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SOME APPLICATIONS OF FRACTIONAL CALCULUS OPERATORS TO THE ANALYTIC PART OF HARMONIC UNIVALENT FUNCTIONS

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Abstract

Recently, Jahangiri [4] studied the harmonic starlike functions of order α, and he defined the class $\mathcal{T}_{\mathcal{H}}(\alpha)$ consisting of functions $f = h + \bar{g}$, where h and g are the analytic and the co-analytic part of the function f, respectively. In [3] the author introduced the class $\mathcal{T}_{\mathcal{H}}(\alpha, \beta)$ of analytic functions and he proved various coefficient inequalities and growth and distortion theorems, and obtained the radius of convexity for the function h if the function f belongs to the classes $\mathcal{T}_{\mathcal{H}}(\alpha)$ and $\mathcal{T}_{\mathcal{H}}(\alpha, \beta)$. In this paper, we derive various distortion theorems for the fractional calculus and the fractional integral operator of the function h, the analytic part of the function f, if the function f belongs to the class $\mathcal{T}_{\mathcal{H}}(\alpha, \beta)$.

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1. Introduction and Definitions

A continuous complex valued function $f = u + iv$ defined in a simply connected complex domain $\mathcal{D}$ is said to be harmonic in $\mathcal{D}$ if both u and v are real harmonic in $\mathcal{D}$. In any simply connected domain we can write $f = h + \bar{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$.

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Let $\mathcal{H}$ denote the family of functions $f = h + \bar{g}$ that are harmonic, univalent and sense preserving in the unit disk $\mathcal{U} = \{z : |z| < 1\}$, and for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in \mathcal{H}$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1.1}$$

The harmonic function $f = h + \bar{g}$ for $g \equiv 0$ reduces to an analytic function $f = h$.

In 1984 Clunie and Sheil-Small [1] investigated the class $\mathcal{H}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several papers related to $\mathcal{H}$ and its subclasses. Recently, Jahangiri *et al.* [5], Jahangiri [4], Silverman [11], Silverman and Silvia [12] studied harmonic starlike functions. Jahangiri [4] defined the class $\mathcal{T}_{\mathcal{H}}(\alpha)$ consisting of functions $f = h + \bar{g}$ such that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \tag{1.2}$$

which satisfy the condition

$$\frac{\partial}{\partial \theta}(\arg f(re^{i\theta})) \geq \alpha, \ 0 \leq \alpha < 1, \ |z| = r < 1. \tag{1.3}$$

Also Jahangiri [4] proved that if $f = h + \bar{g}$ is given by (1.1) and if

$$\sum_{n=1}^{\infty} \left(\frac{n-\alpha}{1-\alpha} |a_n| + \frac{n+\alpha}{1-\alpha} |b_n| \right) \leq 2, \ 0 \leq \alpha < 1, \ a_1 = 1, \tag{1.4}$$

then f is harmonic, univalent, and starlike of order α in $\mathcal{U}$. This condition is proved to be also necessary if $f \in \mathcal{T}_{\mathcal{H}}(\alpha)$. The case when $\alpha = 0$ is given in [12], and for $\alpha = b_1 = 0$, see [11].

A function $f = h + \bar{g} \in \mathcal{T}_{\mathcal{H}}(\alpha)$ is said to be in the class $\mathcal{T}_{\mathcal{H}}(\alpha, \beta)$ if the analytic functions h and g satisfies the condition

$$\operatorname{Re}\left\{ \alpha z h''(z) + \frac{g(z)}{z} \right\} > 1 - |\beta|, \ (\beta \in \mathbb{C}, \ \alpha \geq 0, \ z \in \mathcal{U}). \tag{1.5}$$

The class $\mathcal{T}_{\mathcal{H}}(\alpha, \beta)$ was introduced and studied by Frasin [3].

In the present paper and for $f = h + \bar{g} \in \mathcal{T}_{\mathcal{H}}(\alpha, \beta)$, we give various distortion theorems for the fractional calculus and the fractional integral operator of the function h, the analytic part of the function f .

In order to show our results, we shall need the following lemma.

1.1. Lemma. [3] *Let the function $f = h + \bar{g}$ be so that h and g are given by* (1.2)*. If $f \in \mathcal{T}_{\mathcal{H}}(\alpha, \beta)$, then*

$$\sum_{n=2}^{\infty} \left[\alpha n(n-1) |a_n| - \frac{1-3\alpha}{n+\alpha} \right] \leq |\beta|, \tag{1.6}$$

where $a_1 = b_1 = 1$, $0 \leq \alpha \leq 1/3$ and $\beta \in \mathbb{C}$. The result is sharp.

2. Fractional Calculus

Many essentially equivalent definitions of the fractional calculus (that is, of fractional derivatives and fractional integrals) have been given in the literature (cf., e.g., [2, Chap. 13], [6], [8], [9], [10], [13, p.28 *et.seq.*] and [14]). We find it to be convenient to recall here the following definitions which were used earlier by Owa [7] (and, subsequently, by Srivastava and Owa [15]).

2.1. Definition. The fractional integral of order μ is defined, for a function $h(z)$, by

$$D_z^{-\mu} h(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{h(\zeta)}{(z-\zeta)^{1-\mu}} d\zeta, \tag{2.1}$$

where $\mu > 0$, $h(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{1-\mu}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

2.2. Definition. The fractional derivative of order μ is defined, for a function $f(z)$, by

$$D_z^{\mu} h(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{h(\zeta)}{(z-\zeta)^{\mu}} d\zeta, \tag{2.2}$$

where $0 \le \mu < 1$, $h(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{-\mu}$ is removed as in Definition 2.1 above.

2.3. Definition. Under the hypotheses of Definition 2.2, the fractional derivative of order $j+\mu$ is defined by

$$D_z^{j+\mu} h(z) = \frac{d^j}{dz^j} D_z^{\mu} h(z), \tag{2.3}$$

where $0 \le \mu < 1$ and $j \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

We begin by proving

2.4. Theorem. *Let the function $f = h + \bar{g}$ be so that h and g are given by (1.2). If $f \in \mathfrak{T}_{\mathcal{H}}(\alpha, \beta)$, then*

$$\left| D_z^{-\mu} h(z) \right| \ge \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left\{ 1 - \frac{(2+\alpha)\,|\beta| + 1 - 3\alpha}{(2\alpha+\alpha^2)(2+\mu)} \, |z| \right\} \tag{2.4}$$

and

$$\left| D_z^{-\mu} h(z) \right| \le \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left\{ 1 + \frac{(2+\alpha)\,|\beta| + 1 - 3\alpha}{(2\alpha+\alpha^2)(2+\mu)} \, |z| \right\}, \tag{2.5}$$

for $\mu > 0$ and $z \in \mathcal{U}$. The results (2.4) *and* (2.5) *are sharp.*

Proof. It is easy to show that

$$\Gamma(2+\mu) z^{-\mu} D_z^{-\mu} h(z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\mu)}{\Gamma(n+1+\mu)} \, |a_n| \, z^n = z - \sum_{n=2}^{\infty} \Psi(n) \, |a_n| \, z^n,$$

where

$$\Psi(n) = \frac{\Gamma(n+1)\Gamma(2+\mu)}{\Gamma(n+1+\mu)} \ (n \ge 2).$$

Note that

$$0 < \Psi(n) \le \Psi(2) = \frac{2}{2+\mu}. \tag{2.6}$$

Furthermore, it follows from Lemma 1.1 that

$$\sum_{n=2}^{\infty} |a_n| < \frac{(2+\alpha)\,|\beta| + 1 - 3\alpha}{4\alpha + 2\alpha^2}. \tag{2.7}$$

Therefore, by using (2.6) and (2.7), we obtain

$$\left|\Gamma(2+\mu)z^{-\mu}D_z^{-\mu}h(z)\right| \geq |z| - \Psi(2)\,|z|^2\sum_{n=2}^{\infty}|a_n| \geq |z| - \frac{(2+\alpha)\,|\beta|+1-3\alpha}{(2\alpha+\alpha^2)(2+\mu)}\,|z|^2$$

and

$$\left|\Gamma(2+\mu)z^{-\mu}D_z^{-\mu}h(z)\right| \leq |z| + \Psi(2)\,|z|^2\sum_{n=2}^{\infty}a_n \leq |z| + \frac{(2+\alpha)\,|\beta|+1-3\alpha}{(2\alpha+\alpha^2)(2+\mu)}\,|z|^2\,,$$

which prove the inequalities of Theorem 2.4. Finally, we can easily see that the results (2.4) and (2.5) are sharp for the function $h(z)$ defined by

$$D_z^{-\mu}h(z) = \frac{z^{1+\mu}}{\Gamma(2+\mu)}\left\{1 - \frac{(2+\alpha)\,|\beta|+1-3\alpha}{4\alpha+2\alpha^2}z\right\}. \tag{2.8}$$

□

2.5. Corollary. *Let the function $f = h + \bar{g}$ be so that h and g are given by (1.2). Then $D_z^{-\mu}h(z)$ is included in a disk with its center at the origin and radius r_0 given by*

$$r_0 = \frac{1}{\Gamma(2+\mu)}\left\{1 - \frac{(2+\alpha)\,|\beta|+1-3\alpha}{4\alpha+2\alpha^2}z\right\}. \tag{2.9}$$

2.6. Theorem. *Let the function $f = h + \bar{g}$ be so that h and g are given by (1.2). If $f \in \mathcal{T}_{\mathcal{H}}(\alpha,\beta)$, then*

$$\left|D_z^{\mu}h(z)\right| \geq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)}\left\{1 - \frac{(2+\alpha)\,|\beta|+1-3\alpha}{(2\alpha+\alpha^2)(2-\mu)}\,|z|\right\} \tag{2.10}$$

and

$$\left|D_z^{\mu}h(z)\right| \leq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)}\left\{1 + \frac{(2+\alpha)\,|\beta|+1-3\alpha}{(2\alpha+\alpha^2)(2-\mu)}\,|z|\right\}, \tag{2.11}$$

for $0 \leq \mu < 1$ and $z \in \mathcal{U}$. The results (2.10) and (2.11) are sharp.

Proof. Note that

$$\Gamma(2-\mu)z^{\mu}D_z^{\mu}h(z) = z + \sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\mu)}{\Gamma(n+1-\mu)}\,|a_n|\,z^n = z + \sum_{n=2}^{\infty}\Phi(n)n\,|a_n|\,z^n,$$

where

$$\Phi(n) = \frac{\Gamma(n)\Gamma(2-\mu)}{\Gamma(n+1-\mu)},\ (n \geq 2).$$

It is easy to see that

$$0 < \Phi(n) \leq \Phi(2) = \frac{1}{2-\mu}. \tag{2.12}$$

From Lemma 1.1, we can see that

$$\sum_{n=2}^{\infty}n\,|a_n| \leq \frac{(2+\alpha)\,|\beta|+1-3\alpha}{2\alpha+\alpha^2}, \tag{2.13}$$

consequently, with the aid of (2.12) and (2.13), we have

$$\left|\Gamma(2-\mu)z^{\mu}D_z^{\mu}h(z)\right| \geq |z| - \Phi(2)\,|z|^2\sum_{n=2}^{\infty}n\,|a_n| \geq |z| - \frac{(2+\alpha)\,|\beta|+1-3\alpha}{(2\alpha+\alpha^2)(2-\mu)}\,|z|^2\,,$$

which gives (2.10) and

$$\left|\Gamma(2-\mu)z^{\mu}D_z^{\mu}h(z)\right| \leq |z| + \Phi(2)\,|z|^2\sum_{n=2}^{\infty}n\,|a_n| \leq |z| + \frac{(2+\alpha)\,|\beta|+1-3\alpha}{(2\alpha+\alpha^2)(2-\mu)}\,|z|^2\,,$$

which shows (2.11). Finally, by taking the function $h(z)$ defined by

$$(2.14)\qquad D_z^{\mu} h(z) = \frac{z^{1-\mu}}{\Gamma(2-\mu)}\left\{1 - \frac{(2+\alpha)\,|\beta| + 1 - 3\alpha}{(2\alpha+\alpha^2)(2-\mu)} z\right\},$$

the results (2.10) and (2.11) are easily seen to be sharp. □

2.7. Remark. Letting $\mu = 0$ in Theorem 2.4 and $\mu \longrightarrow 1$ in Theorem 2.6, we have the growth and distortion theorems for the function $h(z)$ obtained in [3].

3. Fractional integral operator

We need the following definition of the fractional integral operator given by Srivastava *et al.*[16].

3.1. Definition. For real numbers $\lambda > 0$, γ and δ, the fractional integral operator $I_{0,z}^{\lambda,\gamma,\delta}$ is defined by

$$(3.1)\qquad I_{0,z}^{\lambda,\gamma,\delta} h(z) = \frac{z^{-\lambda-\gamma}}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1} F(\lambda+\gamma, -\delta; \lambda; 1-t/z) h(t)dt,$$

where the function $h(z)$ is analytic in a simply-connected region of the z-plane containing the origin with the order

$$h(z) = O\big(|z|^{\varepsilon}\big) \quad (z \to 0),$$

with $\varepsilon > \max\{0, \gamma-\delta\} - 1$. Here $F(a, b; c; z)$ is the Gauss hypergeometric function defined by

$$(3.2)\qquad F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n},$$

where $(\nu)_n$ is the Pochhammer symbol defined by

$$(3.3)\qquad (\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & (n=0) \\ \nu(\nu+1)(\nu+2)\cdots(\nu+n-1) & (n \in \mathbb{N}^+), \end{cases}$$

and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log\,(z-t)$ to be real when $z - t > 0$.

3.2. Remark. For $\gamma = -\lambda$, we note that

$$I_{0,z}^{\lambda,-\lambda,\delta} h(z) = D_z^{-\lambda} h(z).$$

In order to prove our result for the fractional integral operator, we have to recall here the following lemma due to Srivastava *et al.* [16].

3.3. Lemma. *If $\lambda > 0$ and $n > \gamma - \delta - 1$, then*

$$(3.4)\qquad I_{0,z}^{\lambda,\gamma,\delta} z^n = \frac{\Gamma(n+1)\Gamma(n-\gamma+\delta+1)}{\Gamma(n-\gamma+1)\Gamma(n+\lambda+\delta+1)} z^{n-\gamma}.$$

With aid of Lemma 3.3, we prove

3.4. Theorem. *Let $\lambda > 0$, $\gamma > 2$, $\lambda + \delta > -2$, $\gamma - \delta < 2$ and $\gamma(\lambda+\delta) \leq 3\lambda$, and let the function $f = h + \bar{g}$ be so that h and g are given by (1.2). If $f \in \mathcal{T}_{\mathcal{H}}(\alpha, \beta)$, then*

$$(3.5)\qquad \left|I_{0,z}^{\lambda,\gamma,\delta} h(z)\right| \geq \frac{\Gamma(2-\gamma+\delta)\,|z|^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\lambda+\delta)}\left\{1 - \frac{[(2+\alpha)\,|\beta| + 1 - 3\alpha](2-\gamma+\delta)}{(2\alpha+\alpha^2)(2-\gamma)(2+\lambda+\delta)}\,|z|\right\}$$

and

$$\left|I_{0,z}^{\lambda,\gamma,\delta}h(z)\right| \leq \frac{\Gamma(2-\gamma+\delta)\,|z|^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\lambda+\delta)}\left\{1+\frac{[(2+\alpha)\,|\beta|+1-3\alpha](2-\gamma+\delta)}{(2\alpha+\alpha^2)(2-\gamma)(2+\lambda+\delta)}\,|z|\right\} \tag{3.6}$$

for $z \in \mathcal{U}_0$, *where*

$$\mathcal{U}_0 = \begin{cases}\mathcal{U} & (\gamma \leq 1),\\ \mathcal{U}-\{0\} & (\gamma > 1).\end{cases} \tag{3.7}$$

The equalities in (3.5) *and* (3.6) *are attained for the function* $h(z)$ *defined by*

$$I_{0,z}^{\lambda,\gamma,\delta}h(z) = \frac{\Gamma(2-\gamma+\delta)z^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\lambda+\delta)}\left\{1-\frac{[(2+\alpha)\,|\beta|+1-3\alpha](2-\gamma+\delta)}{(2\alpha+\alpha^2)(2-\gamma)(2+\lambda+\delta)}z\right\} \tag{3.8}$$

Proof. By using Lemma 3.3, we have

$$\begin{aligned}I_{0,z}^{\lambda,\gamma,\delta}h(z) = {}& \frac{\Gamma(2-\gamma+\delta)}{\Gamma(2-\gamma)\Gamma(2+\lambda+\delta)}z^{1-\gamma}\\ &-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(n-\gamma+\delta+1)}{\Gamma(n-\gamma+1)\Gamma(n+\lambda+\delta+1)}\,|a_n|\,z^{n-\gamma},\ (z \in \mathcal{U}_0).\end{aligned}$$

Letting

$$H(z) = \frac{\Gamma(2-\gamma)\Gamma(2+\lambda+\delta)}{\Gamma(2-\gamma+\delta)}z^{\gamma}I_{0,z}^{\lambda,\gamma,\delta}h(z) = z+\sum_{n=2}^{\infty}\Delta(n)\,|a_n|\,z^n,$$

where

$$\Delta(n) = \frac{(2-\gamma+\delta)_{n-1}(1)_n}{(2-\gamma)_{n-1}(2+\lambda+\delta)_{n-1}},\ (n \geq 2),$$

we can see that the function $\Delta(n)$ is non-increasing for integers $n \geq 2$, and we have

$$0 < \Delta(n) \leq \Delta(2) = \frac{2(2-\gamma+\delta)}{(2-\gamma)(2+\lambda+\delta)}. \tag{3.9}$$

Therefore, by using (2.7) and (3.9), we have

$$\begin{aligned}|H(z)| &\geq |z| - \Delta(2)\,|z|^2\sum_{n=2}^{\infty}|a_n|\\ &\geq |z| - \frac{[(2+\alpha)\,|\beta|+1-3\alpha](2-\gamma+\delta)}{(2\alpha+\alpha^2)(2-\gamma)(2+\lambda+\delta)}\,|z|^2\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}|H(z)| &\leq |z| + \Delta(2)\,|z|^2\sum_{n=2}^{\infty}|a_n|\\ &\leq |z| + \frac{[(2+\alpha)\,|\beta|+1-3\alpha](2-\gamma+\delta)}{(2\alpha+\alpha^2)(2-\gamma)(2+\lambda+\delta)}\,|z|^2\end{aligned} \tag{3.11}$$

for $z \in \mathcal{U}_0$, where $\mathcal{U}_0$ is defined by (3.7). This completes the proof of Theorem 3.4. □

3.5. Remark. Taking $\gamma = -\lambda = -\mu$ in Theorem 3.4, we again obtain the result of Theorem 2.4.

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