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THE ROLE OF EXAMPLES IN THE FORMATION OF MATHEMATICAL CONCEPTS

MATEMATİKSEL KAVRAMLARIN OLUŞUMUNDA ÖRNEKLERİN ROLÜ

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ABSTRACT: This research presents a brief survey of literature on the use of examples and generic examples in concept formation, and the observation of these theoretical ideas in a number of exam papers. Explanatory insight is given to clarify the situations in the exam papers. By focusing on the students' answers and the definition given on generic example definition, mainly two suggestions arise for the usage of generic examples. Firstly, the sequence of the exemplification may be in the order of all the generic examples first and then the practising examples. Secondly, while chosing the examples the non-parallel principle can be taken into consideration besides the parallel principle.

KEY WORDS: *Example, generic examples, practising examples, abstraction, generalisation, clarification*

ÖZET: Bu araştırmanın amacı, öncelikle genel örneklerin kavram oluşturmadaki kullanımı üzerine bir literarür taraması, daha sonra ise bulunan bu kuramsal bilgilerin sınav kağıtları üzerinde incelenmesinden oluşmaktadır. İncelenen sınav kağıtları hakkında literatüre dayalı açıklayıcı bilgiler verilmiştir. Öğrencilerin sınav kağıtlarındaki yanıtları ve yapılan literatür taraması üzerinde yoğunlaştığımızda genel örneklerin kullanılması için şu önerilerde bulunulabilir. Bunlardan birincisi, tüm genel örneklerin verilmesi daha sonra ise alıştırmalara geçilmesi şeklinde olabilir. İkinci ise genel örneklerin seçiminde paralel olma ilkesi yanında paralel olmama ilkesinin de kullanılması olabilir.

ANAHTAR SÖZCÜKLER: *Örnek, genel örnek, alıştırmalar, kuramsallaştırma, genelleme.*

1. INTRODUCTION

Passing from elementary to advanced mathematical thinking involves a significant change from describing to defining, from convincing to proving in a logical manner based on the definitions. This transition is from the coherence of elementary mathematics to the consequence of advanced mathematics, which is based on abstract entities the individual must construct through the deductions from the formal definition [1]. As mentioned in [1], advanced mathematics is usually considered as a collection of abstract subjects. While elementary mathematics gives the process of mathematical thinking, advanced mathematics tends to give students the product of mathematical thought. These two approaches seem so far away from each other with respect to difficulty and also the students' cognitive development. In this respect, mathematics teachers construct the subject with the help of examples through elementary to advanced mathematics. The usage of examples in mathematics lessons carries different purposes. They can be used for practising, generalisation, abstraction or clarification. In this study, explanatory insight is given for the use of examples and generic examples in the concept formation by focusing on situations in the exam papers of the students.

2. THEORETICAL BACKGROUND RELATED TO THE USAGE OF EXAMPLES

There are various kinds of usage of examples. The first kind of usage of examples is practising in which the acquired behaviours are reviewed. For instance, the examples given like (find the solution set of $3x+2=3$, $5x+9=3x-1$, $4x-7=7-3x$) after teaching how to find the

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solution set of the equations in one variable are the kinds of examples used for practising the concept learned.

Abstraction and generalisation are used in mathematics both to denote processes in which concepts are seen in a broader context and also the products of those processes. However, when we look at the cognitive processes involved in generalisation and abstraction, we see subtle differences. Generalisation means the extension of the student's existing cognitive structure, which requires no changes in the current idea [1]. That is, it simply involves an extension of the familiar processes. For instance, while generalising the solution of linear equations in two and three dimensions to n dimensions, the chain of ideas from R^1 to R^2 to R^3, and etc. extend to R^n by applying the usual arithmetic processes to each coordinate system. Abstraction means a reconstruction of the existing cognitive structure, which needs a massive mental reorganisation. For example, the notion of the vector space (V) is abstracted from the solution of linear equations in two and three dimensions. In the abstraction of the notion of the vector space, lists of axioms are defined in which different mental processes are required.

The last purpose of using examples is that of clarification. As mentioned in [2], "The generic example involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of the class". As seen in this definition, the term generic example is used for the examples in which explanations or clarifications needed for the concepts have been just taught. [3] refers to the same thing: "The generic example is taken as structural generalisation which is based on underlying meanings, structures or procedures. It goes beneath to form results to achieve explanatory insight". In [4], the generic example is used as a tool for clarification of the new concepts. He suggests three principles for selecting effective generic examples: (i) The entification principle, (ii) The necessity principle, and (iii) The parallel principle. The entification principle states that the context of the examples from which the new object's properties are to be abstracted must be familiar. The necessity principle states that students must be able to see the reason for the abstraction they are being asked to make. The parallel principle says that the generic example must be treated in a way, which can be paralleled later in a general case.

In the lights of the studies mentioned above, it seems that the generic example is a stage between particular and general where particular denotes the concepts learned previously and the general denotes the concept being learned. It is not just an example but it can be considered as a teaching technique for concept formation [5]. He also stated that the most important quality of the generic example is that only one is required. However, what should be important here is that not the number of generic examples, but the understanding of the concept by the generic examples whatever the number of examples. The number of generic examples should be related to how many is required for clarification of the concept.

3. EXAMPLES FROM EXAM PAPERS

The discussion of the use of examples in the teaching and the learning of mathematics make us look for students' use of examples. The exam papers in which the theoretical ideas were examined were the ones administered by the instructors after the completion of the teaching of the subjects during the fall semester of 1999-2000. One of these exam papers was taken from the course called "Mathematics 1" given at the Technical School of Higher Education (TSHE) in Middle East Technical University. The other was taken from the 9^{th} grade mathematics course given at a private high school (PHS) in Turkey. These groups of students were taught the same subject, namely functions. While investigating the exam papers, the lecture notes were also taken into consideration.

The following is Murat's paper who is a student in TSHE (See Figure 1).

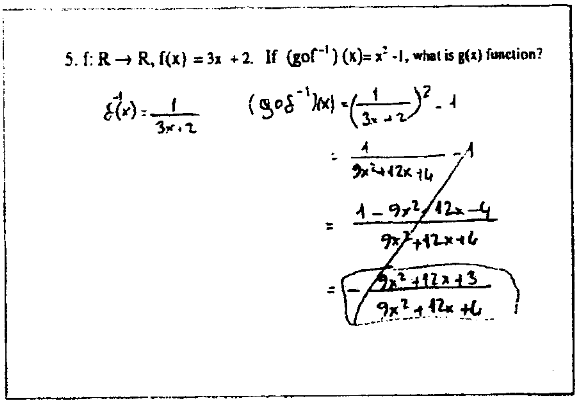

Figure 1

When his answer was examined, it is seen that Murat found the inverse of the function by using the multiplicative inverse rather then the inverse of a function. While finding the multiplicative inverse of any number such as "2", $2^{-1}= 1/2$ is written. So when he saw the inverse notation on the function, he connected this with his previous knowledge and then wrote $f^{-1}(x)=1/(3x+2)$. Here, we see that the familiar sytem and notation influences the learning of the new concept [6]. In other words, features of the example which are not part of the generality it represents have been inputted to that generality.

Another example is from Tolga's paper who is also a student in TSHE. The example given below was also seen in other student answers (See Figure 2).

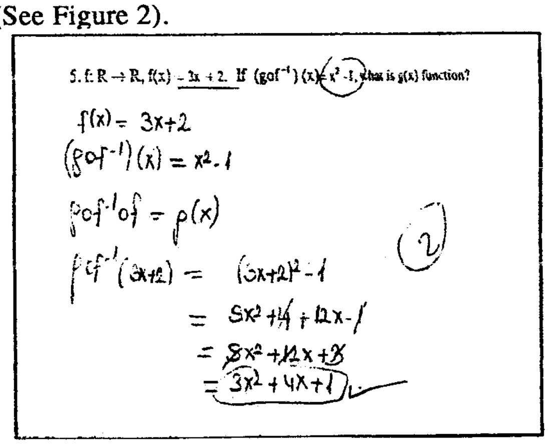

Figure 2

Examination of these students answers indicates that they made the simplification in the equality, $(gof^{-1})(x)=9x^2+12x+3$, by considering that this equality shows a proportion. As it is seen, one side of the equality is the multiple of 3 and $(3x^2+4x+1)$. Then, they made the simplification without considering the other side of the equality. Again we have seen how a familiar system influences the learning of the forthcoming subject.

The following is the last example, which is taken from Efe's exam paper. This student is a student in PHS (See Figure 3).

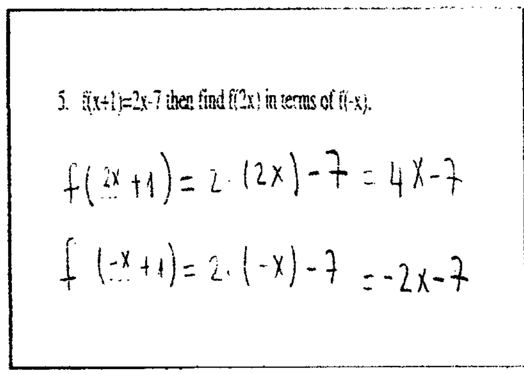

Figure 3

When Efe's answer is compared with the lecture notes, it is easily seen that Efe was putting much more weight on an example than it could. While teaching the function concept, the examples given respectively were f(x)=... and f(ax+b)=....While finding the value of f(4) for the function f(x)=3x+2, 4 is substituted for x.. However, while finding the value of f(4) for the function f(2x+1)=3x, 4 represents the value of 2x+1not x only. In lectures both kinds of examples were given by emphasising the diffrences between f(x) and f(ax+b). It seems that this student took the examples given like f(x)=3x + 2 as a generic example. Although the examples are generic, they are just the representatives of the particular cases, not the general. This is also seen in [7] where the role of example is to help students to see the generality which is represented by the particular. As stated in [5], students need to see the examples as "examples of" some more general statements. That is, students take the particular as the generalisation.

4. CONCLUSION AND IMPLICATION

We conclude this paper with some comments on teaching mathematics in the light of the generic example definition described previously. Our study does not have anything to say about the affective aspects of teaching/learning situation. In particular, we have ignored Piaget's notion of equilibration [8] which for him was the driving force behind the reconstruction of concepts. We have also omitted consideration of various issues such as discovery versus guided learning, and large classes versus individual instruction versus small group problem solving. The main implication of our study is that the main concern of the mathematics teachers should be the students' construction of concepts for understanding. Instruction should be dedicated to inducing students to make constructions and helping them along in the process.

Examples 1 and 2 given in Part 3 indicate how the familiar mathematical systems and notations influence the students' understanding of the new concept. So while giving examples for introducing a new concept, we should not give just examples of the same kind, but instead much more attention should be given to choosing examples which include the description of the process. This should be done since students mainly memorise the type of examples given at the begining. Tall [9] mentions the same thing: "if an individual works in a restricted context in which all the examples considered have a certain property, then, in the absence of counter examples, the mind assumes the known properties to be implicit in other contexts." Besides the parallel principle in selecting the generic example, the non-parallel principle should also be taken into consideration. What we mean by the non-parallel principle is that the generic example must be treated in a way, which can not be paralleled later in a general case. For example, the expression of $2^{-1}=1/2$ can be paralleled to the expression of f^{-1} for the inverse function.

In the third example given in Part 3, it was seen that students took the examples given on some more general statement as the general case. While giving examples, we are, as teachers, giving a lot of similar kind of examples before giving examples in other context. Then it may be useful to give all the generic examples at the beginning while covering the concept and then later some more practicing examples can be given. To sum up, the sequence of the examples can be in the form of all the generic examples and then practicing examples rather than in the form of generic example, practicing examples, generic example, practicing examples etc.

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