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Existence Results for Solutions of Nabla Fractional Boundary Value Problems with General Boundary Conditions

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Abstract

In this article, we consider a particular class of nabla fractional boundary value problems with general boundary conditions, and establish sufficient conditions on existence and uniqueness of its solutions. We present two examples to illustrate the applicability of established results.

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1. Introduction

Despite the existence of a significant mathematical theory for the continuous fractional calculus, there was really no considerable parallel development of nabla fractional calculus until very recently. However, within the past one decade, there has been a surge of interest in the development of nabla fractional calculus. The combined efforts of Gray, Zhang, Eloe, Atici, Anastassiou, Abdeljawad, Baleanu, Zhou, Jia, Erbe, Peterson, Goodrich and many other researchers produced a fairly strong basic theory of nabla fractional difference equations during the past one decade. For a detailed discussion on the basic theory of nabla fractional calculus, we refer to a recent monograph [13] and the references therein.

The nabla fractional sum or difference of any function contains information about the function at earlier points, so it possesses a long memory effect. From science to engineering, there are many processes that involve different space or time scales. In many problems of this context, the dynamics of the system can

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be formulated by fractional difference equations which include the nonlocal effects. We refer to the works of Atici's research group [4, 10, 29, 25, 27, 28] for this purpose. On the other hand, the dynamics of many phenomena in nature, for instance biological species, change only at discrete times. Generally, one assumes that if a natural system could be modeled by a discrete time system instead of a continuous one, many qualitative aspects of such system could be determined.

Recently, scholars of Erbe & Peterson [6, 9] initiated the study of boundary value problems for linear and nonlinear nabla fractional difference equations and used contraction mapping theorem to prove the existence and uniqueness of a positive solution to a forced self-adjoint fractional nabla difference equation. Gholami & Ghanbari [12] studied the existence of solutions for a coupled system of two point nabla fractional difference BVPs. The author [16, 17, 18, 19, 20, 21, 22] established sufficient conditions on existence and uniqueness of solutions and obtained Lyapunov-type inequalities for a particular class of standard two-point Riemann–Liouville type nabla fractional boundary value problems associated with different boundary conditions.

In [9], Brackins considered the following homogeneous nabla fractional boundary value problem

$$\begin{cases} -\big(\nabla_a^{\nu-1}(\nabla u)\big)(t) = 0, \quad t \in \mathbb{N}_{a+2}^b, \\ \alpha u(a+1) - \beta(\nabla u)(a+1) = 0, \\ \gamma u(b) + \delta(\nabla u)(b) = 0, \end{cases} \tag{1}$$

and proved that it has only the trivial solution if, and only if

$$\xi = (\beta - \alpha)\gamma + \alpha\gamma H_{\nu-1}(b,a) + \alpha\delta H_{\nu-2}(b,a) \neq 0. \tag{2}$$

Here a, $b \in \mathbb{R}$ with $b - a \in \mathbb{N}_1$, $1 < \nu < 2$, $\alpha^2 + \beta^2 > 0$ and $\gamma^2 + \delta^2 > 0$. Further, Brackins [9] gave an explicit expression for the Green's function of (1) as follows.

Theorem 1.1 (See [9]). *Assume* (2) *holds. The Green's function for the boundary value problem* (1) *is given by*

$$G(t,s) = \begin{cases} u(t,s), \quad t \leq s-1, \\ v(t,s), \quad t \geq s, \end{cases} \tag{3}$$

where

$$u(t,s) = \frac{1}{\xi}\Big[\alpha\gamma H_{\nu-1}(t,a)H_{\nu-1}(b,\rho(s)) + \alpha\delta H_{\nu-1}(t,a)H_{\nu-2}(b,\rho(s)) \\ + (\beta-\alpha)\gamma H_{\nu-1}(b,\rho(s)) + (\beta-\alpha)\delta H_{\nu-2}(b,\rho(s))\Big], \tag{4}$$

and

$$v(t,s) = u(t,s) - H_{\nu-1}(t,\rho(s)). \tag{5}$$

Recently, the author [21] has shown that this Green's function is nonnegative and obtained an upper bound for its maximum value. In a continuation to this study, in this article, we determine sufficient conditions on existence and uniqueness of solutions for the corresponding nonlinear nabla fractional boundary value problem

$$\begin{cases} -\big(\nabla_a^{\nu-1}(\nabla u)\big)(t) = f(t,u(t)), \quad t \in \mathbb{N}_{a+2}^b, \\ \alpha u(a+1) - \beta(\nabla u)(a+1) = 0, \\ \gamma u(b) + \delta(\nabla u)(b) = 0, \end{cases} \tag{6}$$

using Banach and Brouwer fixed point theorems. We assume that $f : \mathbb{N}_a^b \times \mathbb{R} \to \mathbb{R}$.

2. Preliminaries

We use the following notations, definitions and known results throughout the article. Denote by

$$\mathbb{N}_a = \{a, a+1, a+2, \ldots\},$$

and

$$\mathbb{N}_a^b = \{a, a+1, a+2, \ldots, b\},$$

for any a, $b \in \mathbb{R}$ such that $b - a \in \mathbb{N}_1$.

Definition 2.1 (See [8]). *The backward jump operator $\rho : \mathbb{N}_a \to \mathbb{N}_a$ is defined by*

$$\rho(t) = \begin{cases} a, & t = a, \\ t-1, & t \in \mathbb{N}_{a+1}. \end{cases}$$

Definition 2.2 (See [24, 26]). *The Euler gamma function is defined by*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0.$$

Using its reduction formula, the Euler gamma function can also be extended to the half-plane $\Re(z) \leq 0$ except for $z \in \{\ldots, -2, -1, 0\}$.

Definition 2.3 (See [13]). *For $t \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t + r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, the generalized rising function is defined by*

$$t^{\overline{r}} = \frac{\Gamma(t + r)}{\Gamma(t)}.$$

Also, if $t \in \{\ldots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t + r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, then we use the convention that $t^{\overline{r}} = 0$.

Definition 2.4 (See [13]). *Let $\mu \in \mathbb{R} \setminus \{\ldots, -2, -1\}$. Define the μ^{th}-order nabla fractional Taylor monomial by*

$$H_\mu(t, a) = \frac{(t - a)^{\overline{\mu}}}{\Gamma(\mu + 1)},$$

provided the right-hand side exists. Observe that $H_\mu(a, a) = 0$ and $H_\mu(t, a) = 0$ for all $\mu \in \{\ldots, -2, -1\}$ and $t \in \mathbb{N}_a$.

Definition 2.5 (See [8]). *Let $u : \mathbb{N}_a \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The first order backward (nabla) difference of u is defined by*

$$\big(\nabla u\big)(t) = u(t) - u(t - 1), \quad t \in \mathbb{N}_{a+1},$$

and the N^{th}-order nabla difference of u is defined recursively by

$$\big(\nabla^N u\big)(t) = \Big(\nabla\big(\nabla^{N-1} u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

Definition 2.6 (See [13]). *Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The N^{th}-order nabla sum of u based at a is given by*

$$\big(\nabla_a^{-N} u\big)(t) = \sum_{s=a+1}^{t} H_{N-1}(t, \rho(s)) u(s), \quad t \in \mathbb{N}_a,$$

where by convention $\big(\nabla_a^{-N} u\big)(a) = 0$. We define $\big(\nabla_a^{-0} u\big)(t) = u(t)$ for all $t \in \mathbb{N}_{a+1}$.

Definition 2.7 (See [13]). *Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u based at a is given by*

$$\left(\nabla_a^{-\nu} u\right)(t) = \sum_{s=a+1}^{t} H_{\nu-1}(t, \rho(s))u(s), \quad t \in \mathbb{N}_a,$$

where by convention $\left(\nabla_a^{-\nu} u\right)(a) = 0$.

Definition 2.8 (See [13]). *Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$ and choose $N \in \mathbb{N}_1$ such that $N - 1 < \nu \leq N$. The ν^{th}-order nabla difference of u is given by*

$$\left(\nabla_a^{\nu} u\right)(t) = \left(\nabla^N \left(\nabla_a^{-(N-\nu)} u\right)\right)(t), \quad t \in \mathbb{N}_{a+N}.$$

Proposition 2.9 (See [13]). *Assume the following generalized rising functions and fractional nabla Taylor monomials are well defined.*

1. $\Gamma(t) > 0$ *for* $t > 0$, *and* $\Gamma(t) < 0$ *for* $-1 < t < 0$.
2. $t^{\overline{\nu}}(t + \nu)^{\overline{\mu}} = t^{\overline{\nu+\mu}}$.
3. $\nabla(\nu + t)^{\overline{\mu}} = \mu(\nu + t)^{\overline{\mu-1}}$.
4. $\nabla(\nu - t)^{\overline{\mu}} = -\mu(\nu - \rho(t))^{\overline{\mu-1}}$.
5. $\nabla H_\mu(t, a) = H_{\mu-1}(t, a)$.
6. $H_\mu(t, a) - H_{\mu-1}(t, a) = H_\mu(t, a + 1)$.
7. $\sum_{s=a+1}^{t} H_\mu(s, a) = H_{\mu+1}(t, a)$.
8. $\sum_{s=a+1}^{t} H_\mu(t, \rho(s)) = H_{\mu+1}(t, a)$.

Proposition 2.10 (See [13]). *Let $\nu \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$ such that μ, $\mu + \nu$ and $\mu - \nu$ are nonnegative integers. Then, for all $t \in \mathbb{N}_a$,*

(i) $\nabla_a^{-\nu}(t - a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t - a)^{\overline{\mu+\nu}}$.
(ii) $\nabla_a^{\nu}(t - a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)}(t - a)^{\overline{\mu-\nu}}$.
(iii) $\nabla_a^{-\nu} H_\mu(t, a) = H_{\mu+\nu}(t, a)$.
(iv) $\nabla_a^{\nu} H_\mu(t, a) = H_{\mu-\nu}(t, a)$.

Proposition 2.11 (See [15]). *Let $\mu > -1$ and $s \in \mathbb{N}_a$. Then, the following hold:*

(a) If $t \in \mathbb{N}_{\rho(s)}$, then $H_\mu(t, \rho(s)) \geq 0$, and if $t \in \mathbb{N}_s$, then $H_\mu(t, \rho(s)) > 0$.
(b) If $t \in \mathbb{N}_{\rho(s)}$ and $\mu > 0$, then $H_\mu(t, \rho(s))$ is a decreasing function of s.
(c) If $t \in \mathbb{N}_s$ and $-1 < \mu < 0$, then $H_\mu(t, \rho(s))$ is an increasing function of s.
(d) If $t \in \mathbb{N}_{\rho(s)}$ and $\mu \geq 0$, then $H_\mu(t, \rho(s))$ is a nondecreasing function of t.
(e) If $t \in \mathbb{N}_s$ and $\mu > 0$, then $H_\mu(t, \rho(s))$ is an increasing function of t.
(f) If $t \in \mathbb{N}_{s+1}$ and $-1 < \mu < 0$, then $H_\mu(t, \rho(s))$ is a decreasing function of t.

3. Properties of Green's Function

The author has obtained some properties of the Green's function in [21]. In this section, we derive a few more properties of $G(t, s)$.

Theorem 3.1 (See [21]). *Assume α, β, γ, $\delta \geq 0$ and $\beta \geq \alpha$ such that (2) holds. Then, $G(t, s) \geq 0$ for all $(t, s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+1}^b$.*

Lemma 3.2 (See [21]). *Assume α, β, γ, $\delta \geq 0$ and $\beta \geq \alpha$ such that (2) holds. Then,*

1. *$u(t, s)$ is an increasing function of t for all $(t, s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+1}^b$ such that $t \leq s - 1$.*

2. *$v(t,s)$ is a decreasing function of t for all $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+1}^b$ such that $t \geq s$.*

Lemma 3.3. *Assume α, β, γ, $\delta \geq 0$ and $\beta \geq \alpha$ such that (2) holds. Then,*

$$\max_{(t,s)\in\mathbb{N}_a^b\times\mathbb{N}_{a+1}^b} G(t,s) = \max_{s\in\mathbb{N}_{a+1}^b} u(\rho(s),s). \tag{7}$$

Proof. It follows from Lemma 3.2 that

$$\max_{(t,s)\in\mathbb{N}_a^b\times\mathbb{N}_{a+1}^b} G(t,s) = \max_{s\in\mathbb{N}_{a+1}^b} \big\{u(\rho(s),s), v(s,s)\big\}. \tag{8}$$

We have

$$\begin{aligned} &u(\rho(s),s) \\ &\quad= \frac{1}{\xi}\Big[\alpha\gamma H_{\nu-1}(\rho(s),a)H_{\nu-1}(b,\rho(s)) + \alpha\delta H_{\nu-1}(\rho(s),a)H_{\nu-2}(b,\rho(s)) \\ &\qquad + (\beta-\alpha)\gamma H_{\nu-1}(b,\rho(s)) + (\beta-\alpha)\delta H_{\nu-2}(b,\rho(s))\Big], \quad s \in \mathbb{N}_{a+1}^b, \end{aligned} \tag{9}$$

and

$$\begin{aligned} v(s,s) = \frac{1}{\xi}\Big[&\alpha\gamma H_{\nu-1}(s,a)H_{\nu-1}(b,\rho(s)) + \alpha\delta H_{\nu-1}(s,a)H_{\nu-2}(b,\rho(s)) \\ &+ (\beta-\alpha)\gamma H_{\nu-1}(b,\rho(s)) + (\beta-\alpha)\delta H_{\nu-2}(b,\rho(s))\Big] - 1, \quad s \in \mathbb{N}_{a+1}^b. \end{aligned}$$

Consider

$$\begin{aligned} u(\rho(s),s) - v(s,s) = \frac{1}{\xi}\Big[&\alpha\gamma H_{\nu-1}(b,\rho(s))\Big(H_{\nu-1}(\rho(s),a) - H_{\nu-1}(s,a)\Big) \\ &+ \alpha\delta H_{\nu-2}(b,\rho(s))\Big(H_{\nu-1}(\rho(s),a) - H_{\nu-1}(s,a)\Big) \\ &+ (\beta-\alpha)\gamma + \alpha\gamma H_{\nu-1}(b,a) + \alpha\delta H_{\nu-2}(b,a)\Big]. \end{aligned} \tag{10}$$

From Proposition 2.9, we know that

$$H_{\nu-1}(s,a) - H_{\nu-1}(\rho(s),a) = H_{\nu-2}(s,a), \quad s \in \mathbb{N}_{a+1}^b. \tag{11}$$

Using (11) in (10), we obtain

$$\begin{aligned} u(\rho(s),s) - v(s,s) &= \frac{1}{\xi}\Big[-\alpha\gamma H_{\nu-1}(b,\rho(s))H_{\nu-2}(s,a) \\ &- \alpha\delta H_{\nu-2}(b,\rho(s))H_{\nu-2}(s,a) \\ &+ (\beta-\alpha)\gamma + \alpha\gamma H_{\nu-1}(b,a) + \alpha\delta H_{\nu-2}(b,a)\Big] \\ &= \frac{1}{\xi}\Big[\alpha\gamma\Big(H_{\nu-1}(b,a) - H_{\nu-1}(b,\rho(s))H_{\nu-2}(s,a)\Big) \\ &+ \alpha\delta\Big(H_{\nu-2}(b,a) - H_{\nu-2}(b,\rho(s))H_{\nu-2}(s,a)\Big) + (\beta-\alpha)\gamma\Big] \\ &= \frac{1}{\xi}\Big[\alpha\gamma A_1 + \alpha\delta A_2 + (\beta-\alpha)\gamma\Big], \end{aligned} \tag{12}$$

where

$$\begin{aligned} A_1 &= H_{\nu-1}(b,a) - H_{\nu-1}(b,\rho(s))H_{\nu-2}(s,a), \\ A_2 &= H_{\nu-2}(b,a) - H_{\nu-2}(b,\rho(s))H_{\nu-2}(s,a). \end{aligned}$$

We already know that α, β, γ, δ, $(\beta-\alpha)\geq 0$, and $\xi>0$ for all $t\in\mathbb{N}_a^b$. We show that

$$A_i \geq 0, \quad i=1,2.$$

From Proposition 2.11, we have

$$H_{\nu-1}(b,\rho(s)) \leq H_{\nu-1}(b,a) \text{ and } H_{\nu-2}(s,a)\leq 1, \qquad s\in\mathbb{N}_{a+1}^b. \tag{13}$$

Consequently, we obtain

$$\begin{aligned} A_1 &= H_{\nu-1}(b,a) - H_{\nu-1}(b,\rho(s))H_{\nu-2}(s,a) \\ &\geq H_{\nu-1}(b,a) - H_{\nu-1}(b,\rho(s)) \\ &\geq 0. \end{aligned} \tag{14}$$

Now, consider

$$\begin{aligned} A_2 &= H_{\nu-2}(b,a) - H_{\nu-2}(b,\rho(s))H_{\nu-2}(s,a) \\ &= H_{\nu-2}(b,a)\Big[1-\frac{H_{\nu-2}(b,\rho(s))H_{\nu-2}(s,a)}{H_{\nu-2}(b,a)}\Big] \\ &= H_{\nu-2}(b,a)\Big[1-\frac{(b-s+1)^{\overline{\nu-2}}(s-a)^{\overline{\nu-2}}}{(b-a)^{\overline{\nu-2}}\Gamma(\nu-1)}\Big]. \end{aligned} \tag{15}$$

Obviously, from Proposition 2.11, we have $H_{\nu-2}(b,a)>0$. Denote by

$$f(s) = \frac{(b-s+1)^{\overline{\nu-2}}(s-a)^{\overline{\nu-2}}}{(b-a)^{\overline{\nu-2}}\Gamma(\nu-1)}, \quad s\in\mathbb{N}_{a+1}^b. \tag{16}$$

It is enough to show that

$$f(s)\leq 1, \quad s\in\mathbb{N}_{a+1}^b. \tag{17}$$

Clearly, $f(a+1)=f(b)=1$. Now, consider

$$\begin{aligned} \nabla f(s) &= \frac{1}{(b-a)^{\overline{\nu-2}}\Gamma(\nu-1)}\nabla\Big[(b-s+1)^{\overline{\nu-2}}(s-a)^{\overline{\nu-2}}\Big] \\ &= \frac{(b-s+2)^{\overline{\nu-3}}(s-a)^{\overline{\nu-3}}}{(b-a)^{\overline{\nu-2}}\Gamma(\nu-1)}\big[(\nu-2)(b+a+2-2s)\big]. \end{aligned} \tag{18}$$

Clearly, $\Gamma(\nu-1)>0$, and from Proposition 2.11, we have

$$(b-a)^{\overline{\nu-2}} = \Gamma(\nu-1)H_{\nu-2}(b,a)>0.$$

Also, from Proposition 2.9, we have

$$(b-s+2)^{\overline{\nu-3}} = \frac{\Gamma(b-s+\nu-1)}{\Gamma(b-s+2)}>0, \quad s\in\mathbb{N}_{a+1}^b,$$

and

$$(s-a)^{\overline{\nu-3}} = \frac{\Gamma(s-a+\nu-3)}{\Gamma(s-a)}>0, \quad s\in\mathbb{N}_{a+2}^b.$$

Since $\nu-2<0$, it follows from (18) that

$$\nabla f(s)<0, \quad a+2\leq s\leq \left\lfloor\frac{b+a+2}{2}\right\rfloor,$$

and

$$\nabla f(s) > 0, \quad \left\lfloor \frac{b+a+2}{2} \right\rfloor \leq s \leq b.$$

That is, f is a decreasing function of s if $a+1 \leq s \leq \left\lfloor \frac{b+a+2}{2} \right\rfloor$ and it is an increasing function of s if $\left\lfloor \frac{b+a+2}{2} \right\rfloor \leq s \leq b$. In any case, we obtain (17). Thus, from (12), we have

$$v(s,s) \leq u(\rho(s),s), \quad s \in \mathbb{N}_{a+1}^{b}.$$

Therefore, from (8), we obtain

$$\max_{(t,s)\in \mathbb{N}_a^b \times \mathbb{N}_{a+1}^b} G(t,s) = \max_{s\in \mathbb{N}_{a+1}^b} \left\{ u(\rho(s),s), v(s,s) \right\} = \max_{s\in \mathbb{N}_{a+1}^b} u(\rho(s),s).$$

The proof is complete. □

Lemma 3.4. *Assume α, β, γ, $\delta \geq 0$ and $\beta \geq \alpha$ such that (2) holds. Then,*

$$\sum_{s=a+1}^{b} u(\rho(s),s) = \Lambda, \tag{19}$$

where

$$\Lambda = \frac{1}{\xi}\Big[\alpha\gamma H_{2\nu-1}(b,a+1) + \alpha\delta H_{2\nu-2}(b,a+1) + (\beta-\alpha)\gamma H_{\nu}(b,a) + (\beta-\alpha)\delta H_{\nu-1}(b,a)\Big]. \tag{20}$$

Proof. In view of (9), we have

$$\begin{aligned}
&\sum_{s=a+1}^{b} u(\rho(s),s) \\
&= \frac{1}{\xi}\Big[\alpha\gamma \sum_{s=a+1}^{b} H_{\nu-1}(\rho(s),a)H_{\nu-1}(b,\rho(s)) \\
&+ \alpha\delta \sum_{s=a+1}^{b} H_{\nu-1}(\rho(s),a)H_{\nu-2}(b,\rho(s)) \\
&+ (\beta-\alpha)\gamma \sum_{s=a+1}^{b} H_{\nu-1}(b,\rho(s)) + (\beta-\alpha)\delta \sum_{s=a+1}^{b} H_{\nu-2}(b,\rho(s))\Big] \\
&= \frac{1}{\xi}\Big[\alpha\gamma B_1 + \alpha\delta B_2 + (\beta-\alpha)\gamma B_3 + (\beta-\alpha)\delta B_4\Big].
\end{aligned} \tag{21}$$

where

$$B_1 = \sum_{s=a+1}^{b} H_{\nu-1}(\rho(s),a)H_{\nu-1}(b,\rho(s)),$$

$$B_2 = \sum_{s=a+1}^{b} H_{\nu-1}(\rho(s),a)H_{\nu-2}(b,\rho(s)),$$

$$B_3 = \sum_{s=a+1}^{b} H_{\nu-1}(b,\rho(s)),$$

$$B_4 = \sum_{s=a+1}^{b} H_{\nu-2}(b,\rho(s)).$$

Consider

$$\begin{aligned}B_1 &= \sum_{s=a+1}^{b} H_{\nu-1}(\rho(s),a)H_{\nu-1}(b,\rho(s))\\ &= \sum_{s=a+2}^{b} H_{\nu-1}(b,\rho(s))H_{\nu-1}(\rho(s),a) \quad \text{(Observe that } H_{\nu-1}(a,a)=0)\\ &= \nabla_{a+1}^{-\nu}H_{\nu-1}(b,a+1) = H_{2\nu-1}(b,a+1). \quad \text{(Using Proposition 2.10)} \qquad (22)\end{aligned}$$

Similarly, we obtain

$$B_2 = \sum_{s=a+1}^{b} H_{\nu-1}(\rho(s),a)H_{\nu-2}(b,\rho(s)) = H_{2\nu-2}(b,a+1). \qquad (23)$$

Also, from Proposition 2.10, we get

$$B_3 = \sum_{s=a+1}^{b} H_{\nu-1}(b,\rho(s)) = H_{\nu}(b,a), \qquad (24)$$

$$B_4 = \sum_{s=a+1}^{b} H_{\nu-2}(b,\rho(s)) = H_{\nu-1}(b,a). \qquad (25)$$

Using (22) - (25) in (21), we obtain (19). The proof is complete. □

Theorem 3.5. *Assume α, β, γ, $\delta \geq 0$ and $\beta \geq \alpha$ such that* (2) *holds. Then,*

$$\sum_{s=a+1}^{b} G(t,s) \leq \Lambda. \qquad (26)$$

Proof. The proof follows from Lemmas 3.3 and 3.4. To see this, consider

$$\sum_{s=a+1}^{b} G(t,s) \leq \sum_{s=a+1}^{b} u(\rho(s),s) = \Lambda.$$

□

4. Existence and Uniqueness of Solutions

In this section, we present some existence and uniqueness results for (6) using various fixed point theorems. Brackins [9] expressed the unique solution of the nonhomogeneous boundary value problem

$$\begin{cases} -\big(\nabla_a^{\nu-1}(\nabla u)\big)(t) = h(t), \quad t \in \mathbb{N}_{a+2}^{b},\\ \alpha u(a+1) - \beta(\nabla u)(a+1) = 0,\\ \gamma u(b) + \delta(\nabla u)(b) = 0,\end{cases} \qquad (27)$$

as follows.

Theorem 4.1 (See [9])**.** *Let $h : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$. If* (1) *has only the trivial solution, then the nonhomogeneous boundary value problem* (27) *has a unique solution given by*

$$u(t) = \sum_{s=a+1}^{b} G(t,s)h(s), \quad t \in \mathbb{N}_{a}^{b}. \qquad (28)$$

By Theorem 4.1, we observe that u is a solution of (6) if and only if u is a solution of the summation equation

$$u(t) = \sum_{s=a+1}^{b} G(t,s)f(s,u(s)), \quad t \in \mathbb{N}_a^b. \tag{29}$$

Note that any solution $u : \mathbb{N}_a^b \to \mathbb{R}$ of (6) can be viewed as a real $(b-a+1)$-tuple vector. Consequently,

$$u \in \mathbb{R}^{b-a+1}.$$

Define the operator $T : \mathbb{R}^{b-a+1} \to \mathbb{R}^{b-a+1}$ by

$$(Tu)(t) = \sum_{s=a+1}^{b} G(t,s)f(s,u(s)), \quad t \in \mathbb{N}_a^b. \tag{30}$$

Clearly, u is a fixed point of T if and only if u is a solution of (6). We use the fact that $\mathbb{R}^{b-a+1}$ is a Banach space equipped with the maximum norm defined by

$$\|u\| = \max_{t \in \mathbb{N}_a^b} |u(t)|.$$

A closed ball with radius R centered on the zero vector in $\mathbb{R}^{b-a+1}$ is defined by

$$B_R(0) = B_R = \left\{u \in \mathbb{R}^{b-a+1} : \|u\| \le R\right\}.$$

4.1. Assumptions

Assume

(H1) $f : \mathbb{N}_a^b \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2) f satisfies Lipschitz condition with respect to the second variable with Lipschitz constant K on $\mathbb{N}_a^b \times B_R$. That is, for all (t,u), $(t,v) \in \mathbb{N}_a^b \times B_R$,

$$\|f(t,u) - f(t,v)\| \le K\|u - v\|.$$

(H3) f satisfies Lipschitz condition with respect to the second variable with Lipschitz constant L on $\mathbb{N}_a^b \times \mathbb{R}$. That is, for all (t,u), $(t,v) \in \mathbb{N}_a^b \times \mathbb{R}$,

$$\|f(t,u) - f(t,v)\| \le L\|u - v\|.$$

(H4) For all $(t,u) \in \mathbb{N}_a^b \times B_R$,

$$|f(t,u)| \le M.$$

(H5) f is bounded on $\mathbb{N}_a^b \times \mathbb{R}$.

(H6) Take

$$\max_{t \in \mathbb{N}_a^b} |f(t,0)| = P,$$

and

$$\max_{(t,u) \in \mathbb{N}_a^b \times B_R} |f(t,u)| = Q.$$

(H7) $K\Lambda < 1$.

(H8) $L\Lambda < 1$.

4.2. Existence of Solutions

First, we apply Brouwer fixed point theorem to establish existence of solutions of (6).

Theorem 4.2. *[3] (Brouwer fixed point theorem) Let C be a nonempty bounded closed convex subset of $\mathbb{R}^n$ and $T : C \to C$ be a continuous mapping. Then, T has a fixed point in C.*

Theorem 4.3. *(Local Existence) Assume (H1) and (H4) hold. If we choose*

$$R \geq M\Lambda, \tag{31}$$

then the boundary value problem (6) *has a solution u in B_R.*

Proof. Clearly, B_R is a nonempty bounded closed convex subset of $\mathbb{R}^{b-a+1}$. First, we claim that $T : B_R \to B_R$. To see this, let $u \in B_R$, $t \in \mathbb{N}_a^b$ and consider

$$\begin{aligned}
|(Tu)(t)| &= \left| \sum_{s=a+1}^{b} G(t,s) f(s,u(s)) \right| \\
&\leq \sum_{s=a+1}^{b} G(t,s) |f(s,u(s))| \\
&\leq M \sum_{s=a+1}^{b} G(t,s) \quad \text{(Using (H4))} \\
&\leq M\Lambda \quad \text{(Using (26))} \\
&\leq R,
\end{aligned}$$

implying that $T : B_R \to B_R$. Since f is continuous, T is also continuous. Hence, by Theorem 4.2, the boundary value problem (6) has at least one solution u in B_R. The proof is complete. □

Theorem 4.4. *(Global Existence) Assume (H1) and (H5) hold. Then, the boundary value problem* (6) *has a solution u in $\mathbb{R}^{b-a+1}$.*

Proof. The proof is similar to the proof of Theorem 4.3. □

4.3. Existence and Uniqueness of Solutions

Next, we apply Banach fixed point theorem to establish existence and uniqueness of solutions of (6).

Theorem 4.5. *[3] (Banach fixed point theorem) Let B_r be the closed ball of radius $r > 0$, centred at zero, in $\mathbb{R}^n$ with $T : B_r \to \mathbb{R}^n$ a contraction mapping and $T(\partial B_r) \subseteq B_r$. Then, T has a unique fixed point in B_r.*

Theorem 4.6. *[3] (Banach fixed point theorem) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a contraction mapping. Then, T has a unique fixed point in $\mathbb{R}^n$.*

Theorem 4.7. *(Local Existence & Uniqueness) Assume (H1), (H2), (H6) and (H7) hold. If we choose*

$$R \geq \frac{P\Lambda}{1 - K\Lambda}, \tag{32}$$

or

$$R \geq Q\Lambda, \tag{33}$$

then the boundary value problem (6) *has a unique solution u in B_R.*

Proof. Clearly, $T : B_R \to \mathbb{R}^{b-a+1}$. Now, we claim that T is a contraction mapping. To see this, let u, $v \in B_R$, $t \in \mathbb{N}_a^b$ and consider

$$\begin{aligned}|(Tu)(t) - (Tv)(t)| &= \left|\sum_{s=a+1}^{b} G(t,s)\left[f(s,u(s)) - f(s,v(s))\right]\right| \\ &\le \sum_{s=a+1}^{b} G(t,s)\left|f(s,u(s)) - f(s,v(s))\right| \\ &\le K\|u-v\| \sum_{s=a+1}^{b} G(t,s) \quad \text{(Using (H2))} \\ &\le K\Lambda\|u-v\|, \quad \text{(Using (26))}\end{aligned}$$

implying that

$$\|Tu - Tv\| \le K\Lambda\|u-v\|.$$

Since $K\Lambda < 1$, it follows that T is a contraction mapping. Next, we show that $T(\partial B_R) \subseteq B_R$. To see this, let $u \in \partial B_R$, $t \in \mathbb{N}_a^b$ and consider

$$\begin{aligned}|(Tu)(t)| &= \left|\sum_{s=a+1}^{b} G(t,s) f(s,u(s))\right| \\ &\le \sum_{s=a+1}^{b} G(t,s)|f(s,u(s))| \\ &\le \sum_{s=a+1}^{b} G(t,s)|f(s,u(s)) - f(s,0)| + \sum_{s=a+1}^{b} G(t,s)|f(s,0)| \\ &\le K \sum_{s=a+1}^{b} G(t,s)|u(s)| + P \sum_{s=a+1}^{b} G(t,s) \quad \text{(Using (H2), (H6))} \\ &\le (KR + P)\Lambda \le R, \quad \text{(Using (32))}\end{aligned}$$

implying that $(Tu) \in B_R$. On the other hand, consider

$$\begin{aligned}|(Tu)(t)| &= \left|\sum_{s=a+1}^{b} G(t,s) f(s,u(s))\right| \\ &\le \sum_{s=a+1}^{b} G(t,s)|f(s,u(s))| \\ &\le Q \sum_{s=a+1}^{b} G(t,s) \quad \text{(Using (H6))} \\ &\le Q\Lambda \quad \text{(Using (26))} \\ &\le R,\end{aligned}$$

implying that $(Tu) \in B_R$. Thus, we have $T(\partial B_R) \subseteq B_R$. Hence, by Theorem 4.5, the boundary value problem (6) has unique solution u in B_R. The proof is complete. □

Theorem 4.8. *(Global Existence & Uniqueness) Assume (H1), (H3) and (H8) hold. Then, the boundary value problem* (6) *has a unique solution* u *in* $\mathbb{R}^{b-a+1}$.

Proof. We know that $T : \mathbb{R}^{b-a+1} \to \mathbb{R}^{b-a+1}$. We claim that T is a contraction mapping. To see this, let u, $v \in \mathbb{R}^{b-a+1}$, $t \in \mathbb{N}_a^b$ and consider

$$\begin{aligned}
|(Tu)(t) - (Tv)(t)| &= \left| \sum_{s=a+1}^{b} G(t,s)\left[f(s,u(s)) - f(s,v(s))\right] \right| \\
&\le \sum_{s=a+1}^{b} G(t,s)\left|f(s,u(s)) - f(s,v(s))\right| \\
&\le L\|u-v\| \sum_{s=a+1}^{b} G(t,s) \quad \text{(Using (H3))} \\
&\le L\Lambda\|u-v\|, \quad \text{(Using (26))}
\end{aligned}$$

implying that

$$\|Tu - Tv\| \le L\Lambda\|u-v\|.$$

Since $L\Lambda < 1$, it follows that T is a contraction mapping. Hence, by Theorem 4.6, the boundary value problem (6) has unique solution u in $\mathbb{R}^{b-a+1}$. The proof is complete. □

5. Examples

In this section we provide two examples to demonstrate the applicability of Theorem 4.8 and Theorem 4.7.

Example 5.1. *Consider the nabla fractional boundary value problem*

$$\begin{cases} -\left(\nabla_0^{0.5}(\nabla u)\right)(t) = (0.02)\sin u(t), & t \in \mathbb{N}_2^9, \\ u(0) + u(1) = 0, \\ u(8) + u(9) = 0. \end{cases} \tag{34}$$

Here $\alpha = 2$, $\beta = 1$, $\gamma = 2$ and $\delta = -1$ such that $\alpha^2 + \beta^2 > 0$ and $\gamma^2 + \delta^2 > 0$. Also, $a = 0$, $b = 9$ and $\nu = 1.5$. Clearly, $f(t,u) = (0.02)\sin u$ is continuous on $\mathbb{N}_0^9 \times \mathbb{R}$. Further, $f(t,u)$ satisfies Lipschitz condition with respect to u on $\mathbb{N}_0^9 \times \mathbb{R}$ with Lipschitz constant $L = 0.02$. Consider

$$\begin{aligned}
\xi &= (\beta - \alpha)\gamma + \alpha\gamma H_{\nu-1}(b,a) + \alpha\delta H_{\nu-2}(b,a) \\
&= -2 + 4H_{0.5}(9,0) - 2H_{-0.5}(9,0) \\
&= -2 + 4(3.3386) - 2(0.1964) = 10.9616,
\end{aligned}$$

and

$$\begin{aligned}
\Lambda &= \frac{1}{\xi}\Big[\alpha\gamma H_{2\nu-1}(b,a+1) + \alpha\delta H_{2\nu-2}(b,a+1) \\
&\quad + (\beta-\alpha)\gamma H_{\nu}(b,a) + (\beta-\alpha)\delta H_{\nu-1}(b,a)\Big] \\
&= \frac{1}{10.9616}\Big[4H_2(9,1) - 2H_1(9,1) + 2H_{1.5}(9,0) + H_{0.5}(9,0)\Big] \\
&= \frac{1}{10.9616}\Big[4(36) - 2(8) + 2(21.1443) + (3.3386)\Big] = 15.8396.
\end{aligned}$$

Since $L\Lambda = (0.02)(15.8396) = 0.3168 < 1$, by Theorem 4.8, the boundary value problem (34) has unique solution u in $\mathbb{R}^{10}$.

Example 5.2. *Consider the nabla fractional boundary value problem*

$$\begin{cases} -\left(\nabla_0^{0.5}(\nabla u)\right)(t) = (0.01)u^2(t), \quad t \in \mathbb{N}_2^6, \\ u(0) = u(6) = 0. \end{cases} \tag{35}$$

Here $\alpha = 1$, $\beta = 1$, $\gamma = 1$ and $\delta = 0$ such that $\alpha^2 + \beta^2 > 0$ and $\gamma^2 + \delta^2 > 0$. Also, $a = 0$, $b = 6$ and $\nu = 1.5$. Consider

$$\xi = (\beta - \alpha)\gamma + \alpha\gamma H_{\nu-1}(b, a) + \alpha\delta H_{\nu-2}(b, a) = H_{0.5}(6, 0) = 2.7071,$$

and

$$\begin{aligned} \Lambda &= \frac{1}{\xi}\Big[\alpha\gamma H_{2\nu-1}(b, a+1) + \alpha\delta H_{2\nu-2}(b, a+1) \\ &\quad + (\beta - \alpha)\gamma H_{\nu}(b, a) + (\beta - \alpha)\delta H_{\nu-1}(b, a)\Big] \\ &= \frac{1}{2.7071} H_2(6, 1) = 11.0820. \end{aligned}$$

Also, we have

$$P = \max_{t \in \mathbb{N}_0^6} |f(t, 0)| = 1,$$

and

$$Q = \max_{(t,u) \in \mathbb{N}_a^b \times B_R} |f(t, u)| = (0.01)R^2.$$

Clearly, $f(t, u) = (0.01)u^2$ is continuous on $\mathbb{N}_0^6 \times B_R$. Further, $f(t, u)$ satisfies Lipschitz condition with respect to u on $\mathbb{N}_0^6 \times B_R$ with Lipschitz constant $K = (0.02)R$. To apply Theorem 4.7, we must have

1. $K\Lambda = (0.02)(11.0820)R = (0.2216)R < 1$;
2. $R \geq \frac{11.0820}{1-(0.2216)R}$ or $R \geq (0.1108)R^2$.

Clearly, $R = 4.5$ satisfies the first and the second part of the second inequalities. Thus, by Theorem 4.7, the boundary value problem (35) has unique solution u in $B_{4.5}$.

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