0.0400	0.2485	5.0217
	SS	0.0630	**0.0033**	0.0423	0.1798	3.6094
	H-Classic	0.0501	0.0491	0.0492	0.0505	0.1200
	H-Inverse	**0.0350**	0.0257	**0.0284**	**0.0403**	0.3982
	H-Nazsodi	0.0545	0.0543	0.0518	0.0517	**0.1041**
	H-AS	0.0573	0.0197	0.0312	0.0655	0.1098
	H-SS	**0.0348**	0.0267	**0.0293**	**0.0397**	0.3548
$\beta = 1.00$	Classic	0.0677	0.0381	0.0218	0.0168	0.2942
	Inverse	0.0166	**0.0036**	0.0231	0.0753	1.2720
	Nazsodi	0.0496	0.0318	0.0209	0.0152	0.1328
	AS	0.0757	0.0074	0.0380	0.1790	0.0352
	SS	0.0131	**0.0043**	0.0225	0.0678	1.0773
	H-Classic	0.0118	0.0109	0.0115	0.0112	0.0270
	H-Inverse	**0.0101**	0.0089	**0.0097**	**0.0106**	0.0592
	H-Nazsodi	0.0170	0.0155	0.0151	0.0143	**0.0260**
	H-AS	0.0138	0.0079	0.0137	0.0131	0.0265
	H-SS	**0.0101**	0.0090	**0.0098**	**0.0105**	0.0542
$\beta = 2.00$	Classic	0.0138	0.0078	0.0043	0.0032	0.0572
	Inverse	**0.0023**	**0.0030**	0.0063	0.0108	0.0889
	Nazsodi	0.0125	0.0074	0.0044	0.0032	0.0426
	AS	0.0057	0.0042	0.0257	0.0233	0.0426
	SS	**0.0024**	**0.0030**	0.0062	0.0099	0.0732
	H-Classic	0.0029	0.0027	**0.0027**	**0.0027**	0.0065
	H-Inverse	**0.0025**	**0.0025**	**0.0026**	**0.0026**	0.0076
	H-Nazsodi	0.0075	0.0068	0.0064	0.0059	**0.0056**
	H-AS	0.0029	0.0030	0.0034	**0.0028**	0.0064
	H-SS	**0.0025**	**0.0025**	**0.0026**	**0.0026**	0.0073

Table 5 (Continued). MSE's for estimators in case of outliers ($n = 20$). Design of X: 3 (The Unconstrained Design)

Beta	Estimator	$X_0 = 0.10$	$X_0 = 0.40$	$X_0 = 0.70$	$X_0 = 1.00$	$X_0 = 3.00$
$\beta = 0.20$	Classic	8.4641	3.4128	1.6321	2.5342	65.0250
	Inverse	0.1352	**0.0030**	**0.0537**	0.2994	6.6521
	Nazsodi	0.2972	0.0414	**0.0529**	0.3271	8.9225
	AS	0.1254	**0.0029**	0.0605	0.2981	6.4862
	SS	0.2016	0.0194	**0.0535**	0.3110	7.4949
	H-Classic	88.9677	30.2990	32.9407	88.1869	757.5042
	H-Inverse	**0.1146**	0.0067	**0.0530**	**0.2498**	5.4728
	H-Nazsodi	0.2302	0.2090	0.2221	0.2700	**1.2014**
	H-AS	**0.1195**	0.0046	**0.0534**	**0.2548**	4.6214
	H-SS	0.3621	0.1018	0.1481	0.4569	6.7752
$\beta = 0.50$	Classic	574.9850	169.0646	41.4731	222.3043	8564.4143
	Inverse	0.1163	**0.0024**	0.0600	0.2892	6.2000
	Nazsodi	0.0472	0.0121	0.0598	0.1912	3.1751
	AS	0.1253	**0.0029**	0.0605	0.2981	6.4561
	SS	1.7933	0.5193	0.1862	0.8771	29.5538
	H-Classic	0.0725	0.0591	0.0594	0.0830	0.5967
	H-Inverse	**0.0443**	0.0116	**0.0272**	0.0917	1.7547
	H-Nazsodi	0.0598	0.0541	0.0526	**0.0657**	**0.3004**
	H-AS	0.0637	0.0092	0.0395	0.1016	**0.3106**
	H-SS	**0.0408**	0.0130	**0.0267**	0.0831	1.5252
$\beta = 1.00$	Classic	0.1666	0.0612	0.0291	0.0743	2.3426
	Inverse	0.0607	**0.0010**	0.0440	0.1891	3.7694
	Nazsodi	0.0527	0.0314	0.0193	0.0177	0.2774
	AS	0.1109	0.0027	0.0596	0.2742	0.7184
	SS	0.0458	**0.0012**	0.0397	0.1607	3.0780
	H-Classic	0.0138	0.0120	0.0122	0.0148	0.0876
	H-Inverse	**0.0127**	0.0069	**0.0098**	0.0212	0.3095
	H-Nazsodi	0.0152	0.0133	0.0128	**0.0145**	**0.0748**
	H-AS	0.0166	0.0052	0.0163	0.0164	0.0846
	H-SS	**0.0122**	0.0071	**0.0098**	0.0198	0.2737
$\beta = 2.00$	Classic	0.0194	0.0075	0.0039	0.0086	0.2496
	Inverse	0.0060	**0.0016**	0.0109	0.0341	0.5354
	Nazsodi	0.0152	0.0067	0.0037	0.0059	0.1555
	AS	0.0093	0.0018	0.0307	0.0184	0.2040
	SS	0.0049	**0.0017**	0.0100	0.0295	0.4428
	H-Classic	0.0032	0.0026	0.0028	**0.0034**	0.0177
	H-Inverse	**0.0030**	0.0022	**0.0027**	**0.0036**	0.0342
	H-Nazsodi	0.0049	0.0041	0.0039	**0.0037**	**0.0155**
	H-AS	0.0032	0.0020	0.0033	**0.0036**	0.0175
	H-SS	**0.0029**	0.0022	**0.0027**	**0.0039**	0.0311

Table 6. The bias values for estimators in case of outlier ($n = 6$).
Design of X: 1 (The End-Point Design)

Beta	Estimator	$X_0 = 0.10$	$X_0 = 0.40$	$X_0 = 0.70$	$X_0 = 1.00$	$X_0 = 3.00$
$\beta = 0.20$	Classic	1.3051	0.8826	0.4454	**0.0113**	2.8692
	Inverse	0.4957	0.1825	0.1317	0.4461	2.5393
	Nazsodi	0.6847	0.3454	**0.0031**	0.3395	2.6167
	AS	0.4035	0.1024	0.1985	0.4991	2.5004
	SS	0.6981	0.3575	0.0126	0.3317	2.6218
	H-Classic	1.4372	0.6941	0.8729	0.1163	2.8071
	H-Inverse	0.5008	0.1874	0.1269	0.4414	2.5357
	H-Nazsodi	0.0491	0.1126	0.2534	0.4166	**1.1309**
	H-AS	0.3901	0.0967	0.2013	0.4989	2.4868
	H-SS	**0.0163**	**0.0329**	0.3134	0.3020	1.2000
$\beta = 0.50$	Classic	4.1257	2.8583	1.4295	**0.0185**	8.7557
	Inverse	0.4499	0.1356	0.1789	0.4923	2.5860
	Nazsodi	0.5857	0.2326	0.1215	0.4715	2.8200
	AS	0.4006	0.1003	0.1999	0.5000	2.5030
	SS	1.3689	0.8163	0.2232	0.3739	4.1285
	H-Classic	0.3084	0.2049	**0.0639**	**0.0186**	**0.1059**
	H-Inverse	0.4377	0.1325	0.1736	0.4785	2.5150
	H-Nazsodi	0.3282	0.2967	0.2625	0.2519	**0.0933**
	H-AS	0.3769	0.0943	0.1958	0.4844	2.0996
	H-SS	**0.2461**	**0.0481**	0.1461	0.3492	1.6713
$\beta = 1.00$	Classic	1.9165	1.2514	0.6234	0.0299	4.3766
	Inverse	0.2724	0.0254	0.2239	0.4719	2.1251
	Nazsodi	0.0293	0.1508	0.2801	0.4049	1.2367
	AS	0.3930	0.0984	0.2001	0.4999	2.3424
	SS	0.2748	0.2938	0.3238	0.3464	0.4997
	H-Classic	0.0192	0.0649	**0.0349**	**0.0044**	**0.1019**
	H-Inverse	**0.0109**	**0.0167**	0.0851	0.1829	0.8328
	H-Nazsodi	0.3183	0.2878	0.2709	0.2455	**0.0781**
	H-AS	0.0771	0.0455	0.1245	0.1621	0.1569
	H-SS	0.0559	0.0037	0.0725	0.1361	0.5591
$\beta = 2.00$	Classic	0.4487	0.3008	0.1505	0.0030	1.0116
	Inverse	0.0213	0.0992	0.1760	0.2510	0.7594
	Nazsodi	0.2767	0.2197	0.1609	0.0994	0.2982
	AS	0.1264	0.0580	0.1994	0.4262	0.6179
	SS	0.1281	0.1496	0.1696	0.1875	0.3167
	H-Classic	0.0062	0.0317	**0.0157**	**0.0027**	**0.0144**
	H-Inverse	**0.0009**	**0.0122**	0.0238	0.0333	0.1062
	H-Nazsodi	0.2742	0.2623	0.2488	0.2330	**0.0185**
	H-AS	0.0150	0.0178	0.0575	0.0235	0.1086
	H-SS	0.0109	0.0171	0.0218	0.0243	0.0511

Table 6 (Continued). The bias values for estimators in case of outlier ($n = 6$). Design of X: 3 (The Unconstrained Design)

Beta	Estimator	$X_0 = 0.10$	$X_0 = 0.40$	$X_0 = 0.70$	$X_0 = 1.00$	$X_0 = 3.00$
$\beta = 0.20$	Classic	1.3009	0.3981	0.4620	0.0206	2.9948
	Inverse	0.4055	0.1046	0.1961	0.4970	2.5026
	Nazsodi	0.4212	0.1176	0.1850	0.4886	2.5100
	AS	0.4000	0.1000	0.2000	0.5000	2.5000
	SS	0.6293	0.1779	0.0316	0.3676	2.6256
	H-Classic	**0.1981**	0.0838	**0.0249**	0.0291	**0.3032**
	H-Inverse	0.4024	0.1032	0.1959	0.4960	2.4924
	H-Nazsodi	0.2884	**0.0506**	0.1945	0.4339	2.0691
	H-AS	0.3971	0.0999	0.1978	0.4974	2.4671
	H-SS	0.2523	0.0565	0.1531	0.3498	1.7935
$\beta = 0.50$	Classic	2.4677	2.0415	1.5527	0.6059	3.1619
	Inverse	0.3907	0.0935	0.2038	0.5011	2.4825
	Nazsodi	0.3642	0.0751	0.2144	0.5041	2.4328
	AS	0.3999	0.1000	0.2000	0.5000	2.4997
	SS	0.3239	0.4402	0.5410	0.5273	1.0714
	H-Classic	**0.0765**	0.0732	**0.0473**	0.0268	**0.4626**
	H-Inverse	0.3670	0.0877	0.1919	0.4720	2.3373
	H-Nazsodi	**0.0517**	**0.0394**	0.1350	0.2205	0.8094
	H-AS	0.3622	0.0926	0.1905	0.4613	2.0311
	H-SS	0.2311	0.0475	0.1558	0.3473	1.6373
$\beta = 1.00$	Classic	0.5468	0.4007	0.2735	0.1486	0.7154
	Inverse	0.3395	0.0667	0.2054	0.4782	2.2949
	Nazsodi	0.2021	0.0889	0.2177	0.4285	1.8286
	AS	0.3956	0.0991	0.2000	0.4997	2.3558
	SS	0.1179	0.0502	0.2225	0.3958	1.5424
	H-Classic	**0.0552**	0.0574	**0.0255**	0.0011	**0.1990**
	H-Inverse	0.2491	0.0490	0.1514	0.3529	1.6908
	H-Nazsodi	**0.0584**	0.0777	0.0933	0.1143	0.2341
	H-AS	0.2006	0.0677	0.1566	0.2941	0.4316
	H-SS	0.1656	**0.0224**	0.1199	0.2644	1.2183
$\beta = 2.00$	Classic	0.1520	0.1155	0.0789	0.0407	0.2023
	Inverse	0.1454	0.0004	0.1447	0.2891	1.2564
	Nazsodi	0.0259	0.0664	0.1070	0.1463	0.4172
	AS	0.1523	0.0715	0.1928	0.3347	0.0630
	SS	0.0710	0.0286	0.1283	0.2270	0.8917
	H-Classic	**0.0341**	0.0244	**0.0145**	0.0032	**0.0513**
	H-Inverse	0.0782	**0.0007**	0.0770	0.1535	0.6708
	H-Nazsodi	0.0829	0.0836	0.0842	0.0833	0.0882
	H-AS	**0.0363**	0.0239	0.0766	0.0480	0.0514
	H-SS	0.0501	0.0055	0.0614	0.1159	0.4877

Table 7. The bias values for estimators in case of outlier ($n = 20$). Design of X: 1 (The End-Point Design)

Beta	Estimator	$X_0 = 0.10$	$X_0 = 0.40$	$X_0 = 0.70$	$X_0 = 1.00$	$X_0 = 3.00$
$\beta = 0.20$	Classic	2.7826	3.1660	0.7520	1.0985	4.7063
	Inverse	0.3996	0.0999	0.2002	0.5001	2.4996
	Nazsodi	0.3937	0.0983	0.2035	0.5026	2.4939
	AS	0.4000	0.1000	0.2000	0.5000	2.5000
	SS	0.2228	0.0815	0.2309	0.5334	2.0993
	H-Classic	0.1173	**0.0745**	**0.0067**	0.0521	0.3810
	H-Inverse	0.2922	**0.0791**	0.1343	0.3446	1.7685
	H-Nazsodi	**0.1078**	0.1016	0.0727	**0.0480**	**0.0458**
	H-AS	0.3127	0.0817	0.1656	0.3922	1.0664
	H-SS	0.2694	**0.0705**	0.1272	0.3225	1.6491
$\beta = 0.50$	Classic	0.6095	0.4212	0.1867	0.0210	1.4219
	Inverse	0.2953	0.0458	0.1990	0.4467	2.0945
	Nazsodi	0.2722	0.2475	0.1921	0.1547	0.1096
	AS	0.3935	0.0984	0.2000	0.4985	2.2390
	SS	0.2451	0.0199	0.1983	0.4207	1.8991
	H-Classic	**0.0311**	0.0309	**0.0022**	**0.0102**	0.1036
	H-Inverse	0.0956	**0.0148**	0.0448	0.1174	0.5880
	H-Nazsodi	0.0827	0.0860	0.0611	0.0524	0.0168
	H-AS	0.1090	0.0345	0.0854	0.1212	0.0605
	H-SS	0.0885	**0.0123**	0.0425	0.1103	0.5496
$\beta = 1.00$	Classic	0.2237	0.1477	0.0678	0.0061	0.5003
	Inverse	0.1142	0.0135	0.1397	0.2685	1.1263
	Nazsodi	0.1851	0.1324	0.0760	0.0253	0.3144
	AS	0.2663	0.0841	0.1947	0.4203	0.0539
	SS	0.0954	0.0210	0.1357	0.2532	1.0359
	H-Classic	**0.0160**	0.0094	**0.0002**	**0.0052**	0.0387
	H-Inverse	0.0268	**0.0042**	0.0155	0.0397	0.1993
	H-Nazsodi	0.0746	0.0688	0.0601	0.0560	**0.0281**
	H-AS	0.0229	0.0308	0.0485	0.0268	0.0320
	H-SS	0.0244	**0.0035**	0.0147	0.0372	0.1861
$\beta = 2.00$	Classic	0.1023	0.0679	0.0337	0.0114	0.2229
	Inverse	0.0074	0.0363	0.0651	0.0934	0.2916
	Nazsodi	0.0958	0.0658	0.0358	0.0050	0.1881
	AS	0.0384	0.0559	0.1570	0.1394	0.1872
	SS	0.0127	0.0380	0.0634	0.0881	0.2630
	H-Classic	**0.0005**	0.0068	**0.0030**	**0.0014**	**0.0197**
	H-Inverse	**0.0009**	**0.0033**	0.0075	0.0111	0.0455
	H-Nazsodi	0.0683	0.0648	0.0613	0.0570	**0.0200**
	H-AS	0.0020	0.0156	0.0195	0.0055	**0.0184**
	H-SS	**0.0003**	**0.0035**	0.0073	0.0104	0.0419

Table 7 (Continued). The bias values for estimators in case of outlier ($n = 20$). Design of X: 3 (The Unconstrained Design)

Beta	Estimator	$X_0 = 0.10$	$X_0 = 0.40$	$X_0 = 0.70$	$X_0 = 1.00$	$X_0 = 3.00$
$\beta = 0.20$	Classic	1.4216	0.7547	0.0514	0.6218	5.0309
	Inverse	0.3675	0.0626	0.2422	0.5471	2.5791
	Nazsodi	0.5317	0.1667	0.1957	0.5601	2.9826
	AS	0.3541	0.0540	0.2460	0.5460	2.5468
	SS	0.4261	0.1010	0.2259	0.5512	2.7153
	H-Classic	0.4211	0.0328	0.1166	0.6830	1.6468
	H-Inverse	0.3321	0.0559	0.2215	0.4945	2.3288
	H-Nazsodi	**0.0206**	**0.0258**	**0.0839**	**0.1381**	**0.4988**
	H-AS	0.3410	0.0520	0.2364	0.5181	2.0625
	H-SS	0.2903	0.0510	0.2027	0.4290	2.1080
$\beta = 0.50$	Classic	2.3799	1.0212	0.1643	1.1936	9.1974
	Inverse	0.3408	0.0480	0.2449	0.5377	2.4897
	Nazsodi	0.1568	0.0358	0.2291	0.4209	1.7017
	AS	0.3539	0.0540	0.2460	0.5460	2.5409
	SS	0.1897	**0.0114**	0.2404	0.4415	1.8404
	H-Classic	**0.0657**	0.0319	**0.0005**	0.0463	0.2954
	H-Inverse	0.1803	**0.0262**	0.1285	0.2797	1.2986
	H-Nazsodi	0.0678	0.0513	0.0349	**0.0068**	**0.1303**
	H-AS	0.1765	0.0318	0.1521	0.2191	**0.1098**
	H-SS	0.1666	**0.0230**	0.1213	0.2616	1.2101
$\beta = 1.00$	Classic	0.3599	0.1796	0.0025	0.1888	1.4213
	Inverse	0.2449	0.0180	0.2082	0.4341	1.9409
	Nazsodi	0.1872	0.1235	0.0580	0.1105	0.4599
	AS	0.3327	0.0523	0.2442	0.5235	0.8216
	SS	0.2113	**0.0091**	0.1965	0.3995	1.7541
	H-Classic	**0.0220**	0.0108	**0.0017**	**0.0167**	0.1115
	H-Inverse	0.0737	**0.0094**	0.0535	0.1144	0.5260
	H-Nazsodi	0.0475	0.0393	0.0298	**0.0170**	**0.0569**
	H-AS	0.0370	0.0208	0.0610	0.0254	0.1017
	H-SS	0.0684	**0.0083**	0.0504	0.1071	0.4906
$\beta = 2.00$	Classic	0.1224	0.0627	0.0004	0.0616	0.4742
	Inverse	0.0674	0.0163	0.0978	0.1807	0.7296
	Nazsodi	0.1055	0.0585	0.0084	0.0399	0.3667
	AS	0.0674	0.0386	0.1702	0.1156	0.4245
	SS	0.0568	0.0188	0.0924	0.1672	0.6628
	H-Classic	**0.0014**	0.0062	**0.0007**	**0.0020**	0.0457
	H-Inverse	0.0184	**0.0005**	0.0168	0.0347	0.1498
	H-Nazsodi	0.0426	0.0391	0.0329	0.0283	**0.0068**
	H-AS	**0.0005**	0.0147	0.0147	**0.0012**	0.0442
	H-SS	0.0168	**0.0008**	0.0158	0.0324	0.1389

5. Conclusion

In this study, the calibration estimators are compared by Monte-Carlo simulation with and without the presence of outliers. Two types of estimators are taken as original; least square and robust. Five types of calibration predictors $\widehat{x}_0$ are taken, namely the Classical, Inverse, Nazsodi, Ali and Singh and Srivastava and Singh estimators. In simulation, similar results to other authors are obtained in case of no outlier. The Huber versions of the estimators for linear calibration problems with outliers yielded the smallest MSE's. But these results depend on two factors, namely the β coefficient and the sample size n. For the design of X, the MSE's of all estimators monotonically increase from the end-point design to the mixed design. While the inverse estimators may be preferred with or without outliers in the case of interpolation, especially when x_0 is near $\overline{x}$, in terms of the MSE, the Nazsodi estimators may be used for extrapolation and also for a small bias choice in both interpolation and extrapolation for large β and designs with a small sample size.

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