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General iterative algorithm for solving system of variational inequality problems in real Banach spaces

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Abstract

In this paper, we introduce general approximation method for solving system of variational inequality problems in Banach spaces. The strong convergence of this general iterative method is proved under certain assumptions imposed on the sequence of parameters. Application to quadratic optimization problem is also considered. The results presented in the paper extend and improve some recent results announced in the current literature.

Keywords: System of variational inequality, General iterative method, Accretive mappings, Strong convergence.
2010 MSC: 47H09; 47H10; 47J25.

1. Introduction

Let E be a real Banach space and C be a nonempty, closed and convex subset of E. We denote by J the normalized duality map from E to 2^{E^*} (E^* is the dual space of E) defined by:

$$J(x) := \{x^* \in E^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\}, \forall\, x \in E.$$

Let $S := \{x \in E : \|x\| = 1\}$. E is said to be *smooth* if

$$\lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in S$. E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $x, y \in S$.

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Let E be a normed space with dimE ≥ 2. The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}; \quad \tau > 0.$$

It is known that a normed linear space E is *uniformly smooth* if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

If there exists a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be *q-uniformly smooth*. Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p \ (or\ l_p) \ or \ W_p^m \text{ is}$$

$$\begin{cases} 2-\text{uniformly smooth and } p-\text{uniformly convex} & \text{if} \quad 2 \leq p < \infty; \\ 2-\text{uniformly convex and } p-\text{uniformly smooth} & \text{if} \quad 1 < p < 2. \end{cases} \tag{1.1}$$

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$$L_p \ (or\ l_p) \ or \ W_p^m \text{ is } \begin{cases} 2-\text{uniformly smooth and } p-\text{uniformly convex} & \text{if} \quad 2 \leq p < \infty; \\ 2-\text{uniformly convex and } p-\text{uniformly smooth} & \text{if} \quad 1 < p < 2. \end{cases}$$

Let J_q denote the *generalized duality mapping* from E to 2^{E^*} defined by

$$J_q(x) := \left\{f \in E^* : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}\right\}$$

where $\langle .,. \rangle$ denotes the generalized duality pairing. Notice that for $x \neq 0$,

$$J_q(x) = \|x\|^{q-2} J_2(x), \ q > 1.$$

Following Browder [4], we say that a Banach space has a weakly continuous normalized duality map if J is a single-valued and is weak-to-weak* sequentially continous, i.e., if $(x_n) \subset E$, $x_n \rightharpoonup x$, then $J(x_n) \rightharpoonup J(x)$ in E^*.
Weak continuity of duality map J plays an important role in the fixed point theory for nonlinear operators.

Finally recall that a Banach space E satisfies Opial property (see, e.g., [15]) if $\limsup\limits_{n\to+\infty}\|x_n - x\| < \limsup\limits_{n\to+\infty}\|x_n - y\|$ whenever $x_n \rightharpoonup x$, $x \neq y$.
A Banach space E that has a weakly continuous normalized duality map satisfies Opial's property.

Remark 1. *Note also that a duality mapping exists in each Banach space. We recall from [1] some of the examples of this mapping in $l_p, L_p, W^{m,p}$-spaces, $1 < p < \infty$.*

(i) $l_p: \ Jx = \|x\|_{l_p}^{2-p} y \in l_q, \ x = (x_1, x_2, \cdots, x_n, \cdots), \ y = (x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, \cdots, x_n|x_n|^{p-2}, \cdots),$

(ii) $L_p: \ Ju = \|u\|_{L_p}^{2-p}|u|^{p-2}u \in L_q,$

(iii) $W^{m,p}: \ Ju = \|u\|_{W^{m,p}}^{2-p} \sum_{|\alpha \leq m|}(-1)^{|\alpha|}D^\alpha\Big(|D^\alpha u|^{p-2}D^\alpha u\Big) \in W^{-m,q},$

where $1 < q < \infty$ is such that $1/p + 1/q = 1$.

Finally recall that a Banach space E satisfies Opial property (see, e.g., [15]) if $\limsup_{n\to+\infty}\|x_n - x\| < \limsup_{n\to+\infty}\|x_n - y\|$ whenever $x_n \xrightarrow{w} x$, $x \neq y$. We denote by $Fix(T)$ the set of fixed points of the mapping $T : C \to C$ that is $Fix(T) := \{x \in D(T) : x = Tx\}$. Let $D(T) \subset C$, then T is said to be Lipschitz if there exists an $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \;\; x, y \in D(T).$$

If $L = 1$, T is called nonexpansive. Recall that an operator $A : C \to E$ is said to be accretive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq 0, \;\; \forall x, y \in D(T).$$

An operator $A : C \to E$ is said to be α-inverse strongly accretive if, for some $\alpha > 0$,

$$\langle Ax - Ay, j(x-y)\rangle \geq \alpha\|Ax - Ay\|^2, \;\; \forall x, y \in D(T).$$

It is said to be strongly accretive if there exists a positive constant $k \in (0, 1)$ and such that for all $x, y \in D(A)$, such that

$$\langle Ax - Ay, j(x-y)\rangle \geq k\|x - y\|^2, \;\; \forall x, y \in D(T).$$

In a Hilbert space, the normalized duality map is the identity map. Hence, in Hilbert spaces, *monotonicity* and *accretivity* coincide.

Now, we focus on the following generalized variational inequality in Banach space E: find a point $x^* \in C$ such that, for some $j(x - x^*) \in J(x - y)$,

$$\langle Ax^*, j(x - x^*)\rangle \geq 0, \;\; \forall x \in C. \tag{1.2}$$

This general variational inequality was considered by Aoyama et al. [2]. Throughout, the solution set of variational inequality (1.2) is denoted by $VI(C, A)$, that is,

$$VI(C, A) := \{x^* \in C, \;\; \langle Ax^*, j(x - x^*)\rangle \geq 0, \;\; \forall x \in C\}.$$

For a lot of real-life problems, such as, in signal processing, resource allocation, image recovery and so on, the constraints can be expressed as the variational inequality problem. Hence, the problem of finding solutions of variational inequality has become a flourishing area of contemporary research for numerous mathematicians working in nonlinear operator theory; (see, for example, [6, 21, 3, 18, 22] and the references contained in them). For solving the above variational inequality (1.2), Aoyama et al. [2] introduced an iterative algorithm:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Q_C(I - \lambda_n A)x_n \;\; n \geq 0, \tag{1.3}$$

where Q_C is a sunny nonexpansive retraction from E onto C and $\{\alpha_n\} \subset (0, 1)$, $\{\lambda_n\} \subset (0, \infty)$ are two real number sequences. Aoyama et al. [2] proved the following weak convergence theorem for solving variational inequality (1.2).

Theorem 1. *[2] Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C. Let $A : C \to E$ be an α-inverse strongly accretive operator with $VI(C, A) \neq \emptyset$. If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen so that $\lambda_n \in \left[a, \frac{\alpha}{K^2}\right]$ for some $a > 0$ and $\alpha_n \in [b, c]$ for some b, c with $0 < b < c < 1$, then $\{x_n\}$ converges weakly to a solution of variational inequality $VI(C, A)$, where K is the 2-uniformly smoothness constant of E.*

Recently, many authors studied the following convex feasibility problem (for short, CFP):

$$\textit{finding} \;\;\; \textit{an } x^* \in \bigcap_{i=1}^{m} K_i, \tag{1.4}$$

where $m \geq 1$ is an integer and each K_i is a nonempty closed convex subset of H. There is a considerable investigation on the CFP in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration [11, 5], computer tomography and radiation therapy treatment planning [6].In this paper, we shall consider the case when K_i is the solution set of a finite family of variational inequalities in real Banach spaces. Recently, iterative methods for single-valued nonexpansive mappings have been applied to solve fixed points problems and variational inequality problems in Hilbert spaces, see, e.g.,[14, 19, 13] and the references therein.

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle. \tag{1.5}$$

In [19], Xu proved that the sequence $\{x_n\}$ defined by iterative method below with initial guess $x_0 \in H$ chosen arbitrary:

$$x_{n+1} = \alpha_n b + (I - \alpha_n A)Tx_n, \quad n \geq 0, \tag{1.6}$$

converges strongly to the unique solution of the minimization problem (1.5), where T is a nonexpansive mappings in H and A a strongly positive bounded linear operator. In 2006 Marino and Xu [13] extended Moudafi's results [14] and Xu's results [19] via the following general iteration $x_0 \in H$ and

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \; n \geq 0, \tag{1.7}$$

where$\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1)$, A is bounded linear operator on H and T is a nonexpansive. Under suitable conditions, they proved the sequence $\{x_n\}$ defined by (1.7) converges strongly to the fixed point of T, which is a unique solution of the following variational inequality

$$\langle Ax^* - \gamma f(x^*), x^* - p\rangle \leq 0, \quad \forall p \in Fix(T).$$

Inspired and motivated by current research in this area, we consider the problem of finding an element of $\bigcap_{i=1}^{m} VI(C, A_i)$, where $A_i : C \to E$ is α_i-inverse strongly accretive for $i = 1, 2, ..., m.$. Based on the well-known general iterative method, we introduce a new iterative algorithm for finding an element in $\bigcap_{i=1}^{m} VI(C, A_i)$, that is,

$$\begin{cases} x_0 \in C, \; \textit{choosen arbitrarily,} \\ \\ y_n = \lambda_0 x_n + \displaystyle\sum_{i=1}^{m} \lambda_i Q_C(I - \beta_i A_i)x_n, \\ \\ x_{n+1} = Q_C(\alpha_n \gamma f(x_n) + (I - \alpha_n \eta B)y_n), \end{cases} \tag{1.8}$$

where $\lambda_i \in (0,1)$, $\sum_{i=0}^{m} \lambda_i = 1$, $\beta_i > 0$ and $\{\alpha_n\} \subset (0,1)$. Assume that the above control sequences satisfy the following conditions:

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=0}^{\infty} \alpha_n = \infty$. It is proven that the sequence $\{x_n\}$ generated by algorithm (1.8) converges strongly to

$x^* \in \bigcap_{i=1}^{m} VI(C, A_i)$, which is a unique solution of the following variational inequality

$$\langle \eta Bx^* - \gamma f(x^*), J(x^* - p)\rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^{m} VI(C, A_i), \tag{1.9}$$

where $B : K \to E$ be an k-strongly accretive and L-Lipschitzian operator.

2. Preliminairies

Let C and D be nonempty subsets of a Banach space E such that C is a nonempty closed convex and $D \subset C$, then a mapping $Q : C \to D$ is said to be sunny if $Q(x + t(x - Q(x))) = Q(x)$ whenever $x + t(x - Q(x)) \in C$ for all $x \in C$ and $t > 0$. A mapping $Q : C \to D$ is called a retraction if $Q^2 = Q$. Also, if a mapping Q is a retraction, then we have $Qz = z$ for all z in the range of Q.

Lemma 1. *Let E be a smooth Banach space and let C be a nonempty subset of E. Let $Q : E \to C$ be a retraction and let J be the normalized duality mapping on E. Then the following statements are equivalent:*

(i) *Q is sunny and nonexpansive;*
(ii) $\langle Qx - Qy, J(x - y)\rangle \leq \|Qx - Qy\|^2 \ \forall y, x \in C$;
(iii) $\|(x - y) - (Qx - Qy)\|^2 \leq \|x - y\|^2 - \|Qx - Qy\|$;
(iv) $\langle x - Qx, J(y - Qx)\rangle \leq 0$.

The demiclosedness of a nonlinear operator T usually plays an important role in dealing with the convergence of fixed point iterative algorithms.

Definition 1. *Let K be a nonempty, closed convex subset of a real Hilbert space H and let $T : K \to K$ be a single-valued mapping. $I - T$ is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $\|x_n - Tx_n\|$ converges to zero, then $p \in Fix(T)$.*

Lemma 2. *[10] Let E be a real Banach space satisfying Opial's property, K be a closed convex subset of E, and $T : K \to K$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$. Then $I - T$ is demiclosed*

Theorem 2. *[9] Let $q > 1$ be a fixed real number and E be a smooth Banach space. Then the following statements are equivalent:*
(i) E is q-uniformly smooth.
(ii) There is a constant $d_q > 0$ such that for all $x, y \in E$

$$\|x + y\|^q \leq \|x\|^q + q\langle y \ , \ J_q(x)\rangle + d_q\|y\|^q.$$

(iii) There is a constant $c_1 > 0$ such that

$$\langle x - y \ , \ J_q(x) - J_q(y)\rangle \ \leq c_1\|x - y\|^q \quad \forall \ \ x, y \in E.$$

Lemma 3 (Xu, [20])**.** *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*
(a) $\displaystyle\sum_{n=0}^{\infty} \alpha_n = \infty$, *(b)* $\displaystyle\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\displaystyle\sum_{n=0}^{\infty} |\sigma_n\alpha_n| < \infty$. *Then* $\displaystyle\lim_{n\to\infty} a_n = 0$.

Lemma 4. *[12] Let t_n be a sequence of real numbers that does not decrease at infinity in a sense that there exists a subsequence t_{n_i} of t_n such that t_{n_i} such that $t_{n_i} \leq t_{n_{i+1}}$ for all $i \geq 0$. For sufficiently large numbers $n \in \mathbb{N}$, an integer sequence $\{\tau(n)\}$ is defined as follows:*

$$\tau(n) = \max\{k \leq n : \ t_k \leq t_{k+1}\}.$$

Then, $\tau(n) \to \infty$ as $n \to \infty$ and

$$\max\{t_{\tau(n)}, \ \ t_n\} \leq t_{\tau(n)+1}.$$

Lemma 5 (Chang et al. [8]). *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B(0)_r := \{x \in E : ||x|| \leq r\}$, a closed ball with center 0 and radius $r > 0$. For any given sequence $\{u_1, u_2, ..., u_m\} \subset B(0)_r$ and for $i = 1, 2, ..., m$, any positive real numbers $\{\lambda_1, \lambda_2, ..., \lambda_m\}$ with $\sum_{k=1}^{m} \lambda_k = 1$, then there exists a continuous, strictly increasing and convex function*

$$g : [0, 2r] \to \mathbb{R}^+,\ g(0) = 0,$$

such that for any integer i, j with $i < j$,

$$\|\sum_{k=1}^{m} \lambda_k u_k\|^2 \leq \sum_{k=1}^{m} \lambda_k \|u_k\|^2 - \lambda_i \lambda_j g(\|u_i - u_j\|).$$

Lemma 6. *[2] Let C be a nonempty closed convex subset of a smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let A be an accretive operator of C into E. Then for all $\lambda > 0$,*

$$VI(C, A) = Fix(Q_C(I - \lambda A)). \tag{2.1}$$

Lemma 7. *[2] Let $q > 1$ be a fixed real number and E be a smooth Banach space. Let C be a nonempty closed convex subset of a 2-uniformly smooth Banach space E. Let $\alpha > 0$ and let A be an α-inverse strongly accretive operator of C into E. If $0 < \lambda \leq \frac{\alpha}{K^2}$, then $I - \lambda A$ is a nonexpansive mapping of C into E, where K is the 2-uniformly smoothness constant of E.*

Lemma 8. *[17] Let $q > 1$ be a fixed real number and E be a q-uniformly smooth real Banach space with constant d_q. Let K be a nonempty, closed convex subset of E and $A : K \to E$ be a k-strongly accretive and L-Lipschitzian operator with $k > 0$, $L > 0$. Assume that $0 < \eta < \Big(\frac{kq}{d_q L^q}\Big)^{\frac{1}{q-1}}$ and $\tau = \eta\Big(k - \frac{d_q L^q \eta^{q-1}}{q}\Big)$. Then for each $t \in \Big(0, min\{1, \frac{1}{\tau}\}\Big)$, we have*

$$\|(I - t\eta A)x - (I - t\eta A)y\| \leq (1 - t\tau)\|x - y\|,\ \forall\, x, y \in K.$$

3. Main result

In this section, we study Algorithms (1.8) and we will show that our explicit algorithms work well with strong convergence.

Theorem 3. *Let E be a 2-uniformly smooth and uniformly convex Banach space with weakly sequentially continuous duality mapping and C be a nonempty, closed convex subset of E. Let $f : C \to E$ be an b-Lipschitzian mapping with a constant $b \geq 0$. Let $B : C \to E$ be an μ-strongly accretive and L-Lipschitzian operator with $0 < \eta < \frac{2\mu}{d_2 L^2}$ and $0 \leq \gamma b < \tau$, where $\tau = \eta(\mu - \frac{d_2 L^2 \eta}{2})$. Let $A_i : C \to E$ is α_i-inverse strongly accretive for $i = 1, 2, ..., m$, such that $\bigcap_{i=1}^{m} VI(C, A_i) \neq \emptyset$. Assume that $\beta_i \in \Big[a, \frac{\alpha_i}{K^2}\Big]$ for some $a > 0$ where K is the 2-uniformly smoothness constant of E. Then, the sequence $\{x_n\}$ generated by (1.8) converges strongly to $x^* \in \bigcap_{i=1}^{m} VI(C, A_i)$, which is the unique solution of variational inequality (1.9).*

Proof. From the choice of η and γ, $(\eta A - \gamma f)$ is strongly accretive, then the variational inequality (1.9) has a unique solution in $\bigcap_{i=1}^{m} VI(C, A_i)$. Without loss of generality, we can assume $\alpha_n \in \Big(0, min\{1\ , \frac{1}{\tau}\}\Big)$. In what follows, we denote x^* to be the unique solution of (1.9). Fixing $p \in \bigcap_{i=1}^{m} VI(C, A_i)$. We prove that the sequence $\{x_n\}$ is bounded. From (1.8) and Lemmas 5 and 7, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\lambda_0 x_n + \sum_{i=1}^{m} \lambda_i Q_C(I - \beta_i A_i)x_n - p\|^2 \\
&\leq \lambda_0\|x_n - p\|^2 + \sum_{i=1}^{m} \lambda_i \|Q_C(I - \beta_i A_i)x_n - p\|^2 - \lambda_0\lambda_i g(\|Q_C(I - \beta_i A_i)x_n - x_n\|) \\
&\leq \|x_n - p\|^2 - \lambda_0\lambda_i g(\|Q_C(I - \beta_i A_i)x_n - x_n\|).
\end{aligned}$$

Hence,

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \lambda_0\lambda_i g(\|Q_C(I - \beta_i A_i)x_n - x_n\|). \tag{3.1}$$

Since $\lambda_i \in (0, 1)$, for $i = 0, 1, ..., m$, we obtain

$$\|y_n - p\Big\| \leq \|x_n - p\Big\|. \tag{3.2}$$

By Lemma 8 and inequality (3.2), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|Q_C(\alpha_n\gamma f(x_n) + (I - \eta\alpha_n B)y_n) - p\| \\
&\leq \alpha_n\gamma\|f(x_n) - f(p)\| + (1 - \tau\alpha_n)\|y_n - p\| + \alpha_n\|\gamma f(p) - \eta Bp\| \\
&\leq (1 - \alpha_n(\tau - b\gamma))\|x_n - p\| + \alpha_n\|\gamma f(p) - \eta Bp\| \\
&\leq \max\{\|x_n - p\|, \frac{\|\gamma f(p) - \eta Bp\|}{\tau - b\gamma}\}.
\end{aligned}$$

By induction, it is easy to see that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|\gamma f(p) - \eta Bp\|}{\tau - b\gamma}\}, \quad n \geq 1.$$

Hence, $\{x_n\}$ is bounded also are $\{f(x_n)\}$, and $\{Bx_n\}$.
Thanks (1.8) and (3.1), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|\alpha_n\gamma f(x_n) + (I - \eta\alpha_n B)y_n - p\|^2 \\
&\leq \|y_n - p + \alpha_n\gamma f(x_n) - \alpha_n\eta By_n\|^2 \\
&\leq \|y_n - p\|^2 - 2\alpha_n\langle \eta By_n - \gamma f(x_n), J(y_n - p)\rangle + d_2\Big\|\alpha_n\gamma f(x_n) + \alpha_n\eta By_n\Big\|^2 \\
&\leq \|y_n - p\|^2 + 2\alpha_n\|\eta By_n - \gamma f(x_n)\|\|y_n - p\| + d_2\Big\|\alpha_n\gamma f(x_n) - \alpha_n\eta By_n\Big\|^2 \\
&\leq \Big\|x_n - p\|^2 - \lambda_0\lambda_i g(\|Q_C(I - \beta_i A_i)x_n - x_n\|) + 2\alpha_n\|\eta By_n - \gamma f(x_n)\|\|y_n - p\| \\
&\quad + d_2{\alpha_n}^2\|\gamma f(x_n) - \alpha_n\eta By_n\|^2.
\end{aligned}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, then there exists a constant $D > 0$ such that

$$\lambda_0\lambda_i g(\|Q_C(I - \beta_i A_i)x_n - x_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n D. \tag{3.3}$$

Now we prove that $\{x_n\}$ converges strongly to x^*.
We divide the proof into two cases.
Case 1. Assume that there is $n_0 \in N$ such that $\{\|x_n - x^*\|\}$ is decreasing for all $n \geq n_0$. Since $\{\|x_n - x^*\|\}$ is monotonic and bounded, $\{\|x_n - x^*\|\}$ is convergent. Clearly, we have

$$\lim_{n\to\infty}\Big(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\Big) = 0. \tag{3.4}$$

It then implies from (3.3) that

$$\lim_{n\to\infty} g(\|Q_C(I-\beta_i A_i)x_n - x_n\|) = 0. \tag{3.5}$$

By using properties of g, we get

$$\lim_{n\to\infty} \|Q_C(I-\beta_i A_i)x_n - x_n\| = 0. \tag{3.6}$$

Now, we show that $\limsup\limits_{n\to+\infty}\langle \eta Bx^* - \gamma f(x^*), J(x^*-x_n)\rangle \le 0$. Since E is reflexive and $\{x_n\}_{n\ge 0}$ is bounded there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges weakly to a in C and

$$\limsup_{n\to+\infty}\langle \eta Bx^* - \gamma f(x^*), J(x^*-x_n)\rangle = \lim_{j\to+\infty}\langle \eta Bx^* - \gamma f(x^*), J(x^*-x_{n_j})\rangle.$$

From (3.6) and Lemma 2, we obtain $a \in \bigcap\limits_{i=1}^{m} Fix(Q_C(I-\beta_i A_i))$. Using Lemma 6, we have $a \in \bigcap\limits_{i=1}^{m} VI(C, A_i)$. On the other hand, by using x^* solves (1.9) and the assumption that the duality mapping J is weakly continuous, we have

$$\begin{aligned}
\limsup_{n\to+\infty}\langle \eta Bx^* - \gamma f(x^*), J(x^*-x_n)\rangle &= \lim_{j\to+\infty}\langle \eta Bx^* - \gamma f(x^*), J(x^*-x_{n_j})\rangle \\
&= \langle \eta Bx^* - \gamma f(x^*), J(x^*-a)\rangle \le 0.
\end{aligned}$$

Finally, we show that $x_n \to x^*$. Applying Lemma 1, we get

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|Q_C(\alpha_n\gamma f(x_n) + (I-\eta\alpha_n B)y_n) - x^*\|^2 \\
&\le \langle \alpha_n\gamma f(x_n) + (I-\eta\alpha_n B)y_n - x^*, J(x_{n+1}-x^*)\rangle \\
&= \langle \alpha_n\gamma f(x_n) + (I-\eta\alpha_n B)y_n - x^* - \alpha_n\gamma f(x^*) + \alpha_n\gamma f(x^*) - \alpha_n\eta Bx^* + \alpha_n\eta Bx^*, \\
&\quad J(x_{n+1}-x^*)\rangle \\
&\le \Big(\alpha_n\gamma\|f(x_n)-f(x^*)\| + \|(I-\alpha_n\eta B)(y_n-x^*)\|\Big)\|x_{n+1}-x^*\| \\
&\quad +\alpha_n\langle \eta Bx^* - \gamma f(x^*), J(x_{n+1}-x^*)\rangle \\
&\le (1-\alpha_n(\tau-b\gamma))\|x_n-x^*\|\|x_{n+1}-x^*\| + \alpha_n\langle \eta Bx^* - \gamma f(x^*), J(x_{n+1}-x^*)\rangle \\
&\le (1-\alpha_n(\tau-b\gamma))\|x_n-x^*\|^2 + 2\alpha_n\langle \eta Bx^* - \gamma f(x^*), J(x_{n+1}-x^*)\rangle.
\end{aligned}$$

Hence, by Lemma 3, we conclude that the sequence $\{x_n\}$ converge strongly to the point $x^* \in \bigcap\limits_{i=1}^{m} VI(C, A_i)$.

Case 2. Assume that the sequence $\{\|x_n - x^*\|\}$ is not monotonically decreasing. Set $\Gamma_n = \|x_n - x^*\|$ and $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \ge n_0$ (for some n_0 large enough) by $\tau(n) = \max\{k \in \mathbb{N} : k \le n,\ \Gamma_k \le \Gamma_{k+1}\}$. Obviously, $\{\tau(n)\}$ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ for $n \ge n_0$. From (3.3), we have

$$\lambda_0\lambda_i g(\|Q_C(I-\beta_i A_i)x_{\tau(n)} - x_{\tau(n)}\|) \le \alpha_{\tau(n)} D.$$

Furthermore, we have

$$\lim_{n\to\infty} \|Q_C(I-\beta_i A_i)x_{\tau(n)} - x_{\tau(n)}\| = 0. \tag{3.7}$$

By same argument as in case 1, we can show that $x_{\tau(n)}$ and $y_{\tau(n)}$ are bounded in E and $\limsup\limits_{\tau(n)\to+\infty}\langle \eta Bx^* - \gamma f(x^*), J_\varphi(x^*-x_{\tau(n)})\rangle \le 0$. We have for all $n \ge n_0$,

$$0 \le \|x_{\tau(n)+1}-x^*\|^2 - \|x_{\tau(n)}-x^*\|^2 \le \alpha_{\tau(n)}[-(\tau-b\gamma)\|x_{\tau(n)}-x^*\|^2 + \langle \eta Bx^* - \gamma f(x^*), J(x^*-x_{\tau(n)+1})\rangle],$$

which implies that

$$\|x_{\tau(n)}-x^*\|^2 \le \frac{1}{\tau-b\gamma}\langle \eta Bx^* - \gamma f(x^*), J(x^*-x_{\tau(n)+1})\rangle.$$

Then, we have

$$\lim_{n\to\infty} \|x_{\tau(n)} - x^*\|^2 = 0.$$

Therefore,

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$$

Thus, by Lemma 4, we conclude that

$$0 \le \Gamma_n \le \max\{\Gamma_{\tau(n)},\ \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty} \Gamma_n = 0$, that is $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Corollary 1. *Let H be a real Hilbert space and C be a nonempty, closed convex subset of H. Let $f : C \to H$ be an b-Lipschitzian mapping with a constant $b \ge 0$. Let $B : C \to H$ be an μ-strongly monotone and L-Lipschitzian operator with $0 < \eta < \dfrac{2\mu}{L^2}$ and $0 \le \gamma b < \tau$, where $\tau = \eta(\mu - \dfrac{L^2\eta}{2})$. Let $A_i : C \to H$ is α_i-inverse strongly monotone for $i = 1, 2, ..., m$, such that $\bigcap_{i=1}^{m} VI(C, A_i) \ne \emptyset$. Assume that $\beta_i \in [0, 2\alpha_i]$. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_0 \in K, \ \textit{choosen arbitrarily}, \\ \\ y_n = \lambda_0 x_n + \displaystyle\sum_{i=1}^{m} \lambda_i P_C(I - \beta_i A_i)x_n, \\ \\ x_{n+1} = P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n \eta B)y_n), \end{cases} \tag{3.8}$$

where $\lambda_i \in (0, 1)$, $\sum_{i=0}^{m} \lambda_i = 1$ and $\{\alpha_n\} \subset (0, 1)$. Assume that the above control sequences satisfy the following conditions:
(a) $\lim_{n\to\infty} \alpha_n = 0$;
(b) $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, the sequence $\{x_n\}$ generated by (3.12) converges strongly to $x^* \in \bigcap_{i=1}^{m} VI(C, A_i)$, which is a unique solution of the following variational inequality:

$$\langle \eta Bx^* - \gamma f(x^*), x^* - p\rangle \le 0, \quad \forall p \in \bigcap_{i=1}^{m} VI(C, A_i). \tag{3.9}$$

Proof. Since Hilbert spaces are 2-uniformly convex and uniformly convex, then the proof follows from Theorem 3. □

Finally, we consider the following quadratic optimization problem:

$$\min_{x\in\Gamma} g(x) := \frac{\eta}{2}\langle Bx, x\rangle - \langle b, x\rangle, \tag{3.10}$$

where $B : C \to H$ be a *strongly positive bounded* linear operator, $\Gamma := \bigcap_{i=1}^{m} VI(C, A_i)$ and b be a fixed real.The set of solutions of (3.10) is denoted by Ω.

Lemma 9. *Let K be a nonempty, closed convex subset of E be normed linear space and let $g : K \to \mathbb{R}$ a real valued differentiable convex function. Then x^* is a minimizer of g over K if and only if x^* solves the following variational inequality $\langle \nabla g(x^*), y - x^*\rangle \ge 0$ for all $y \in K$.*

Remark 2. *By Lemma 9, $x^* \in \Omega$ if and only if x^* solves the following variational inequality:*

$$\langle \eta Bx^* - b, x^* - p\rangle \leq 0, \ \forall p \in \Gamma. \tag{3.11}$$

Hence, one has the following result.

Theorem 4. *Let H be a real Hilbert space and C be a nonempty, closed convex subset of H. Let $B : C \to H$ be strongly bounded linear operator with coefficient $\mu > 0$. Let $A_i : C \to H$ be α_i-inverse strongly monotone for $i = 1, 2, ..., m$, such that $\bigcap_{i=1}^{m} VI(C, A_i) \neq \emptyset$. Assume that $\beta_i \in [0, 2\alpha_i]$, $0 < \eta < \dfrac{2\mu}{\|B\|^2}$. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_0 \in K, \ \text{choosen arbitrarily}, \\ \\ y_n = \lambda_0 x_n + \displaystyle\sum_{i=1}^{m} \lambda_i P_C(I - \beta_i A_i)x_n, \\ \\ x_{n+1} = P_C(\alpha_n b + (I - \alpha_n \eta B)y_n), \end{cases} \tag{3.12}$$

where $\lambda_i \in (0, 1)$, $\sum_{i=0}^{m} \lambda_i = 1$ and $\{\alpha_n\} \subset (0, 1)$. Assume that the above control sequences satisfy the following conditions:
(a) $\lim_{n\to\infty} \alpha_n = 0$;
(b) $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Then, the sequence $\{x_n\}$ generated by (3.12) converges strongly to a solution of Problem (3.10).*

Proof. We note that strongly positive bounded linear operator B is a $\|B\|$-Lipschitzian and μ- strongly monotone operator. Using Remark 2, the proof follows Theorem 3 with $f \equiv b$. □

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