Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 48 (6) (2019), 1620 – 1625
DOI : 10.15672/HJMS.2018.594

Research Article

On regular bipartite divisor graph for the set of irreducible character degrees

Roghayeh Hafezieh

Gebze Technical University, Department of Mathematics, 41400, Gebze/Kocaeli, Turkey

Abstract

Given a finite group G, the *bipartite divisor graph*, denoted by $B(G)$, for its irreducible character degrees is the bipartite graph with bipartition consisting of $cd(G)^*$, where $cd(G)^*$ denotes the nonidentity irreducible character degrees of G and the $\rho(G)$ which is the set of prime numbers that divide these degrees, and with $\{p, n\}$ being an edge if $\gcd(p, n) \neq 1$. In [Bipartite divisor graph for the set of irreducible character degress, Int. J. Group Theory, 2017], the author considered the cases where $B(G)$ is a path or a cycle and discussed some properties of G. In particular she proved that $B(G)$ is a cycle if and only if G is solvable and $B(G)$ is either a cycle of length four or six. Inspired by 2-regularity of cycles, in this paper we consider the case where $B(G)$ is an n-regular graph for $n \in \{1, 2, 3\}$. In particular we prove that there is no solvable group whose bipartite divisor graph is $C_4 + C_6$.

Mathematics Subject Classification (2010). 00Z99, 99A00

Keywords. bipartite divisor graph, irreducible character degrees, regular graph

1. Introduction

Given a finite group G, it is an area of research to convey nontrivial information about the structure of G through some sets of invariants associated to G such as the set of degrees of the irreducible complex characters of G and would be interesting to distinguish the group structure of G influenced by these sets.

It is well known that the set of irreducible characters of G, denoted by $Irr(G)$, can be used to obtain information about the structure of the group G. In this paper we are interested in the set of irreducible character degrees of G, that is, $cd(G) = \{\chi(1) : \chi \in Irr(G)\}$. When studying problems on character degrees, it is useful to attach the following graphs, which have been widely studied, to the sets $\rho(G)$ and $cd(G) \setminus \{1\}$.

(i) Prime degree graph, namely $\Delta(G)$, which is an undirected graph whose set of vertices is $\rho(G)$; there is an edge between two different vertices p and q if pq divides some degree in $cd(G)$.

(ii) Common divisor degree graph, namely $\Gamma(G)$, which is an undirected graph whose set of vertices is $cd(G) \setminus \{1\}$; there is an edge between two different vertices m and k if $gcd(m, k) \neq 1$.

Email address: roghayeh@gtu.edu.tr
Received: 14.01.2018; Accepted: 18.05.2018

We refer the reader to the overview [8] in which the author discussed many remarkable connections among these graphs by analysing properties of these graphs for arbitrary positive integer subsets. Inspired by the survey of Lewis, Praeger and Iranmanesh in [3] introduced the notion of bipartite divisor graph $B(X)$ for a finite set X of positive integers as an undirected bipartite graph with vertex set $\rho(X) \cup (cd(X) \setminus \{1\})$; there is an edge between vertices p of $\rho(G)$ and m of $cd(G) \setminus \{1\}$ if p divides m. Furthermore, they studied some basic invariants of this graph such as the diameter, girth, number of connected components and clique number.

One of the main questions in this area of research is classifying the groups whose bipartite divisor graphs have special graphical shapes. For instance, in [1], the author of this paper has considered the cases where the bipartite divisor graph is a path, a union of paths and a cycle. In particular she proved that for a finite group G, bipartite divisor graph for the set of irreducible character degrees is a cycle if and only if G is solvable and $B(G)$ is either a cycle of length four or six. Since cycles are 2-regular connected graphs, inspired by the results in [1], in this paper we consider the cases where $B(G)$ is k-regular for $k \in \{1, 2, 3\}$ and obtain some classification on G.

Notation 1.1. For positive integers m and n, we denote the greatest common divisor of m and n by $gcd(m, n)$; the number of connected components of a graph $\mathcal{G}$ by $n(\mathcal{G})$; the diameter of a graph $\mathcal{G}$ by $diam(\mathcal{G})$ (where by the diameter we mean the maximum distance between vertices in the same connected component of the graph). If $\mathcal{G}$ is a disconnected graph with $\mathcal{G}_1$ and $\mathcal{G}_2$ as its connected components, then we denote $\mathcal{G}$ by $\mathcal{G}_1 + \mathcal{G}_2$. If α is a vertex of the graph $\mathcal{G}$, then $deg_{\mathcal{G}}(\alpha)$ is the number of vertices adjacent to α in $\mathcal{G}$. If the graph is well-understood, then we denote it by $deg(\alpha)$. By length of a path or a cycle, we mean the number of edges in the path or in the cycle. Also, by P_n and C_n we mean a path of length n and a cycle of length n, respectively. Let G be a finite solvable group. We denote the first and second Fitting subgroups of G by $F = F(G)$ and $E = E(G)$, respectively. As usual we write $dl(G)$ and $h(G)$ to denote the derived length and Fitting height of G, respectively. Also in a finite group G, $cd(G)^*$ denotes $cd(G) \setminus \{1\}$. Other notations throughout the paper are standard.

2. 1-regular bipartite divisor graph

Considering the structure of the bipartite divisor graph of a finite group, it is clear that 0-regularity makes no sense. So we may start by discussing the influence of 1-regularity of this graph on the group structure.

Theorem 2.1. *Suppose G is a finite group and $B(G)$ is 1-regular. Then one of the following cases occurs:*

(1) *If G is nonsolvable, then $n(B(G)) = 3$, $G \simeq A \times PSL(2, 2^n)$, where A is abelian and $n \in \{2, 3\}$.*

(2) *If G is solvable, then one of the following properties holds:*

  (i) *for a prime p either $G \simeq P \times A$, where P is a non-abelian p-group with $cd(P) = \{1, p^\alpha\}$ for some $\alpha \geq 1$ and A is abelian, or $F(G)$ is the only abelian subgroup of G with $[G : F(G)] = p^\alpha \in cd(G)$, for some $\alpha \geq 1$. In particular $cd(G) = \{1, p^\alpha\}$.*

  (ii) *$h(G) \in \{2, 3\}$ and G, with respect to its Fitting height, has one of the two structures mentioned in [6, Lemma 4.1]. In particular:*

    (a) *If $h(G) = 3$, then $cd(G) = \{1, [G : E], [E : F]\}$, where $[G : E]$ is a prime s and $\frac{E}{F}$ is a cyclic t-group for a prime $t \neq s$.*

    (b) *If $h(G) = 2$, we have $cd(G) = \{[G : F]\} \cup cd(F)$, where $\frac{G}{F}$ is a cyclic t-group for a prime t and $|cd(F)| = 2$.*

Proof. 1-regularity of $B(G)$ implies that each connected component is a path of length one. First suppose that G is nonsolvable. As, in this case, $|cd(G)| > 3$ and $n(B(G)) \leq 3$ by [8, Theorem 6.4(1)], 1-regularity of $B(G)$ implies that $B(G)$, (so does $\Delta(G)$), has exactly three connected components. Now [8, Theorem 6.4(2)] verifies that $G \simeq A \times PSL(2, 2^n)$, where A is abelian and $n \geq 2$. As $|\rho(PSL(2, 2^n))| = |\rho(G)| = 3$, by [2] we conclude that $n \in \{2, 3\}$.

Now suppose that G is solvable. Then $B(G)$ has at most two connected components. If $B(G)$ is connected, then we have $cd(G) = \{1, p^\alpha\}$, for a prime p and a positive integer $\alpha \geq 1$. This implies that either $G \simeq P \times A$, where P is a non-abelian p-group and A is abelian, or $F(G)$ is the only abelian subgroup of G with $[G : F(G)] = p^\alpha \in cd(G)$, by [11, Lemma 1.6].

Consider the case where $n(B(G)) = 2$; then by the structure of $B(G)$ we can see that $cd(G) = \{1, p^\alpha, q^\beta\}$ for distinct primes p and q and for some positive integers α and β. Since $p^\alpha q^\beta \notin cd(G)$, G is not nilpotent, so $h(G) \geq 2$. On the other hand, by [4, Corollary 12.21], we have $h(G) \leq |cd(G)| = 3$. Hence $h(G) \in \{2, 3\}$ and G, with respect to its Fitting height, has one of the two structures mentioned in [6, Lemma 4.1]. In particular if $h(G) = 3$, then $cd(G) = \{1, [G : E], [E : F]\}$, where $[G : E]$ is a prime $s \in \{p, q\}$ and $\frac{E}{F}$ is a cyclic t-group for some $t \in \{p, q\}$, where $t \neq s$. While $h(G) = 2$, we have $cd(G) = \{[G : F]\} \cup cd(F)$, where $\frac{G}{F}$ is a cyclic t-group for some $t \in \{p, q\}$ and $|cd(F)| = 2$. □

3. 2-regular and 3-regular bipartite divisor graph

Lemma 3.1. *Let G be a finite group whose $B(G)$ is a connected 2-regular graph. Then G is solvable with $dl(G) \leq 4$ and $B(G)$ is either a cycle of length four or six.*

Proof. As a connected 2-regular graph is a cycle, by [1, Theorem 8, Corollary 9] the proof is complete. □

Theorem 3.2. *Suppose that G is a finite group whose $B(G)$ is 2-regular. Then G is solvable and $B(G)$ is a cycle of length four or six.*

Proof. Assume that $B(G)$ is a 2-regular disconnected graph. As 2^n is a vertex of degree one in $B(PSL(2, 2^n))$ and $n(\Delta(G)) = n(B(G))$, by [8, Theorem 6.4] we conclude that $n(B(G)) \neq 3$. Hence $B(G)$ has two connected components and each component is a cycle. Since $n(\Delta(G)) = n(B(G)) = 2$, [8, Theorem 6.4] implies that while G is nonsolvable, there is a prime $p \in \rho(G)$ which is an isolated vertex of $\Delta(G)$, which contradicts the structure of $B(G)$. Therefore G is solvable. [8, Corollary 4.2] verifies that each connected component of $\Delta(G)$ is a complete graph. As $B(G)$ is 2-regular, the connected components of $\Delta(G)$ belongs to $\{K_2, K_3\}$. Thus we have the following cases for $B(G)$:

$$\{C_4 + C_4, C_4 + C_6, C_6 + C_6\}$$

By [8, Theorem 4.3, Theorem 7.1], none of the cases $C_4 + C_4$ and $C_6 + C_6$ is possible. Suppose that G is a solvable group whose $B(G)$ is $C_4 + C_6$. Hence G is one of the six-type groups mentioned in [7]. Using the lemmas in the third chapter of [7], we can see that for groups of types 1, 2, 3 and 5, $\Delta(G)$ has an isolated vertex which is impossible. Suppose that G is a group of type 4. This implies that G is a semi-direct product of a subgroup H acting on an elementary abelian p-group for some prime p. Let K be the Fitting subgroup of H, $m = [H : K] > 1$, $F = F(G)$, and $E/F = F(G/F)$. Then [7, Lemma 3.4] verifies that $\{1, m, [E : F]\} \subseteq cd(G)$, where $\pi(m)$ and $\pi([E : F])$ are the connected components of $\Delta(G)$. Hence either m or $[E : F]$ is divisible by three primes which contradicts the structure of $B(G)$. So G is not a group of type 4. Now assume that G is a group of type 6; where G is a semi-direct product of an abelian subgroup D acting coprimely on a subgroup T so that $[T, D]$ is a Frobenius group with a Frobenius kernel $A = T' = [T, D]'$, where A is a non-abelian p-group for a prime p and a Frobenius complement B with

$[B, D] \subseteq B$. Let $m = [D : C_D(A)]$ and q is a power of p so that $[A : A^{'}] = q^m$. Then by [7, Lemma 3.6], $G/A^{'}$ has the properties of groups of type 4 and $\frac{F}{A^{'}}$ is the Fitting subgroup of $\frac{G}{A^{'}}$. As $\frac{E}{F} = F(\frac{G}{F}) \simeq F(\frac{G/A^{'}}{F/A^{'}})$ and $\frac{G}{A^{'}}$ has the properties of groups of type 4, applying [7, Lemma 3.4 (vi)] we can see that $\{1, m, [E : F]\} \subseteq cd(G/A^{'}) \subseteq cd(G)$, where $\pi(m)$ and $\pi([E : F]) \cup \{p\}$ are the connected components of $\Delta(G)$. Now 2-regularity of $B(G)$ implies that $|\pi(m)| = |\pi([E : F])| = 2$. Without loss of generality we may assume that $\pi(m) = \{r, s\}$ and $\pi([E : F]) = \{q, t\}$. Let $\eta \in Irr(G|A^{'})$ be nonlinear. [7, Lemma 3.6 (vi)] implies that $p|B|$ divides $\eta(1)$ and $\eta(1)$ divides $|P|[E : F]$. Since $B(G)$ is 2-regular, we deduce that that $|B|$ is divisible by either q or t. Without loss of generality, assume that $|B| = q^{\gamma}$, for a nonzero positive integer γ. Now we have $\eta(1) = p^{\alpha}q^{\beta}$, for some positive integers $\alpha, \beta \geq 1$. As $cd(G) = cd(G/A^{'}) \cup cd(G|A^{'})$, we conclude that there is no irreducible character degree of G which is divisible by the primes p and t, a contradiction. Hence there exists no solvable group G whose $B(G)$ is $C_4 + C_6$.

At last we conclude that $B(G)$ is a connected 2-regular graph. Now Lemma 3.1 verifies that G is solvable with $dl(G) \leq 4$ and $B(G)$ is either a cycle of length four or six. □

Corollary 3.3. *Suppose that G is a finite group whose $B(G)$ is a 2-regular graph. Then its diameter is at most 3. In particular, if $diam(B(G)) = 2$, then there exists a normal abelian Hall subgroup N of G such that $cd(G) = \{[G : I_G(\lambda)] : \lambda \in Irr(N)\}$.*

Proof. The proof is clear by Theorem 3.2 and [1, Theorem 11]. □

Theorem 3.4. *Suppose that $B(G)$ is 3-regular for a finite group G. Then $B(G)$ is connected.*

Proof. Let G be a finite group whose $B(G)$ is 3-regular. Neither $\Delta(G)$ nor $\Gamma(G)$ have an isolated vertex which implies that $n(B(G))$ is neither 3 nor 2 while G is nonsolvable. Suppose that $n(B(G)) = 2$ and G is solvable. Since $n(\Delta(G)) = n(B(G))$, [7] implies that G is one the groups of types one to six with respect to the notations of [7]. As for any group of types 1, 2, 3 and 5, $\Delta(G)$ has an isolated vertex, we deduce that G is either a group of type four or six. G is not a group of type four, since [7] implies that $\Gamma(G)$ has an isolated vertex, which contradicts the structure of $B(G)$. Suppose that G is a group of type six. By [7, Lemma 3.4, Lemma 3.6], with respect to the notations there, and similar to the proof of Theorem 3.2, we deduce that $\{1, m, [E : F]\} \subseteq cd(G)$, where $\pi(m)$ and $\pi([E : F]) \cup \{p\}$ are the connected components of $\Delta(G)$. Since $B(G)$ is 3-regular, we conclude that $|\pi(m)| = |\pi([E : F])| = 3$. Thus one of the components of $\Delta(G)$ has 3 vertices, while the other one has 4 vertices, which contradicts [8, Theorem 4.3]. All together, we conclude that $n(B(G)) \neq 2$ which implies that $B(G)$ is connected. □

Theorem 3.5. *Let G be a group whose $B(G)$ is a 3-regular graph. If $\Delta(G)$ is n-regular for $n \in \{2, 3\}$, then G is solvable and $\Delta(G) \simeq K_{n+1} \simeq \Gamma(G)$.*

Proof. As $B(G)$ is 3-regular, Theorem 3.4 implies that $n(B(G)) = 1$.

Let G be a solvable group. Suppose $\Delta(G)$ is a 2-regular graph. Since a connected 2-regular graph is a cycle, [12, Theorem C] implies that $|\rho(G)| \leq 4$, therefore $\Delta(G)$ is either a triangle or a square. First suppose that $\Delta(G)$ is a square; [9] verifies that $G \simeq H \times K$, where both $\Delta(H)$ and $\Delta(K)$ are disconnected graphs with two isolated vertices, furthermore $\rho(H) \cap \rho(K) = \varnothing$. This contradicts 3-regularity of $B(G)$, so $\Delta(G)$ is a triangle. It is easy to see that 3-regularity of $B(G)$, forces $\Gamma(G)$ to be a triangle. Now suppose that $\Delta(G)$ is 3-regular. [13, Theorem 3.2] implies that $\Delta(G)$ is isomorphic with K_4, in particular $|\rho(G)| = 4$. We may assume that $\rho(G) = \{p, q, r, s\}$. As $deg_{B(G)}(p) = 3$, there exists $\{m_1, m_2, m_3\} \subseteq cd(G)$ with a common prime divisor p. Since the number of odd vertices in a graph is even, $|cd(G) \setminus \{1\}| \geq 4$. By the structure of $B(G)$ we can see that $|cd(G) \setminus \{1\}| = 4$ which implies that there exists $m_4 \in cd(G) \setminus \{1\}$ such that

$cd(G) = \{1, m_1, m_2, m_3, m_4\}$. We can see that $\pi(m_4) = \{q, r, s\}$. By symmetry we may assume that $\pi(m_1) = \{p, q, r\}$. The case $\pi(m_2) = \{p, q, r\}$ is impossible. Now it is easy to see that $\Gamma(G) \simeq K_4$.

Let G be a nonsolvable group. First suppose that $\Delta(G)$ is 2-regular. By [12, Theorem C] we conclude that $\Delta(G)$ is either a triangle or a square. Now [10, Theorem B] verifies that square is not the prime degree graph of a nonsolvable group, hence $\Delta(G)$ is a triangle. Similar to the previous part, one can see that 2-regularity of $\Delta(G)$ forces $\Gamma(G)$ to be a triangle. Now [8] implies that G is solvable, a contradiction. Thus $\Delta(G)$ is not 2-regular. If $\Delta(G)$ is 3-regular, then [13] implies that $\Delta(G)$ is isomorphic with K_4. Now similar to the solvable case, we can see that $\Gamma(G)$ is complete, hence G is solvable which is a contradiction. Thus if G is a nonsolvable group whose $B(G)$ is 3-regular, then $\Delta(G)$ is neither 2-regular, nor 3-regular. □

Corollary 3.6. *Let G be a solvable group whose $B(G)$ is a 3-regular graph. If at least one of $\Delta(G)$ or $\Gamma(G)$ is not complete, then $\Delta(G)$ is neither 2-regular, nor 3-regular.*

Proof. The proof is clear by Theorem 3.5. □

Corollary 3.7. *Let G be a solvable group whose $B(G)$ is a 3-regular graph. If $\Delta(G)$ is regular, then it is a complete graph. Furthermore, if $\Gamma(G)$ is not complete, then $\Delta(G)$ is isomorphic with K_n, for $n \geq 5$.*

Proof. As $B(G)$ is 3-regular, Theorem 3.4 implies that $n(B(G)) = 1$. If $\Delta(G)$ is a regular graph which is not complete, then it has no complete vertices. Now the main theorem of [5] implies that $G \simeq \prod M_i$, where for each i, $M_i = P_iQ_i$ with $P_i \in Syl_{p_i}(G)$ is normal non-abelian, and $Q_i \in Syl_{q_i}(G)$ is not normal in G. This contradicts three regularity of $B(G)$, hence $\Delta(G)$ is a complete graph. In particular, if $\Gamma(G)$ is not complete, by Corollary 3.6, we deduce that $\Delta(G)$ isomorphic with K_n, for $n \geq 5$. □

Acknowledgment. The author would like to thank the referees for their valuable comments which helped to improve the manuscript. Also, she would like to thank Prof. Ali Iranmanesh for his helpful conversations during his visit which was supported by TUBITAK project number 2221.

References

[1] R. Hafezieh, *Bipartite divisor graph for the set of irreducible character degress*, Int. J. Group Theory **6** (4), 41-51, 2017.

[2] B. Huppert and W. Lempken, *Simple groups of order divisible by at most four primes*, Proceeding of F. Scorina Gemel State University **16** (3), 64-75, 2000.

[3] M.A. Iranmanesh and C.E. Praeger, *Bipartite divisor graphs for integer subsets*, Graphs Combin. **26**, (2010), 95-105.

[4] I.M. Isaacs, *Character theory of finite groups*, Academic Press, New York, 1976.

[5] D.M. Kasyoki, *Finite Solvable Groups with 4-Regular Prime Graphs*, African Institute for Mathematical Sciences, Master Thesis, 2013.

[6] M.L. Lewis, *Determining group structure from sets of irreducible character degrees*, J. Algebra **206**, 235-260, 1998.

[7] M.L. Lewis, *Solvable groups whose degree graphs have two connected components*, J. Group Theory **4**, 255-275, 2001.

[8] M.L. Lewis, *An overview of graphs associated with character degrees and conjugacy class sizes in finite groups*, Rocky Mountain J. Math. **38**, 175-211, 2008.

[9] M.L. Lewis and Q. Meng, *Square character degree graphs yield direct products*, J. Algebra **349**, 185-200, 2012.

[10] M.L. Lewis and D. L. White, *Four-vertex degree graphs of nonsolvable groups*, J. Algebra **378**, 1-11, 2013.

[11] T. Noritzsch, *Groups having three complex irreducible character degrees*, J. Algebra, **175**, 767-798, 1995.

[12] H.P. Tong-Viet, *Groups whose prime graphs have no triangles*, J. Algebra **378**, 196-206, 2013.

[13] H.P. Tong-Viet, *Finite groups whose prime graphs are regular*, J. Algebra **397**, 18-31, 2014.