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Research Article

An efficient numerical method for solving nonlinear astrophysics equations of arbitrary order

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Abstract

The Lane-Emden type equations of arbitrary (fractional and integer) order and the white dwarf equation are employed in the modeling of several phenomena in the areas of mathematical physics and astrophysics. In this paper, an efficient numerical algorithm based on the generalized fractional order of the Chebyshev orthogonal functions (GFCFs) and the collocation method to solve these well-known differential equations is presented. The operational matrices of the fractional derivative and the product of order α in the Caputo's definition for the GFCFs are used. The obtained results are compared with other results to verify the accuracy and efficiency of the presented method. The obtained numerical results are better than other proposed methods.

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Keywords. fractional order of the Chebyshev functions, Lane-Emden type equations, operational matrix, Tau-Collocation method, white dwarf equation

1. Introduction

In this section, Spectral methods and some preliminary which are useful for our method have been introduced.

1.1. Spectral methods

Spectral methods have been developed rapidly in the past two decades. They have been successfully applied to numerical simulations in many fields, such as heat conduction, fluid dynamics, quantum mechanics, etc. These methods are powerful tools to solve differential equations. The key components of their formulation are the trial functions and the test functions. The trial functions, which are the linear combinations of suitable trial basis functions, are used to provide an approximate representation of the solution. The test functions are used to ensure that the differential equation and perhaps some boundary conditions are satisfied as closely as possible by the truncated series expansion. This is achieved by minimizing the residual function that is produced by using the truncated expansion instead of the exact solution with respect to a suitable norm [4, 29, 36, 48, 54].

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1.2. Fractional calculus

The original ideas of fractional calculus can be traced back to the end of the 17th century, when the classical differential and integral calculus theories were created and developed by Newton and Leibniz in 1695 [25], but for many reasons were not used in sciences for many years, for example, there were the various definitions of the fractional derivative [39] and there were no the exact geometrical interpretation for them [40]. A review of some definitions and applications of fractional derivatives is given in [24] and [12]. In recent years, many physicists and mathematicians have considered on this subject, and found the various applications for the fractional calculus [13, 41]. For example, the nonlinear oscillation of earthquake [18], and the fractional optimal control problems for dynamic systems [11, 22, 53]. Several methods have been used to solve fractional differential equations, such as Bernoulli polynomials [22], Legendre multi-wavelet collocation method [53], the modified Jacobi polynomials [11], Adomian's decomposition method [30], fractional-order Legendre functions [21], fractional-order Chebyshev functions of the second kind [8], Homotopy analysis method [17], Bessel functions and spectral methods [31], Legendre and Bernstein polynomials [42], and other methods [14, 43].

One of the most important definitions of fractional derivatives is the fractional derivative in the Caputo sense. This definition is very important for some reasons, such as simplicity in comparison with the other definitions, similarity to ordinary derivative, apply the initial conditions in solution to easily. For these reasons, many researchers have used this definition in your papers [2, 46, 47].

The paper is organized as follows: in Section 1.3, the Lane-Emden type equations of fractional order are described. In Section 2, some basic definitions and theorems of fractional calculus are defined. In Section 3, the GFCFs and their properties are expressed. Section 4 is devoted to applying the GFCFs operational matrices of the fractional derivative and the product to obtain the solution of fractional differential equations. In Section 5, the work method is explained. Applications of the method are shown in Section 6. Finally, a conclusion is provided.

1.3. The Lane-Emden type equations

The study of singular initial value problems modeled by second-order nonlinear ordinary differential equations has attracted many mathematicians and physicists. One of the equations in this category is the following Lane-Emden type equation:

$$y''(t) + \frac{k}{t}y'(t) + f(t, y(t)) = g(t), \qquad k, t > 0,$$

with the initial conditions:

$$y(0) = d_0, \qquad y'(0) = d_1,$$

where k, d_0 and d_1 are real constants and $f(t, y)$ and $g(t)$ are some given continuous real-valued functions. For special forms of $f(t, y)$, the well-known Lane-Emden equations occur in several models of non-Newtonian fluid mechanics, mathematical physics, astrophysics, etc. For example, when $f(t, y) = q(y)$, the Lane-Emden equations occur in modeling several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas sphere and theory of thermionic currents [7, 9, 27, 32].

Recently, the Lane-Emden equations of the fractional order have been investigated by some researchers. In this paper, we have considered the Lane-Emden equations of fractional order as follows [1, 28]:

$$D^{\gamma}y(t) + \frac{k}{t^{\gamma-\beta}}D^{\beta}y(t) + f(t, y(t)) = g(t), \tag{1.1}$$

with the initial conditions:

$$y(0) = d_0, \qquad y'(0) = d_1, \tag{1.2}$$

where $0 < t \leq 1,\ 1 < \gamma \leq 2,\ 0 < \beta \leq 1$, and $k,\ d_0,\ d_1$ are real constants, $f(t, y)$ and $g(t)$ are some given continuous real-valued functions.

Some researchers have obtained approximations for Lane-Emden equations of fractional order, for example Saeed (2017) by using the Haar Adomian method [45], Akgul et al. (2015) by using the reproducing kernel method [1], Mechee and Senu (2012) by using the collocation method [28], Marasi et al (2015) by using the modified differential transform method [26], Yuzbasi (2011) by using the Bessel collocation method [55], Ibrahim (2011), Fazly and Wei (2015), and Davila et al (2014) by using the analytical methods [10, 15, 19].

There are equations that are convertible into the Lane-Emden type equations, for example, Van Gorder has examined the relation between Lane-Emden solutions and radial solutions to the elliptic Heavenly equation on a disk [51].

2. Basic definitions

In this section, some basic definitions and theorems which are useful for our method have been introduced.

Definition 2.1. For any real function $f(t)$, $t > 0$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0, \infty)$, is said to be in space C_μ, $\mu \in \Re$, and it is in the space C_μ^n if and only if $f^{(n)} \in C_\mu,\ n \in N$.

Definition 2.2. The fractional derivative of $f(t)$ in the Caputo sense by the Riemann-Liouville fractional integral operator of order $\alpha > 0$ is defined as follows [23]

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} D^m f(s) ds, \qquad \alpha > 0,$$

for $m - 1 < \alpha \leq m,\ m \in N,\ t > 0$ and $f \in C_{-1}^m$.

So, for $f \in C_\mu, \mu \geq -1,\ \alpha, \beta \geq 0,\ \gamma \geq -1,\ N_0 = \{0, 1, 2, ...\}$ and constant C, we have:

(i) $D^\alpha C = 0$,

(ii) $D^\alpha D^\beta f(t) = D^{\alpha+\beta} f(t)$,

(iii)

$$D^\alpha t^\gamma = \begin{cases} 0 & \gamma \in N_0 \text{ and } \gamma < \alpha, \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, & \text{Otherwise.} \end{cases} \tag{2.1}$$

(iv)

$$D^\alpha (\sum_{i=1}^n c_i f_i(t)) = \sum_{i=1}^n c_i D^\alpha f_i(t), \qquad \text{where } c_i \in R. \tag{2.2}$$

Definition 2.3. Suppose that $f(t) \in C(0, \eta]$ and $w(t)$ is a weight function, then

$$\| f(t) \|_w^2 \ = \ \int_0^\eta f^2(t) w(t) dt.$$

Theorem 2.4. *Suppose that $\{P_i(t)\}$ be a sequence of orthogonal polynomials, $w(t)$ is a weight function for $\{P_i(t)\}$, and $q(t)$ is a polynomial of degree at most $n-1$, then for $p_n(t) \in \{P_i(t)\}$ we have: $\langle p_n(t), q(t) \rangle_w = 0$.*

Proof. See Section 2.3 in [50]. □

3. Generalized fractional order of the Chebyshev functions

Chebyshev polynomials have many properties, for example orthogonal, recursive, simple real roots, complete in the space of polynomials. For these reasons, many authors have used these functions in their works [3,16,20,44].

Using some conversions, the number of the researchers extended Chebyshev polynomials to semi-infinite or infinite interval, for example by using $x = \frac{t-L}{t+L}, L > 0$ the rational Chebyshev functions on the semi-infinite interval [33,34], by using $x = \frac{t}{\sqrt{t^2+L}}, L > 0$ the rational Chebyshev functions on infinite interval [5], and by using $x = 1 - 2(\frac{t}{\eta})^\alpha$, $\alpha, \eta > 0$ the generalized fractional order of the Chebyshev functions (GFCF) on the finite interval $[0, \eta]$ [37,38] are introduced.

Darani and Nasiri in [8] have been introduced the fractional-order Chebyshev functions of the second kind, then just constructed the derivative operational matrix for them, and used it to solve linear fractional differential equations.

In the present work, the transformation $x = 1 - 2(\frac{t}{\eta})^\alpha$, $\alpha, \eta > 0$ on the Chebyshev polynomials of the first kind is used. The transformation was introduced in [37] and can use to solve nonlinear and linear fractional differential equations, such as the Lane-Emden type equations.

The GFCFs are defined on interval $[0, \eta]$, are shown by ${}_\eta FT_n^\alpha(t) = T_n(1 - 2(\frac{t}{\eta})^\alpha)$, and have the following analytical form [37]:

$$ {}_\eta FT_n^\alpha(t) = \sum_{k=0}^{n} \beta_{n,k,\eta,\alpha}\ t^{\alpha k}, \qquad t \in [0, \eta], \tag{3.1}$$

where

$$\beta_{n,k,\eta,\alpha} = (-1)^k \frac{n 2^{2k}(n+k-1)!}{(n-k)!(2k)!\eta^{\alpha k}} \qquad \textit{and} \qquad \beta_{0,k,\eta,\alpha} = 1.$$

The GFCFs are orthogonal with respect to the weight function $w(t) = \dfrac{t^{\frac{\alpha}{2}-1}}{\sqrt{\eta^\alpha - t^\alpha}}$ on the interval $(0, \eta)$:

$$\int_0^\eta {}_\eta FT_n^\alpha(t)\ {}_\eta FT_m^\alpha(t) w(t) dt = \frac{\pi}{2\alpha} c_n \delta_{mn}, \tag{3.2}$$

where δ_{mn} is Kronecker delta, $c_0 = 2$, and $c_n = 1$ for $n \geq 1$.

Any function $y(t) \in C[0, \eta]$ can be expanded as follows:

$$y(t) = \sum_{n=0}^{\infty} a_n\ {}_\eta FT_n^\alpha(t),$$

and using the property of orthogonality in the GFCFs:

$$a_n = \frac{2\alpha}{\pi c_n} \int_0^\eta {}_\eta FT_n^\alpha(t) y(t) w(t) dt, \qquad n = 0, 1, 2, \cdots,$$

but in the numerical methods, we have to use first m-terms of the GFCFs and approximate $y(t)$:

$$y(t) \approx y_m(t) = \sum_{n=0}^{m-1} a_n\ {}_\eta FT_n^\alpha(t) = A^T \Phi(t), \tag{3.3}$$

with

$$A = [a_0, a_1, ..., a_{m-1}]^T, \tag{3.4}$$

$$\Phi(t) = [\ {}_\eta FT_0^\alpha(t),\ {}_\eta FT_1^\alpha(t), ...,\ {}_\eta FT_{m-1}^\alpha(t)]^T. \tag{3.5}$$

The following theorem shows that by increasing m, the approximation solution $f_m(t)$ is convergent to $f(t)$ exponentially.

Theorem 3.1. *Suppose that $D^{k\alpha}f(t) \in C[0,\eta]$ for $k = 0, 1, ..., m$, and ${}_\eta F_m^\alpha$ is the generated subspace by $\{{}_\eta FT_0^\alpha(t), {}_\eta FT_1^\alpha(t), \cdots, {}_\eta FT_{m-1}^\alpha(t)\}$. If $f_m(t) = A^T\Phi(t)$ (in Eq. (3.3)) is the best approximation to $f(t)$ from ${}_\eta F_m^\alpha$, then the error bound is presented as follows*

$$\| f(t) - f_m(t) \|_w \leq \frac{\eta^{m\alpha} M_\alpha}{2^m \Gamma(m\alpha + 1)} \sqrt{\frac{\pi}{\alpha\ m!}},$$

where $M_\alpha \geq |D^{m\alpha} f(t)|, \quad t \in [0, \eta]$.

Proof. See [37]. □

Theorem 3.2. *The generalized fractional order of the Chebyshev function ${}_\eta FT_n^\alpha(t)$, has precisely n real zeros on interval $(0, \eta)$ in the form*

$$t_k = \eta\left(\frac{1 - \cos(\frac{(2k-1)\pi}{2n})}{2}\right)^{\frac{1}{\alpha}}, \qquad k = 1, 2, \cdots, n.$$

Moreover, $\frac{d}{dt}{}_\eta FT_n^\alpha(t)$ has precisely $n - 1$ real zeros on interval $(0, \eta)$ in the following points:

$$t'_k = \eta\left(\frac{1 - \cos(\frac{k\pi}{n})}{2}\right)^{\frac{1}{\alpha}}, \qquad k = 1, 2, \cdots, n - 1.$$

Proof. See [37]. □

4. Operational matrices of the GFCFs

In this section, operational matrices of the fractional derivative and the product for the GFCFs are constructed, these matrices can be used to solve the linear and nonlinear fractional differential equations.

4.1. The fractional derivative operational matrix of GFCFs

In the next theorem, the operational matrix of the Caputo fractional derivative of order $\alpha > 0$ for the GFCFs is generalized, which can be expressed by:

$$D^\alpha \Phi(t) = \mathbf{D}^{(\alpha)} \Phi(t). \tag{4.1}$$

Theorem 4.1. *Let $\Phi(t)$ be GFCFs vector in Eq. (3.5) and $\boldsymbol{D}^{(\alpha)}$ be an $m \times m$ operational matrix of the Caputo fractional derivatives of order $\alpha > 0$, then:*

$$\boldsymbol{D}_{i,j}^{(\alpha)} = \begin{cases} \frac{2}{\sqrt{\pi} c_j} \sum_{k=1}^{i} \sum_{s=0}^{j} \beta_{i,k,\eta,\alpha} \beta_{j,s,\eta,\alpha} \frac{\Gamma(\alpha k+1)\Gamma(s+k-\frac{1}{2})\eta^{\alpha(k+s-1)}}{\Gamma(\alpha k-\alpha+1)\Gamma(s+k)}, & i > j \\ 0, & \textit{otherwise} \end{cases} \tag{4.2}$$

for $i, j = 0, 1, ..., m - 1$.

Proof. Using Eq. (4.1):

$$\begin{bmatrix} \mathbf{D}_{0,0} & \cdots & \mathbf{D}_{0,j} & \cdots & \mathbf{D}_{0,m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{D}_{i,0} & \cdots & \mathbf{D}_{i,j} & \cdots & \mathbf{D}_{i,m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{D}_{m-1,0} & \cdots & \mathbf{D}_{m-1,j} & \cdots & \mathbf{D}_{m-1,m-1} \end{bmatrix} \begin{bmatrix} \Phi_0 \\ \vdots \\ \Phi_j \\ \vdots \\ \Phi_{m-1} \end{bmatrix} = \begin{bmatrix} D^\alpha\Phi_0 \\ \vdots \\ D^\alpha\Phi_i \\ \vdots \\ D^\alpha\Phi_{m-1} \end{bmatrix}.$$

By orthogonality property of the GFCFs and Eqs. (2.1) and (3.1), for $i, j = 0, 1, ..., m-1$, we have:

$$\mathbf{D}_{i,j}^{(\alpha)} = \frac{2\alpha}{\pi c_j} \int_0^\eta D^\alpha({}_\eta FT_i^\alpha(t))({}_\eta FT_j^\alpha(t)) w(t) dt.$$

Since $D^\alpha FT_0^\alpha(t)=0$, therefore $\mathbf{D}_{0,j}^{(\alpha)}=\int_0^\eta D^\alpha FT_0^\alpha(t)FT_j^\alpha(t)w(t)dt=0$.

If $i\le j$ then $deg(D^\alpha({}_\eta FT_i^\alpha(t)))<deg({}_\eta FT_j^\alpha(t))$, therefore by Theorem 2.4, $\mathbf{D}_{i,j}^{(\alpha)}=0$ for any $i\le j$. Now for $i>j$ we have:

$$\begin{aligned}\mathbf{D}_{i,j}^{(\alpha)} &= \frac{2\alpha}{\pi c_j}\int_0^\eta \sum_{k=1}^{i}\beta_{i,k,\eta,\alpha}\frac{\Gamma(\alpha k+1)t^{\alpha k-\alpha}}{\Gamma(\alpha k-\alpha+1)}\sum_{s=0}^{j}\beta_{j,s,\eta,\alpha}t^{\alpha s}\frac{t^{\frac{\alpha}{2}-1}}{\sqrt{\eta^\alpha-t^\alpha}}dt\\ &= \frac{2\alpha}{\pi c_j}\sum_{k=1}^{i}\sum_{s=0}^{j}\beta_{i,k,\eta,\alpha}\beta_{j,s,\eta,\alpha}\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k-\alpha+1)}\int_0^\eta\frac{t^{\alpha(k+s-\frac{1}{2})-1}}{\sqrt{\eta^\alpha-t^\alpha}}dt,\end{aligned}$$

by integration of the above equation, the theorem can be proved. □

Remark 4.2. The fractional derivative operational matrix of the GFCFs for $\alpha=\eta=1$ is same as the operational matrix of the shifted Chebyshev polynomials [6].

Remark 4.3. By Theorem 4.1, we can see that the fractional derivative operational matrix of the GFCFs is a lower triangular matrix, so at least $50(1+\frac{1}{m})\%$ of the matrix elements are zero, that this reduces the computational and storage costs.

4.2. The product operational matrix of GFCFs

The following property of the product of two GFCFs vectors will also be applied:

$$\Phi(t)\Phi(t)^T A\approx\widehat{\mathbf{A}}\Phi(t), \tag{4.3}$$

where $\widehat{\mathbf{A}}$ is an $m\times m$ product operational matrix for the vector $A=\{a_i\}_{i=0}^{m-1}$.

Theorem 4.4. *Let $\Phi(t)$ be GFCFs vector in Eq. (3.5) and A be a vector, then the elements of $\widehat{\mathbf{A}}$ are obtained as*

$$\widehat{\mathbf{A}}_{ij}=\sum_{k=0}^{m-1}a_k\widehat{g_{ijk}}, \tag{4.4}$$

where

$$\widehat{g}_{ijk}=\begin{cases}\frac{c_k}{2c_j}, & i\neq 0 \textit{ and } j\neq 0 \textit{ and } (k=i+j \textit{ or } k=|i-j|)\\ \frac{c_k}{c_j}, & (j=0 \textit{ and } k=i) \textit{ or } (i=0 \textit{ and } k=j)\\ 0, & \textit{otherwise}\end{cases}$$

Proof. Using Eq. (4.3):

$$\begin{bmatrix}\Phi_0\\ \vdots\\ \Phi_i\\ \vdots\\ \Phi_{m-1}\end{bmatrix}[\Phi_0\cdots\Phi_k\cdots\Phi_{m-1}]\begin{bmatrix}a_0\\ \vdots\\ a_k\\ \vdots\\ a_{m-1}\end{bmatrix}\approx\begin{bmatrix}\widehat{\mathbf{A}}_{0,0} & \cdots & \widehat{\mathbf{A}}_{0,m-1}\\ \vdots & \cdots & \vdots\\ \widehat{\mathbf{A}}_{i,0} & \cdots & \widehat{\mathbf{A}}_{i,m-1}\\ \vdots & \cdots & \vdots\\ \widehat{\mathbf{A}}_{m-1,0} & \cdots & \widehat{\mathbf{A}}_{m-1,m-1}\end{bmatrix}\begin{bmatrix}\Phi_0\\ \vdots\\ \Phi_j\\ \vdots\\ \Phi_{m-1}\end{bmatrix}.$$

By the orthogonal property of Eq. (3.2), the elements $\{\widehat{\mathbf{A}}_{ij}\}_{i,j=0}^{m-1}$ can be calculated from

$$\widehat{\mathbf{A}}_{ij}=\frac{2\alpha}{\pi c_j}\sum_{k=0}^{m-1}a_k g_{ijk}, \tag{4.5}$$

where g_{ijk} is given by

$$g_{ijk}=\int_0^\eta {}_\eta FT_i^\alpha(t)\ {}_\eta FT_j^\alpha(t)\ {}_\eta FT_k^\alpha(t)w(t)dt.$$

Using property: ${}_\eta FT_i^\alpha(t)\ {}_\eta FT_j^\alpha(t) = \frac{1}{2}(\ {}_\eta FT_{i+j}^\alpha(t) +\ {}_\eta FT_{|i-j|}^\alpha(t))$ and substituting it in g_{ijk}, we have:

$$g_{ijk} = \begin{cases} \frac{\pi c_k}{4\alpha}, & i \neq 0 \ and \ j \neq 0 \ and \ (k = i+j \ or \ k = |i-j|) \\ \frac{\pi c_k}{2\alpha}, & (j = 0 \ and \ k = i) \ or \ (i = 0 \ and \ k = j) \\ 0, & otherwise \end{cases}$$

Now by using Eq. (4.5), the theorem can be proved. □

Remark 4.5. The product operational matrix of GFCFs is same as the shifted Chebyshev polynomials [6]. In whole, it can be said that the components of $\widehat{\mathbf{A}}$ are independent of values of α and η.

Remark 4.6. By Theorem 4.4 it can be shown that, any product operational matrix of GFCFs can be made to the sum of two simple matrices: $\widehat{\mathbf{A}} = \frac{1}{2}\widehat{\mathbf{B}} + \frac{1}{2}\widehat{\mathbf{C}}$ where

(1) $\widehat{\mathbf{B}}$ is a symmetric matrix, that the diagonals of $\widehat{\mathbf{B}}$ are the elements of vector A, and for $i, j = 0, 1, ..., m-1$, matrix elements are calculated as follows:

$$\widehat{\mathbf{B}_{i,j}} = \begin{cases} 2a_0, & i = j \\ a_k, & i \neq j, \ |i-j| = k. \end{cases}$$

(2) $\widehat{\mathbf{C}}$ is a sparse matrix, that at least $50(1 + \frac{1}{m})\%$ of the matrix elements are zero, and for $i, j = 0, 1, ..., m-1$, matrix elements are calculated as follows:

$$\widehat{\mathbf{C}_{i,j}} = \begin{cases} a_{i+j}, & j \neq 0, \ 1 \leq i+j \leq m-1 \\ 0, & otherwise. \end{cases}$$

For example, with $m = 5$, at least 60% of the elements in $\widehat{\mathbf{C}}$ are zero, and the product operational matrix of GFCFs is as follows:

$$\begin{aligned} \widehat{\mathbf{A}} &= \frac{1}{2}\begin{bmatrix} 2a_0 & 2a_1 & 2a_2 & 2a_3 & 2a_4 \\ a_1 & 2a_0 + a_2 & a_1 + a_3 & a_2 + a_4 & a_3 \\ a_2 & a_1 + a_3 & 2a_0 + a_4 & a_1 & a_2 \\ a_3 & a_2 + a_4 & a_1 & 2a_0 & a_1 \\ a_4 & a_3 & a_2 & a_1 & 2a_0 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 2a_0 & a_1 & a_2 & a_3 & a_4 \\ a_1 & 2a_0 & a_1 & a_2 & a_3 \\ a_2 & a_1 & 2a_0 & a_1 & a_2 \\ a_3 & a_2 & a_1 & 2a_0 & a_1 \\ a_4 & a_3 & a_2 & a_1 & 2a_0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ 0 & a_2 & a_3 & a_4 & 0 \\ 0 & a_3 & a_4 & 0 & 0 \\ 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

We can see that the computational cost of production is very low.

5. Application of the GFCFs collocation method

In this section, the GFCFs collocation method to solve the Lane-Emden type equations of fractional order for various values of $f(t, y)$, $g(t)$, d_0, d_1 and k is applied.

To apply the collocation method, the residual function by substituting $y_m(t) = A^T\Phi(t)$ in Eq. (3.3) for $y(t)$ in Lane-Emden type Eq. (1.1) is constructed:

$$Res(t) = A^T\mathbf{D}^{(\gamma)}\Phi(t) + \frac{k}{t^{\gamma-\beta}}A^T\mathbf{D}^{(\beta)}\Phi(t) + f(t, A^T\Phi(t)) - g(t), \tag{5.1}$$

where $\mathbf{D}^{(\gamma)}$, $\mathbf{D}^{(\beta)}$ are defined in Eq. (4.1).

We now must choose the value of α such that γ and β be multiples of α. By using the properties of the operator D^α (Definition 2.2), we can calculate the values $\mathbf{D}^{(\gamma)}$ and $\mathbf{D}^{(\beta)}$:

$$D^\gamma y(t) \approx \sum_{n=0}^{m-1} a_n D^\gamma({}_\eta FT_n^\alpha(t)) = A^T \mathbf{D}^{(\gamma)} \Phi(t),$$
$$D^\beta y(t) \approx \sum_{n=0}^{m-1} a_n D^\beta({}_\eta FT_n^\alpha(t)) = A^T \mathbf{D}^{(\beta)} \Phi(t).$$

The equations for obtaining the coefficient $\{a_i\}_{i=0}^{m-1}$ arise from equalizing $Res(t)$ to zero on $m-2$ collocation points:

$$Res(t_i) = 0, \qquad i = 1, 2, ..., m-2, \tag{5.2}$$

and the initial conditions

$$A^T \Phi(0) = d_0,$$
$$A^T \mathbf{D}^{(1)} \Phi(0) = d_1. \tag{5.3}$$

In this study, the roots of the GFCFs in the interval $[0, \eta]$ (Theorem 3.2) as collocation points are used. By solving the obtained set of equations, the approximating function $y_m(t)$ is obtained.

Tau-Collocation algorithm

To obtain the Spectral coefficients $\{a_i\}_{i=0}^{m-1}$ in Eq. (3.3) and an approach of $y(t)$, the Tau-Collocation algorithm is employed. In this algorithm, to solve equation $Ly(t) = g(t)$, where L is the operator of the differential or integral equation, we do:

BEGIN

1. Calculate the operational matrix $\mathbf{D}^{(\alpha)}$ by Eq. (4.1).
2. Calculate the operational matrix $\widehat{\mathbf{A}}$ by Eq. (4.3) (If necessary).
3. Construct a series (3.3).
4. Insert the constructed series of step 3 into the equation $Ly(t) = g(t)$.
5. Construct the residual function as follows: $Res(t; a_0, a_1, ..., a_{m-1}) = L\widehat{y_m}(t) - g(t)$. We now have m unknown coefficients $\{a_i\}_{i=0}^{m-1}$. To obtain these unknown coefficients, we need m equations.
6. Choose m points $t_i, i = 0, 1, ..., m-1$ in the domain of the problem as collocation points and substituting them in $Res(t; a_0, a_1, ..., a_{m-1}) = L\widehat{y_m}(t) - g(t)$, and using the initial conditions, we construct a system which contains m nonlinearly or linearly independent equations.
7. Solve this system of equations by a suitable method (e.g. Newton's method) to find the $\{a_i\}_{i=0}^{m-1}$.

END.

In steps 1 and 2, according to Eqs. (4.1) and (4.3), the orders of complexity are $O(m^4)$ and $O(m^2)$, respectively. In step 4, due to the presence of the matrix multiplication and nonlinear functions, the order of complexity is at least $O(m^3)$, it should be noted that the order of complexity will be changed by changing the nonlinear functions. The order of complexity in step 6 is $O(m)$. The order of complexity in step 7 is dependent on the method of choice. It is worthwhile to note that it is common to solve a system of nonlinear equations, is applying Newton's method. The main difficulty with such a system is how we can choose an initial approximation to handle Newton's method. We have had reason to believe that the best way to discover the proper initial approximation (or initial approximations) is to solve the system analytically for the very small m (by means of symbolic software programs, such as Mathematica or Maple) and, then, we can find proper initial approximations, and particularly the multiplicity of solutions of such system. This action has been done by starting from proper initial approximations with the maximum number of ten iterations. In the present method, due to be added the fractional power, the order of complexity increases, but in many differential equations, the accuracy of computations increases with m less. Thus, the order of complexity in the above algorithm is at least $O(m^4)$.

6. Illustrative examples

In this section, by using the present method we solve some well-known examples to show efficiently and applicability GFCFs method based on Spectral methods. We apply the present method to solve the Lane-Emden equations of fractional order, and their outputs are compared with the corresponding analytical solutions. furthermore, consider that all of the computations have been done by Maple 2015.

Example 6.1. We consider linear Lane-Emden equation of fractional order as follows [1, 28]

$$D^{\gamma}y(t) + \frac{k}{t^{\gamma-\beta}}D^{\beta}y(t) + \frac{1}{t^{\gamma-2}}y(t) = g(t), \tag{6.1}$$

with the initial conditions

$$y(0) = 0, \qquad y'(0) = 0, \tag{6.2}$$

where

$$g(t) = t^{2-\gamma}\left(6t\left(\frac{t^2}{6} + \frac{\Gamma(4-\beta)+k\Gamma(4-\gamma)}{\Gamma(4-\beta)\Gamma(4-\gamma)}\right) - 2\left(\frac{t^2}{2} + \frac{\Gamma(3-\beta)+k\Gamma(3-\gamma)}{\Gamma(3-\beta)\Gamma(3-\gamma)}\right)\right),$$

and $\gamma = \frac{3}{2}$, $\beta = \frac{1}{2}$, $k = 2$. The exact solution of Eqs. (6.1) and (6.2) is $y(t) = t^3 - t^2$ [1, 28].

By applying the technique described in the last section, the residual function is as follows:

$$Res(t) = A^T\mathbf{D}^{(\frac{3}{2})}\Phi(t) + \frac{2}{t}A^T\mathbf{D}^{(\frac{1}{2})}\Phi(t) + \frac{1}{t^{-\frac{1}{2}}}A^T\Phi(t) - g(t).$$

For $\alpha = 0.5$, we have $\mathbf{D}^{(\gamma)} = \mathbf{D}^{(0.5)}\mathbf{D}^{(0.5)}\mathbf{D}^{(0.5)}$ and $\mathbf{D}^{(\beta)} = \mathbf{D}^{(0.5)}$.

Choosing $m-2$ the roots of the GFCFs in the interval $[0,\eta]$, as collocation points and substituting them in $Res(t)$, and using of initial conditions, a set of m algebraic equations is generated.

With $m = 7$ and $\alpha = 0.50$, we can obtain the exact solution with:

$$A = \begin{bmatrix} -0.0478515625 \\ 0.05078125000 \\ 0.02294921875 \\ -0.0449218750 \\ 0.02441406250 \\ -0.0058593750 \\ 0.00048828125 \end{bmatrix}$$

Table 1 shows the comparison of the absolute error obtained by the present method, the reproducing kernel method (RKM) [1] and the collocation method [28]. It can be seen that the present method is more accurate than other methods.

Table 1. Comparison of present method, the reproducing kernel method [1], and the collocation method [28] of the absolute errors for example 6.1

t	Akgul [1]	Mechee [28]	Present	Akgul [1]	Mechee [28]	Present
	$m=5$			$m=10$		
0.25	8.7370e-4	1.3345e-3	3.9203e-3	8.4636e-6	1.3232e-5	2.742e-47
0.50	9.9000e-4	1.5000e-3	2.4963e-2	2.9000e-6	2.6000e-5	2.373e-47
0.75	7.6702e-4	5.0673e-3	1.7490e-2	8.5754e-6	1.5634e-6	1.035e-47
1.00	5.4736e-4	3.6339e-3	5.3771e-2	5.4345e-6	4.1443e-5	4.083e-48

Example 6.2. We consider linear Lane-Emden equation of fractional order as follows [26]

$$D^{\gamma}y(t) + \frac{2}{t}y'(t) + y(t) = 0, \qquad 0 < \gamma \leq 2, \tag{6.3}$$

with the initial conditions

$$y(0) = 1, \qquad y'(0) = 0, \tag{6.4}$$

The exact solution of Eqs. (6.3) and (6.4) for $\gamma = 2$ is given as [26]

$$y(t) = \frac{\sin(t)}{t}, \tag{6.5}$$

and for $\gamma = 1$ is given as

$$y(t) = (1 + \frac{t}{2})^2 e^{-t}. \tag{6.6}$$

By applying the technique described in the last section, the residual function is as follows:

$$Res(t) = A^T\mathbf{D}^{(\gamma)}\Phi(t) + \frac{2}{t}A^T\mathbf{D}^{(1)}\Phi(t) + A^T\Phi(t).$$

Choosing $m-2$ the roots of the GFCFs in the interval $[0,\eta]$, as collocation points and substituting them in $Res(t)$, and using of initial conditions, a set of m nonlinear algebraic equations is generated.

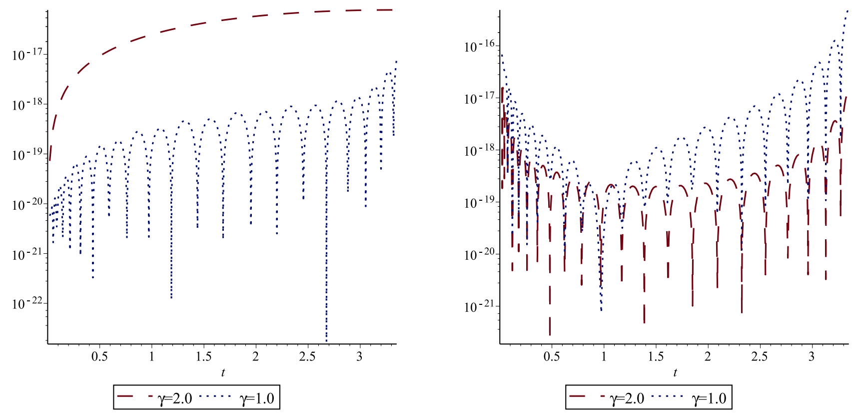
Table 2 shows the absolute errors and the residual errors by the present method, for $\gamma = 1$ and 2 with $\alpha = 0.5,\ m = 30$.

Table 3 shows the values obtained by the present method, for $\gamma = 1.75,\ 1.25$, and 0.80 with $m = 20$.

Fig. 1 shows the absolute errors of approximate solutions with the exact solutions and the residual errors, for $\gamma = 1$ and 2 with $\alpha = 0.5,\ m = 30$.

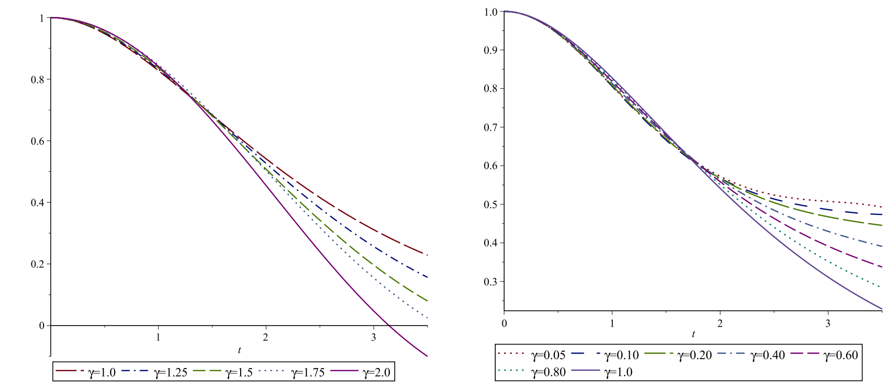
Fig. 2(a) shows the approximate solutions for the various values $1 \leq \gamma \leq 2$ with $m = 20$, Definitely, when γ tends to 2, the approximate solutions of $y(t)$ will converge to the exact solution in Eq. (6.5), and when γ tends to 1, from the right hand side, the approximate solutions of $y(t)$ will converge to the exact solution in Eq. (6.6).

Fig. 2(b) shows the approximate solutions for the various values $0 < \gamma \leq 1$ with $m = 20$, Definitely, when γ tends to 1, from the left hand side, the approximate solutions of $y(t)$ will converge to the exact solution in Eq. (6.6).



(a) The graphs of the absolute errors (b) The graphs of the residual errors

Figure 1. Graphs of the absolute and the residual errors in Example 6.2.



(a) Approximate solutions for $1 \leq \gamma \leq 2$ (b) Approximate solutions for $0 < \gamma \leq 1$

Figure 2. Graphs of the approximate solutions for $0 < \gamma \leq 2$ in Example 6.2 for the various values γ.

Table 2. The absolute and the residual errors by the present method, for $\gamma = 1$ and 2 in Example 6.2.

t	Abs. Err.	Res(t)	Abs. Err.	Res(t)
	$\gamma = 1$		$\gamma = 2$	
0.25	5.621e-20	1.550e-18	3.294e-18	3.480e-19
0.50	1.423e-19	6.168e-19	9.239e-18	1.815e-19
0.75	4.109e-20	3.026e-19	1.669e-17	1.700e-19
1.00	1.271e-19	3.000e-20	2.503e-17	1.085e-19
1.25	3.119e-19	5.138e-19	3.365e-17	1.883e-19
1.50	3.494e-19	1.106e-18	4.242e-17	1.990e-19
1.75	4.512e-19	1.773e-18	5.062e-17	2.025e-19
2.00	3.941e-19	2.653e-18	5.827e-17	2.154e-19

Table 3. Obtained values by the present method, for $\gamma = 1.75,\ 1.25,$ and 0.80 in Example 6.2

t	Appr. Sol.	Res(t)	Appr. Sol.	Res(t)	Appr. Sol.	Res(t)
	$\gamma = 0.80$		$\gamma = 1.25$		$\gamma = 1.75$	
0.25	0.98528225	6.791e-6	0.98634231	1.278e-9	0.98899406	7.002e-7
0.50	0.94589292	6.512e-6	0.95025175	2.171e-8	0.95802956	4.019e-7
0.75	0.88955041	6.319e-6	0.89741111	4.289e-8	0.90955493	4.767e-7
1.00	0.82330884	1.315e-6	0.83256082	3.791e-8	0.84615693	3.196e-7
1.25	0.75282676	9.875e-6	0.75980429	4.708e-8	0.77067967	3.294e-7
1.50	0.68225380	1.806e-6	0.68265934	9.033e-8	0.68615295	5.496e-7
1.75	0.61439158	1.449e-5	0.60407496	8.005e-8	0.59568399	4.359e-7
2.00	0.55095494	1.106e-5	0.52645239	1.495e-7	0.50234237	7.433e-7

Example 6.3. We consider the nonlinear Lane-Emden equation of fractional order as follows

$$D^{\gamma}y(t) + \frac{k}{t^{\gamma-\beta}}D^{\beta}y(t) + (y(t))^{p} = 0, \tag{6.7}$$

with the initial conditions

$$y(0) = 1, \qquad y'(0) = 0, \tag{6.8}$$

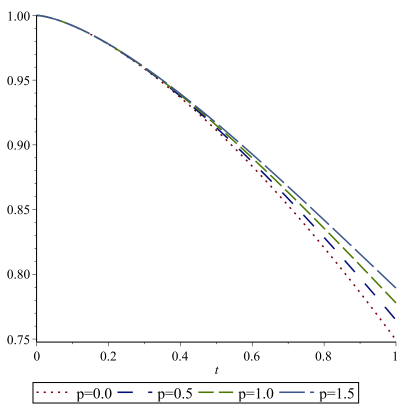
where $\gamma = \frac{3}{2},\ \beta = \frac{1}{2},\ k = 2$ and p is real constant.
By applying the technique described in the last section, the residual function is as follows:

$$Res(t) = A^{T}\mathbf{D}^{(\frac{3}{2})}\Phi(t) + \frac{2}{t}A^{T}\mathbf{D}^{(\frac{1}{2})}\Phi(t) + (A^{T}\Phi(t))^{p}.$$

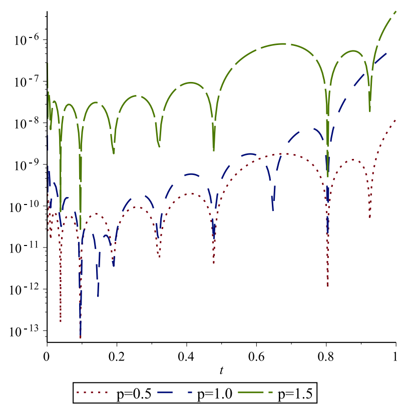
Choosing $m-2$ the roots of the GFCFs in the interval $[0,\eta]$, as collocation points and substituting them in $Res(t)$, and using of initial conditions, a set of m nonlinear algebraic equations is generated.

Table 4 shows the approximate solutions and the residual errors obtained by the present method for $m = 12$, $\alpha = 0.50$, and the various values t.

The approximate solutions and the residual errors with $m = 12$, $\alpha = 0.50$ and the various values p are displayed in Fig. 3.



(a) Graphs of the approximate solutions



(b) Graphs of the residual errors

Figure 3. Graphs of the approximate solutions and the residual errors for example 6.3.

Table 4. Obtained values of $y(t)$ for example 6.3 by the present method.

t	p =0.5		p =1.0		p =1.5	
	Appro. Sol.	Res(t)	Appro. Sol.	Res(t)	Appro. Sol.	Res(t)
0.2	0.97769562	2.303e-11	0.97781738	2.523e-11	0.97793750	1.097e-08
0.4	0.93755231	1.922e-10	0.93850210	5.644e-10	0.93941622	8.986e-08
0.6	0.88679596	1.270e-09	0.88989715	1.668e-09	0.89279101	5.699e-07
0.8	0.82848267	1.003e-10	0.83554765	1.349e-09	0.84191184	4.528e-08
1.0	0.76469285	1.220e-08	0.77787553	7.121e-07	0.78930457	4.849e-06

Example 6.4. We consider the nonlinear Lane-Emden equation (the white-dwarf equation) as follows [32, 35, 49, 52]

$$y''(t) + \frac{k}{t}y'(t) + (y^2 - C)^{\frac{3}{2}} = 0, \tag{6.9}$$

with the initial conditions

$$y(0) = 1, \qquad y'(0) = 0, \tag{6.10}$$

where C is real constant.

Eq. (6.9) has introduced by Davis [9] and Chandrasekhar [7] in their study of the gravitational potential of the degenerate white-dwarf stars.

If $C = 0$, Eq. (6.9) reduces to standard Lane-Emden equation of index $M = 3$. For a thorough discussion of the "white-dwarf" formula (6.9), see [7].

For $C = 0$, Wazwaz [52] has obtained a series solution using ADM as follows:

$$y(t) = 1 - \frac{1}{6}t^2 + \frac{1}{40}t^4 - \frac{19}{7!}t^6 + \frac{619}{3.9!}t^8 - \frac{17117}{5.11!}t^{10} + \cdots. \tag{6.11}$$

Singh et al [49] have obtained a series solution using MHAM as follows:

$$y(t) = 1 - \frac{1}{6}\omega^3 t^2 + \frac{1}{40}\omega^4 t^4 - \frac{1}{7!}\omega^5[5\omega^2 + 14]t^6 + \cdots. \tag{6.12}$$

where $\omega = \sqrt{1 - C}$.

By applying the technique described in the last section, the residual function is as follows:

$$Res(t) = A^T\mathbf{D}^{(2)}\Phi(t) + \frac{2}{t}A^T\mathbf{D}^{(1)}\Phi(t) + \left((A^T\Phi(t))^2 - C\right)^{\frac{3}{2}}.$$

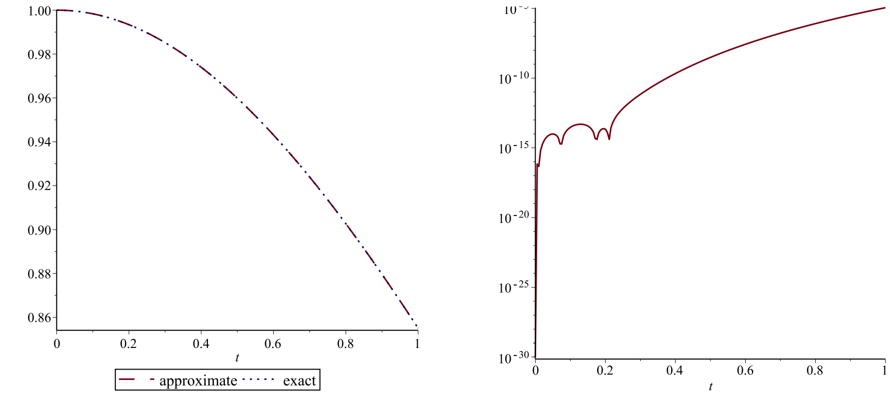
Choosing $m - 2$ the roots of the GFCFs in the interval $[0, \eta]$, as collocation points and substituting them in $Res(t)$, and using of initial conditions, a set of m nonlinear algebraic equations is generated.

The approximate solution, the analytical solution in Eq. (6.11), and the absolute error with $m = 12$ and $C = 0$ are displayed in Fig. 4.

Graphs of the approximate solutions Eq. (6.9) for various values C with $m = 10$ are displayed in Fig. 5.

Table 5 shows the approximate solution, the absolute error, and the residual error obtained by the present method for $C = 0$.

Table 6 shows the comparison of the approximate solution obtained by the present method, the modified Homotopy analysis method (HHAM) [49] and the indirect compactly supported radial basis function (ICSRBF) [32]. We can see that, with m lesser, the same answer is obtained.



(a) Graph of the approximate solution (b) Graph of the absolute error

Figure 4. Graphs of the approximate solution and the analytical solution by Wazwaz [52], and the absolute error, for Example 6.4 with $m = 12$ and $\alpha = 1$.

Figure 5. Graphs of the approximate solutions for various values C with $\alpha = 1$ for Example 6.4.

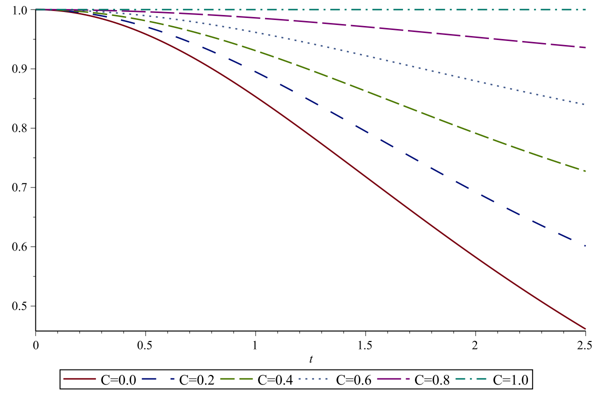


Table 5. Comparison of obtained values of $y(t)$ for the white-dwarf equation by the present method and the analytical solution Wazwaz [52] for Example 6.4 (with $m = 12,\ \alpha = 1$ and $C = 0$).

t	Approximate solution	Absolute error	Residual error
0.1	0.9983358295691	2.94803e-14	8.3350e-13
0.2	0.9933730935103	2.18628e-14	2.4744e-12
0.3	0.9851997885919	6.86704e-12	9.4507e-12
0.4	0.9739582559198	2.11995e-10	6.4552e-11
0.5	0.9598390699443	3.04297e-09	1.7279e-10
0.6	0.9430731727025	2.67095e-08	2.6826e-10
0.7	0.9239228380229	1.66727e-07	1.8506e-10
0.8	0.9026720891297	8.10698e-07	3.2142e-10
0.9	0.8796171670541	3.25573e-06	2.0274e-09
1.0	0.8550575689885	1.12409e-05	6.4882e-07

Table 6. Comparison of obtained values of $y(t)$ for the white-dwarf equation by the present method, Parand and Hemami [32], and Singh et al [49] for Example 6.4 (with $\alpha = 1$ and various values C).

t	C	Present (m=10)	ICSRBF [32] (m=20)	MHAM [49]
0.0001	0.1	0.99999999	0.999999	0.999999
0.01		0.99998576	0.999985	0.999985
0.1		0.99857899	0.998579	0.998578
0.2		0.99434012	0.994340	0.994340
0.4		0.97773869	0.977738	0.977738
0.6		0.95127016	0.951270	0.951263
0.7		0.93482310	0.934823	0.934801
0.9		0.89667360	0.896673	0.896522
0.0001	0.2	0.99999999	0.999999	0.999999
0.01		0.99998807	0.999988	0.999988
0.1		0.99880902	0.998809	0.998809
0.2		0.99525519	0.995255	0.995251
0.4		0.98132027	0.981320	0.981320
0.6		0.95904987	0.959049	0.959045
0.7		0.94517923	0.945179	0.945165
0.9		0.91291418	0.912913	0.912812
0.0001	0.3	0.99999999	0.999999	0.999999
0.01		0.99999023	0.999990	0.999990
0.1		0.99902512	0.999025	0.999025
0.2		0.99611509	0.996115	0.996115
0.4		0.98469022	0.984690	0.984690
0.6		0.96638409	0.966384	0.966381
0.7		0.95495358	0.954953	0.954944
0.9		0.92828056	0.928280	0.928216

Example 6.5. R.A. Van Gorder has examined the relation between Lane-Emden solutions and radial solutions to the elliptic Heavenly equation on a disk [51]. He considered the elliptic Heavenly equation in real variables which reads:

$$u_{xx} + u_{yy} = k(e^u)_{tt}, \tag{6.13}$$

where k is a non-zero parameter, and $u : D \times [0, \infty)$, where D is an unit disk. By using changing the variables of $u = \ln(v)$, $v(x, y, t) = f(t)\Psi(x, y)$ where $f(t) = \frac{\alpha_2}{2}t^2 + \alpha_1 t + \alpha_0$, and $\Psi(x, y) = exp(-k\alpha_2\Phi(x, y))$, he achieved [51]:

$$\Phi_{xx} + \Phi_{yy} + exp(-k\alpha_2\Phi) = 0, \tag{6.14}$$

Finally, by assuming a radial solution $\Phi(x, y) = \phi(r)$ where $r = \sqrt{x^2 + y^2}$, he obtained

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + exp(-k\alpha_2\phi) = 0, \tag{6.15}$$

with initial conditions

$$\phi'(0) = 0, \qquad \phi(1) = 1. \tag{6.16}$$

where is a Lane-Emden type equation of the second kind in two dimensions.

For $k\alpha_2 = 1$, a series solution obtained by Van Gorder [51] as follows:

$$\begin{aligned}\phi(r) = \quad & 1.086204345 - 0.08437252357r^2 - 0.001780496632r^4 \\ & -0.00004958392363r^6 - 0.000001740401376r^8.\end{aligned} \tag{6.17}$$

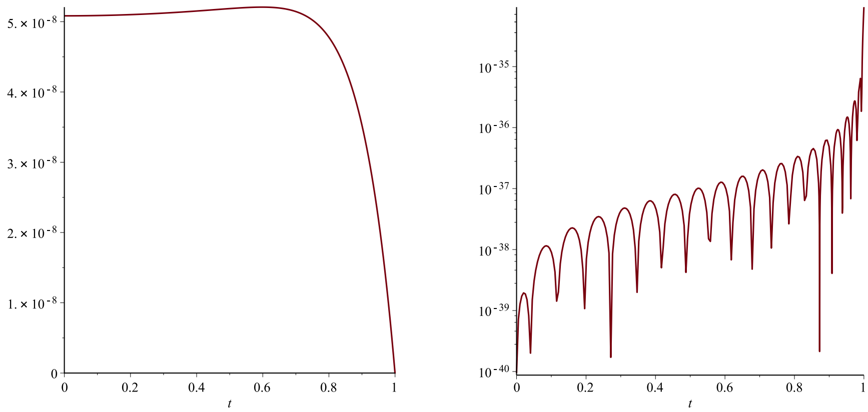
By applying the technique described in the last section, the residual function is as follows:

$$Res(r) = r\mathbf{D}^{(2)}\phi + \mathbf{D}^{(1)}\phi + r\ exp(-k\alpha_2\phi) = 0.$$

For $\alpha = 2$, we choose $m-1$ the roots of the GFCFs in the interval $[0, 1]$, as collocation points and substituting them in $Res(r)$, and using of initial condition $\phi(1) = 1$ (because $\alpha = 2$ so the condition $\phi'(0) = 0$ is satisfy), a set of m nonlinear algebraic equations is generated.

Figs. 6 shows the absolute error between the approximate solution and the analytical solution in Eq. (6.17), and the residual error with $m = 20$ and $\alpha = 2$.

Table 7 shows the approximate solution, the absolute error, and the residual error obtained by the present method.



(a) Graph of the absolute error (b) Graph of Log residual error

Figure 6. Graphs of the absolute error and Log residual error for Example 6.5.

Table 7. Comparison of obtained values of $\phi(r)$ by the present method and the analytical solution for Example 6.5 (with $m = 20$, and $\alpha = 2$)

t	Approximate solution	Absolute error	Residual error
0.0	1.086205127836084	5.0836e-8	0.00e-00
0.1	1.085361212772163	5.0878e-8	9.13e-39
0.2	1.082827328896571	5.1007e-8	5.78e-39
0.3	1.078597042058229	5.1223e-8	4.28e-38
0.4	1.072659568017476	5.1521e-8	4.87e-38
0.5	1.064999680325521	5.1867e-8	4.69e-38
0.6	1.055597579042094	5.2076e-8	1.09e-37
0.7	1.044428717891840	5.1445e-8	1.80e-37
0.8	1.031463586647157	4.7786e-8	2.62e-37
0.9	1.016667444608545	3.5249e-8	4.75e-37

7. Conclusion

In this paper, the generalized fractional order of the Chebyshev functions of the first kind are expressed, then the operational matrices of the fractional derivative and the product are obtained for these orthogonal functions. These matrices can be used to solve the linear and nonlinear Lane-Emden type equations of fractional order. The fractional derivative operational matrix of the GFCFs is a lower-triangular matrix, and the product operational matrix of the GFCFs is a formulation matrix and has an explicit formula, so they can reduce the costs of computation and storage. The obtained results are compared with other results to verify the accuracy and efficiency of the scheme presented. The obtained numerical results are better than the ones provided by other methods. Examples show that the GFCF collocation method can use to solve ODEs such as the white-dwarf equation.

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