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COLOMBEAU TYPE ALGEBRA OF PSEUDO ALMOST PERIODIC GENERALIZED FUNCTIONS

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Abstract. The aim of this work is to introduce and to study an algebra of pseudo almost periodic generalized functions containing the classical pseudo almost periodic functions as well as pseudo almost periodic distributions.

1. Introduction and preliminaries

The concept of pseudo almost periodicity is a generalization of Bohr almost periodicity, it has been introduced by C. Zhang, see [8]. The algebra $\mathcal{G}$ of generalized functions of Colombeau give an answer to the problem of multiplication of distributions. For a detailed study of these generalized functions see the book [6]. An algebra of almost periodic generalized functions of Colombeau type containing classical Bohr almost periodic functions and almost periodic Schwartz distributions has been introduced and studied in [3]. As mentioned in the abstract, the first aim of this work is to introduce and to study an algebra of pseudo almost periodic generalized functions of Colombeau type containing Zhang pseudo almost periodic functions as well as pseudo almost periodic Schwartz distributions. In section 1, we recall the basic definitions and results that we shall use in this work. The main results of paper are given in the next section. First, we construct the space of smooth pseudo almost periodic functions and we recall the algebra $\mathcal{G}_{L^\infty}$ of bounded generalized functions in which we study the pseudo almost periodicity. Next, we define the space $\mathcal{M}_{pap}$ of pseudo almost periodic moderate elements and the space $\mathcal{N}_{pap}$ of pseudo almost periodic negligible elements. The main properties of $\mathcal{M}_{pap}$ and $\mathcal{N}_{pap}$ are summarized in Proposition 2. The new algebra $\mathcal{G}_{pap}$ of pseudo almost periodic generalized functions of Colombeau type is given in Definition 5. A characterization of elements of $\mathcal{G}_{pap}$ similar to the classical result for pseudo almost periodic Schwartz distributions is given by Proposition 4. By means of convolution with a mollifier $\rho \in \Sigma$, we show that the space of pseudo almost periodic Schwartz distributions $\mathcal{B}'_{pap}$. can be embedded by the map i_{pap} into the algebra $\mathcal{B}_{pap}$. By defining the canonical embedding σ_{pap} between $\mathcal{B}_{pap}$ and $\mathcal{G}_{pap}$, Proposition 6, shows

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that we have two ways to embed the space $\mathcal{B}_{pap}$ into $\mathcal{G}_{pap}$ by i_{pap} and by σ_{pap} . Finally, another result is Proposition 7, in which we give an extension of the classical Bohl-Bohr's Theorem. We refer the reader to [3], [4] and [5] from which the results of this paper were inspired. In this paper we consider functions and distributions defined on $\mathbb{R}$. Recall $(\mathcal{C}_b, \| \quad \|_{L^\infty})$ the Banach algebra of bounded and continuous complex valued functions on $\mathbb{R}$ endowed with the norm $\| \quad \|_{L^\infty}$ of uniform convergence on $\mathbb{R}$. The space $\mathcal{C}_{ap}$ of almost periodic functions on $\mathbb{R}$, which was introduced by H. Bohr, is the closed subalgebra of $(\mathcal{C}_b, \| \quad \|_{L^\infty})$ that contains all the functions f, satisfying: for any $\varepsilon > 0$, the set

$$\left\{ \tau \in \mathbb{R} : \sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| < \varepsilon \right\}, \tag{1.1}$$

is relatively dense in $\mathbb{R}$. In [8], C. Zhang introduced an extension of the almost periodic functions. Set

$$\mathcal{C}_0 = \left\{ f \in \mathcal{C}_b : \lim_{t\longrightarrow+\infty} \frac{1}{2t} \int_{-t}^{t} |f(x)|\, dx = 0 \right\}. \tag{1.2}$$

Definition 1.1. *A function $f \in \mathcal{C}_b$ is called pseudo almost periodic if it can be written as $f = g + h$, where $g \in \mathcal{C}_{ap}$ and $h \in \mathcal{C}_0$.*

The above decomposition is unique, so the functions g and h are called respectively the almost periodic component and the ergodic perturbation of the pseudo almost periodic function f. Denote by $\mathcal{C}_{pap}$ the set of all such functions. Then we have $\mathcal{C}_{ap} \subset \mathcal{C}_{pap} \subset \mathcal{C}_b$.

Now, we recall Schwartz almost periodic distributions, see [7]. Let $p \in [1, +\infty]$, the space

$$\mathcal{D}_{L^p} := \left\{ \varphi \in \mathcal{C}^\infty : \varphi^{(j)} \in L^p, \forall j \in \mathbb{Z}_+ \right\}, \tag{1.3}$$

endowed with the topology defined by the countable family of norms

$$|\varphi|_{k,p} := \sum_{j\leq k} \left\| \varphi^{(j)} \right\|_{L^p}, k \in \mathbb{Z}_+, \tag{1.4}$$

is a differential Frechet subalgebra of $\mathcal{C}^\infty$. The topological dual of $\mathcal{D}_{L^1}$, denoted by $\mathcal{D}'_{L^\infty}$, is called the space of bounded distributions.

Definition 1.2. *A distribution $T \in \mathcal{D}'_{L^\infty}$ is called almost periodic if the set $\{\tau_h T, h \in \mathbb{R}\}$ of translated of T is relatively compact in $\mathcal{D}'_{L^\infty}$. The space of Schwartz almost periodic distributions is denoted by $\mathcal{B}'_{ap}$.*

Define

$$\mathcal{B}'_0 := \left\{ T \in \mathcal{D}'_{L^\infty} : \lim_{t\longrightarrow+\infty} \frac{1}{2t} \int_{-t}^{t} |(T * \varphi)(x)|\, dx = 0, \forall \varphi \in \mathcal{D} \right\}. \tag{1.5}$$

Definition 1.3. *A distribution $T \in \mathcal{D}'_{L^\infty}$ is called pseudo almost periodic if it can be written as $T = R + S$, where $R \in \mathcal{B}'_{ap}$ and $S \in \mathcal{B}'_0$. The space of all such distributions is denoted by $\mathcal{B}'_{pap}$.*

The above decomposition is unique and we have $\mathcal{B}'_{ap} \subset \mathcal{B}'_{pap} \subset \mathcal{D}'_{L^\infty}$.

Theorem 1.1. *Let $T \in \mathcal{D}'_{L^\infty}$, the following statements are equivalent :*
(i) $T \in \mathcal{B}'_{pap}$.
(ii) $T * \varphi \in \mathcal{C}_{pap}, \forall\ \varphi \in \mathcal{D}$.
(iii) $\exists k \in \mathbb{Z}_+, \exists\, (f_j)_{j \leq k} \subset \mathcal{C}_{pap} : T = \sum_{j\leq k} f_j^{(j)}$.

2. Results

In this section, we introduce the algebra of pseudo almost periodic generalized functions of Colombeau type and we give their main properties.

Definition 2.1. *The space of smooth pseudo almost periodic functions on $\mathbb{R}$, is denoted and defined by*

$$\mathcal{B}_{pap} := \left\{ \varphi \in \mathcal{D}_{L^\infty} : \varphi^{(j)} \in \mathcal{C}_{pap}, \forall j \in \mathbb{Z}_+ \right\}. \tag{2.1}$$

We give some, easy to prove, properties of the space $\mathcal{B}_{pap}$.

Proposition 2.1. (i) *$\mathcal{B}_{pap}$ is a closed subalgebra of $\mathcal{D}_{L^\infty}$ stable by derivation.*
(ii) *If $T \in \mathcal{B}'_{pap}$ and $\varphi \in \mathcal{B}_{pap}$, then $\varphi T \in \mathcal{B}'_{pap}$.*
(iii) $\mathcal{B}_{pap} * \mathcal{D}'_{L^1} \subset \mathcal{B}_{pap}$.
(iv) $\mathcal{B}_{pap} = \mathcal{D}_{L^\infty} \cap \mathcal{C}_{pap}$.

Let $I =]0,1]$, $\varepsilon \in I$, and

$$\mathcal{M}_{L^\infty} = \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^\infty})^I : \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{-m}\right),\ \varepsilon \longrightarrow 0 \right\}, \tag{2.2}$$

$$\mathcal{N}_{L^\infty} = \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^\infty})^I : \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+,\ |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{m}\right),\ \varepsilon \longrightarrow 0 \right\}. \tag{2.3}$$

The algebra of bounded generalized functions on $\mathbb{R}$, is denoted and defined by the quotient algebra

$$\mathcal{G}_{L^\infty} := \frac{\mathcal{M}_{L^\infty}}{\mathcal{N}_{L^\infty}}, \tag{2.4}$$

An element u of $\mathcal{G}_{L^\infty}$ is an equivalence class, that is, $u = [(u_\varepsilon)_\varepsilon] = (u_\varepsilon)_\varepsilon + \mathcal{N}_{L^\infty}$. Following the construction of the algebra $\mathcal{G}_{ap}$ of almost periodic generalized functions, see [3], we define the space of pseudo almost periodic moderate elements

$$\mathcal{M}_{pap} = \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{pap})^I, \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{-m}\right), \varepsilon \longrightarrow 0 \right\}, \tag{2.5}$$

and the space of pseudo almost periodic negligible elements

$$\mathcal{N}_{pap} = \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{pap})^I, \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{m}\right), \varepsilon \longrightarrow 0 \right\}. \tag{2.6}$$

The main properties of $\mathcal{M}_{pap}$ and $\mathcal{N}_{pap}$ are summarized in the following proposition.

Proposition 2.2. (i) *The space $\mathcal{M}_{pap}$ is a subalgebra of $(\mathcal{B}_{pap})^I$.*
(ii) *The space $\mathcal{N}_{pap}$ is an ideal of $\mathcal{M}_{pap}$.*

Proof. (i) It follows from the fact that $\mathcal{B}_{pap}$ is a differential algebra.

(ii) Let $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{pap}$ and $(v_\varepsilon)_\varepsilon \in \mathcal{M}_{pap}$, then $\forall k \in \mathbb{Z}_+, \exists m' \in \mathbb{Z}_+, \exists c_1 > 0, \exists \varepsilon_0 \in I, \forall \varepsilon < \varepsilon_0, |v_\varepsilon|_{k,\infty} < c_1 \varepsilon^{-m'}$. Take $m \in \mathbb{Z}_+$, then for $m'' = m + m', \exists c_2 > 0$ such that $|u_\varepsilon|_{k,\infty} < c_2 \varepsilon^{m''}$. Since the family of the norms $|\ \ |_{k,\infty}$ is compatible with the algebraic structure of $\mathcal{D}_{L^\infty}$, then $\forall k \in \mathbb{Z}_+, \exists c_k > 0$ such that $|u_\varepsilon v_\varepsilon|_{k,\infty} \leq c_k |u_\varepsilon|_{k,\infty} |v_\varepsilon|_{k,\infty}$, consequently $|u_\varepsilon v_\varepsilon|_{k,\infty} < c_k c_2 \varepsilon^{m''} c_1 \varepsilon^{-m'} \leq c\varepsilon^m$, where $c = c_1 c_2 c_k$. Hence $(u_\varepsilon v_\varepsilon)_\varepsilon \in \mathcal{N}_{pap}$. □

Definition 2.2. *The algebra of pseudo almost periodic generalized functions is defined as the quotient algebra*

$$\mathcal{G}_{pap} := \frac{\mathcal{M}_{pap}}{\mathcal{N}_{pap}}. \tag{2.7}$$

We have the following results.

Proposition 2.3. $\mathcal{G}_{ap} \hookrightarrow \mathcal{G}_{pap} \hookrightarrow \mathcal{G}_{L^\infty}$.

A characterization of elements of $\mathcal{G}_{pap}$ is given by the following result.

Proposition 2.4. *Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{L^\infty}$, the following assertions are equivalent :*
(i) *u is pseudo almost periodic.*
(ii) $u_\varepsilon * \varphi \in \mathcal{B}_{pap}, \forall \varepsilon \in I, \forall \varphi \in \mathcal{D}$.

Proof. $(i) \Longrightarrow (ii)$: If $u \in \mathcal{G}_{pap}$, then for every $\varepsilon \in I$ we have $u_\varepsilon \in \mathcal{B}_{pap}$, the result (iii) of Proposition (2.1) gives $u_\varepsilon * \varphi \in \mathcal{B}_{pap}, \forall \varepsilon \in I, \forall \varphi \in \mathcal{D}$.

$(ii) \Longrightarrow (i)$: Let $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{L^\infty}$ and $u_\varepsilon * \varphi \in \mathcal{B}_{pap}$, $\forall \varepsilon \in I, \forall \varphi \in \mathcal{D}$, then from Theorem $(1.1) - (ii)$ it follows that $u_\varepsilon \in \mathcal{B}_{pap}$, it suffices to show that

$$\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{-m}\right), \varepsilon \longrightarrow 0, \tag{2.8}$$

which follows from the fact that $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{L^\infty}$. If $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{L^\infty}$ and $u_\varepsilon * \varphi \in \mathcal{B}_{pap}$, $\forall \varepsilon \in I, \forall \varphi \in \mathcal{D}$, we obtain the same result, because $\mathcal{N}_{L^\infty} \subset \mathcal{M}_{L^\infty}$. □

Remark. *The characterization (ii) does not depend on representatives.*

The space $\mathcal{B}_{pap}$ is canonically embedded into $\mathcal{G}_{pap}$, i.e.

$$\begin{array}{rccc} \sigma_{pap} : & \mathcal{B}_{pap} & \longrightarrow & \mathcal{G}_{pap} \\ & f & \longrightarrow & [(f)_\varepsilon] = (f)_\varepsilon + \mathcal{N}_{pap} \end{array} \tag{2.9}$$

Set $\Sigma = \left\{ \rho \in \mathcal{S} : \int_{\mathbb{R}} \rho(x)\, dx = 1 \text{ and } \int_{\mathbb{R}} x^\alpha \rho(x)\, dx = 0, \forall \alpha \geq 1 \right\}$ and $\rho_\varepsilon(.) = \frac{1}{\varepsilon}\rho\left(\frac{\cdot}{\varepsilon}\right), \varepsilon > 0$.

Proposition 2.5. *For $\rho \in \Sigma$, the map*

$$\begin{array}{rccc} i_{pap} : & \mathcal{B}'_{pap} & \longrightarrow & \mathcal{G}_{pap} \\ & T & \longrightarrow & (T * \rho_\varepsilon)_\varepsilon + \mathcal{N}_{pap}, \end{array} \tag{2.10}$$

is a linear embedding which commutes with derivatives.

Proof. Let $T \in \mathcal{B}'_{pap}$, from Theorem $(1.1) - (iii)$, $\exists (f_\beta)_\beta \subset \mathcal{C}_{pap}$ such that $T = \sum_{\beta \leq m} f_\beta^{(\beta)}$, so $\forall \alpha \in \mathbb{Z}_+$,

$$\left|\left(T^{(\alpha)} * \rho_\varepsilon\right)(x)\right| \leq \sum_{\beta \leq m} \frac{1}{\varepsilon^{\alpha+\beta}} \|f_\beta\|_{L^\infty} \int_{\mathbb{R}} \left|\rho^{(\alpha+\beta)}(y)\right| dy,$$

consequently, $\exists c > 0$ such that $\sup\limits_{x\in\mathbb{R}} \left| \left(T^{(\alpha)} * \rho_\varepsilon\right)(x)\right| \leq \frac{c}{\varepsilon^{\alpha+m}}$, hence, $\exists c' > 0$ such that

$$|T * \rho_\varepsilon|_{m',\infty} = \sum_{\alpha\leq m'} \sup_{x\in\mathbb{R}} \left| \left(T^{(\alpha)} * \rho_\varepsilon\right)(x)\right| \leq \frac{c'}{\varepsilon^{m+m'}}, \text{ where } c' = \sum_{\alpha\leq m'} \frac{c}{\varepsilon^{\alpha}}, \quad (2.11)$$

which shows that $(T * \rho_\varepsilon)_\varepsilon \in \mathcal{M}_{pap}$. Let $(T * \rho_\varepsilon)_\varepsilon \in \mathcal{N}_{pap}$, then $\lim\limits_{\varepsilon\longrightarrow 0} T * \rho_\varepsilon = 0$ in $\mathcal{D}'_{L^\infty}$, but we have also $\lim\limits_{\varepsilon\longrightarrow 0} T*\rho_\varepsilon = T$ in $\mathcal{D}'_{L^\infty}$, this mean that i_{pap} is an embedding. The linearity of i_{pap} it results from the fact that the convolution is linear and that $i_{pap}\left(T^{(j)}\right) = \left(T^{(j)} * \rho_\varepsilon\right)_\varepsilon = (T * \rho_\varepsilon)_\varepsilon^{(j)} = (i_{pap}(T))^{(j)}$. □

The following result shows that there are tow ways to embed the space $\mathcal{B}_{pap}$ into $\mathcal{G}_{pap}$.

Proposition 2.6. *The following diagram*

$$\begin{array}{ccc} \mathcal{B}_{pap} & \longrightarrow & \mathcal{B}'_{pap} \\ & \sigma_{pap} \searrow & \downarrow i_{pap} \\ & & \mathcal{G}_{pap} \end{array} \quad (2.12)$$

is commutative.

Proof. Let $f \in \mathcal{B}_{pap}$, we must show that $(f * \rho_\varepsilon - f)_\varepsilon \in \mathcal{N}_{pap}$. Indeed, by Taylor‘s formula and the fact that $\rho \in \Sigma$, we have

$$\|f * \rho_\varepsilon - f\|_{L^\infty} \leq \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} \left| \frac{(-y)^m}{m!} f^{(m)}(x - \theta\varepsilon y)\, \rho(y)\, dy \right| \varepsilon^m.$$

Then $\exists C_m > 0$ such that $\|f * \rho_\varepsilon - f\|_{L^\infty} \leq C_m \left\|f^{(m)}\right\|_{L^\infty} \|y^m \rho\|_{L^1} \varepsilon^m$. The same result can be obtained for all the derivatives of f. Hence $(f * \rho_\varepsilon - f)_\varepsilon \in \mathcal{N}_{pap}$. □

We have the following generalized version of the classical Bohl-Bohr's Theorem.

Proposition 2.7. *A primitive of a pseudo almost periodic generalized function is pseudo almost periodic if and only if it is bounded generalized function.*

Proof. Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{pap}$ and U its primitive, i.e. $U = [(U_\varepsilon)_\varepsilon]$. where $U_\varepsilon(x) = \int\limits_{x_0}^{x} u_\varepsilon(t)\, dt$ and $x_0 \in \mathbb{R}$. If $U \in \mathcal{G}_{pap}$, then by Proposition (2.3) , $U \in \mathcal{G}_{L^\infty}$. Conversely, let $x_0 \in \mathbb{R}$ and assume that $U = [(U_\varepsilon)_\varepsilon] \in \mathcal{G}_{L^\infty}$, then by definition $\forall\varepsilon \in I, \forall x \in \mathbb{R}, U_\varepsilon(x) = \int\limits_{x_0}^{x} u_\varepsilon(t)\, dt \in \mathcal{D}_{L^\infty}$, which show that U_ε is a bounded primitive of $u_\varepsilon \in \mathcal{C}_{pap}$. From the classical case, we deduce that $U_\varepsilon \in \mathcal{C}_{pap}$, i.e. $\forall\varepsilon \in I, U_\varepsilon \in \mathcal{C}_{pap} \cap \mathcal{D}_{L^\infty} = \mathcal{B}_{pap}$, Proposition $(2.1) - (iv)$. Moreover, $(U_\varepsilon)_\varepsilon \in \mathcal{M}_{L^\infty}$, i.e. $\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |U_\varepsilon|_{k,\infty} = O(\varepsilon^{-m}), \varepsilon \longrightarrow 0$. Thus, $(U_\varepsilon)_\varepsilon \in \mathcal{M}_{pap}$ and $U \in \mathcal{G}_{pap}$. The result is independent on representatives. □

3. Conclusion

This work has allowed us to lift the concept of pseudo almost periodicity to the level of generalized functions.The results obtained are the first steps to go on studying other problems. Some of them, the uniqueness of the decomposition of a pseudo almost periodic generalized function, the composition of tempered generalized function with pseudo almost periodic generalized function, the convolution and some results of existence for the linear differential equations in the framework of pseudo almost periodic generalized functions.

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