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ON THE STABILIZATION FOR A CLASS OF DISTRIBUTED BILINEAR SYSTEMS USING BOUNDED CONTROLS

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Abstract. This paper considers the question of the output stabilization for a class of infinite dimensional bilinear systems evolving on a spatial domain Ω. Then, we give sufficient conditions for exponential, strong and weak stabilization of the output of such systems. Examples and simulations illustrate the efficiency of such controls.

1. Introduction

In this paper, we consider the following bilinear system

$$\begin{cases} \dot{z}(t) = Az(t) + v(t)Bz(t), \quad t \geq 0, \\ z(0) = z_0, \end{cases} \tag{1.1}$$

where $A : D(A) \subset H \to H$ generates a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on a Hilbert space H, endowed with norm and inner product denoted, respectively, by $\|.\|$ and $\langle ., . \rangle$, $v(.) \in V_{ad}$ (the admissible controls set) is a scalar valued control and $B : H \to H$ is a linear bounded operator. The problem of feedback stabilization of distributed system (1.1) was studied in many works that lead to various results. In [1], it was shown that the control

$$v(t) = -\langle z(t), Bz(t) \rangle, \tag{1.2}$$

weakly stabilizes system (1.1) provided that B be a weakly sequentially continuous operator such that, for all $\psi \in H$, we have

$$\langle BS(t)\psi, S(t)\psi \rangle = 0, \quad \forall \ t \geq 0 \Longrightarrow \psi = 0, \tag{1.3}$$

and if (1.3) is replaced by the following assumption

$$\int_0^T |\langle BS(s)\psi, S(s)\psi \rangle| ds \geq \gamma \|\psi\|^2, \quad \forall \ \psi \in H, \ \text{(for some } \gamma, T > 0), \tag{1.4}$$

then control (1.2) strongly stabilizes system (1.1) (see [2]). In [3], the authors show that when the resolvent of A is compact, B self-adjoint and monotone, then strong

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stabilization of system (1.1) is proved using bounded controls. Let the output state space Y be a Hilbert space with inner product $\langle ., . \rangle_Y$ and the corresponding norm $\|.\|_Y$, and let $C \in \mathcal{L}(H, Y)$ be an output operator. The system (1.1) is augmented with the output

$$w(t) := Cz(t). \tag{1.5}$$

The output stabilization means that $w(t) \to 0$ as $t \to +\infty$ using suitable controls. In the case when $Y = H$ and $C = I$, one obtains the classical stabilization of the state. When $C \neq I$, the output stabilization for distributed systems was studied in many works: in [14], authors considered the output exponential stabilization for one-dimensional wave equations with boundary control. In [4], authors considered output stabilization for Kirchhoff-type equation with boundary control. They studied the existence and uniqueness of solution of system and the strong stabilization of such equation was proved. In [6], authors established the output stabilization for a class of nonlinear systems with boundary control. They investigated the existence of solution and the exponential stabilization of such systems. In [7], author studied weak and strong output stabilization for semilinear systems using controls that do not take into account the output operator. In [11], authors considered exponential, strong and weak output stabilization of semilinear systems. If $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ be the system evolution domain and $\omega \subset \Omega$, when $C = \chi_\omega$, the restriction operator to a subregion ω of Ω, one is concerned with the behaviour of the state only in a subregion of the system evolution domain. This is what we call regional stabilization. The notion of regional stabilization is useful in systems theory since there exist systems which are not stabilizable on the whole domain but stabilizable on some subregion ω. Moreover stabilizing a systems on a subregion is cheaper than stabilizing it on the whole domain [12]. In [13], regional stabilization for bilinear systems was studied using decomposition of system (1.1) into regional stable and regional unstable subsystems, therefore regional stabilization of system (1.1) turns out to stabilizing its unstable part. In [10], authors proved regional strong and weak stabilization of bilinear systems with unbounded control operator. In [9], authors considered regional weak, strong and exponential stabilization of bilinear systems with control operator B assumed to be bounded with respect to the graph norm of the operator A. In this paper, we study the exponential, strong and weak stabilization of the output (1.5) using bounded controls. Then, we develop sufficient conditions that allow exponential, strong and weak stabilization of the output of such system. Illustrations by examples and simulations are given. The approach is based on the decay of an adapted function, the exact and weak observability conditions, and semigroup properties. The paper is organized as follows. The second section discusses sufficient conditions to achieve exponential, strong and weak stabilization of the output (1.5). In the third section, we give illustrating examples. The fourth section is devoted to simulations.

2. Output stabilization

In this section, we develop sufficient conditions that allow exponential, strong and weak stabilization of the output (1.5).

Definition 2.1. *The output (1.5) is said to be:*
1. weakly stabilizable, if there exists a control $v(.) \in V_{ad}$ such that for any initial condition $z_0 \in H$, the corresponding solution $z(t)$ of system (1.1) is global and

satisfies

$$\langle Cz(t), \psi\rangle_Y \to 0, \quad \forall \psi \in Y, \quad \text{as } t \to \infty,$$

2. strongly stabilizable, if there exists a control $v(.) \in V_{ad}$ *such that for any initial condition* $z_0 \in H$*, the corresponding solution* $z(t)$ *of system (1.1) is global and verifies*

$$\|Cz(t)\|_Y \to 0, \quad \text{as } t \to \infty,$$

3. exponentially stabilizable, if there exists a control $v(.) \in V_{ad}$ *such that for any initial condition* $z_0 \in H$*, the corresponding solution* $z(t)$ *of system (1.1) is global and there exist* $\alpha, \beta > 0$ *such that*

$$\|Cz(t)\|_Y \leq \alpha e^{-\beta t}\|z_0\|, \quad \forall t > 0.$$

Remark. *It is clear that exponential stability of (1.5) implies strong stability of (1.5) implies weak stability of (1.5).*

2.1. **Exponential stabilization.** In this subsection, we develop sufficient conditions for exponential stabilization of the output (1.5).
The following result concerns the exponential stabilization of (1.5).

Theorem 2.1. *Let* A *generate a semigroup* $(S(t))_{t\geq 0}$ *of contractions on* H *and* B *is a bounded control operator. If the conditions:*
1. $\mathcal{R}e(\langle C^*CAy, y\rangle) \leq 0$, $\forall y \in D(A)$,
2. $\mathcal{R}e(\langle C^*CBy, y\rangle\langle By, y\rangle) \geq 0$, $\forall y \in H$,
3. there exist $T, \gamma > 0$*, such that*

$$\int_0^T |\langle C^*CBS(t)y, S(t)y\rangle| dt \geq \gamma \|Cy\|_Y^2, \ \ \forall\, y \in H, \tag{2.1}$$

hold, then the control

$$v(t) = \begin{cases} -\frac{\langle C^*CBz(t), z(t)\rangle}{\|z(t)\|^2} & \text{if } z(t) \neq 0 \\ 0 & \text{if } z(t) = 0, \end{cases} \tag{2.2}$$

exponentially stabilizes the output (1.5).

Proof. System (1.1) has a unique weak solution $z(t)$ (see [8]) defined on a maximal interval $[0, t_{\max}]$ by

$$z(t) = S(t)z_0 + \int_0^t g(z(s))S(t-s)Bz(s)ds, \tag{2.3}$$

where

$$g(z(t)) = \begin{cases} -\frac{\langle C^*CBz(t), z(t)\rangle}{\|z(t)\|^2} & \text{if } z(t) \neq 0 \\ 0 & \text{if } z(t) = 0. \end{cases}$$

Since $(S(t))_{t\geq 0}$ is a semigroup of contractions, we deduce

$$\frac{d}{dt}\|z(t)\|^2 \leq 2g(z(t))\langle Bz(t), z(t)\rangle.$$

Integrating this inequality over the interval $[0, t]$, we have

$$\|z(t)\|^2 - \|z(0)\|^2 \leq 2\int_0^t g(z(s))\langle Bz(s), z(s)\rangle ds.$$

Using hypothesis 2 of Theorem 2.1, it follows that

$$\|z(t)\| \leq \|z_0\|. \tag{2.4}$$

For all $z_0 \in H$ and $t \geq 0$, we have

$$\begin{aligned}\langle C^*CBS(t)z_0, S(t)z_0\rangle &= \langle C^*CBz(t), z(t)\rangle - \langle C^*CBz(t), z(t) - S(t)z_0\rangle \\ &+ \langle C^*CBS(t)z_0 - C^*CBz(t), S(t)z_0\rangle.\end{aligned}$$

Since B is bounded, then

$$|\langle C^*CBS(t)z_0, S(t)z_0\rangle| \leq |\langle C^*CBz(t), z(t)\rangle| + 2\alpha\|B\|\|z(t) - S(t)z_0\|\|z_0\|, \quad (2.5)$$

where α is a positive constant.
Using (2.4), we deduce

$$|\langle C^*CBz(t), z(t)\rangle| \leq |g(z(t))|\|z(t)\|\|z_0\|, \ \forall t \in [0, T]. \quad (2.6)$$

While from (2.3) and using Schwartz's inequality, we obtain

$$\|z(t) - S(t)z_0\| \leq \|B\| \left(T\int_0^T |g(z(t))|^2\|z(t)\|^2 dt\right)^{\frac{1}{2}}. \quad (2.7)$$

Integrating (2.5) over the interval $[0, T]$ and taking into account (2.6) and (2.7), we have

$$\begin{aligned}\int_0^T |\langle C^*CBS(t)z_0, S(t)z_0\rangle| dt &\leq 2\alpha T^{\frac{3}{2}}\|B\|^2\|z_0\| \left(\int_0^T |g(z(t))|^2\|z(t)\|^2 dt\right)^{\frac{1}{2}} \\ &+ T^{\frac{1}{2}}\|z_0\| \left(\int_0^T |g(z(t))|^2\|z(t)\|^2 dt\right)^{\frac{1}{2}}. \quad (2.8)\end{aligned}$$

Let us consider the nonlinear semigroup $U(t)z_0 := z(t)$ (see [1]). Replacing z_0 by $U(t)z_0$ in (2.8), and using the superposition properties of the semigroup $(U(t))_{t\geq 0}$, we deduce that

$$\begin{aligned}\int_0^T |\langle C^*CBS(s)U(t)z_0, S(s)U(t)z_0\rangle| ds &\leq 2\alpha T^{\frac{3}{2}}\|B\|^2\|U(t)z_0\| \quad (2.9) \\ &\times \left(\int_t^{t+T} |g(U(s)z_0)|^2\|U(s)z_0\|^2 ds\right)^{\frac{1}{2}} \\ &+ T^{\frac{1}{2}}\|U(t)z_0\| \left(\int_t^{t+T} |g(U(s)z_0)|^2\|U(s)z_0\|^2 ds\right)^{\frac{1}{2}}.\end{aligned}$$

Thus, by using (2.1) and (2.9), it follows that

$$\gamma\|CU(t)z_0\|_Y \leq M \left(\int_t^{t+T} |g(U(s)z_0)|^2\|U(s)z_0\|^2 ds\right)^{\frac{1}{2}}, \quad (2.10)$$

where $M = (2\alpha T\|B\|^2 + 1)T^{\frac{1}{2}}$ is a positive constant depending on $\|z_0\|$ and T.
From hypothesis 1 of Theorem 2.1, we have

$$\frac{d}{dt}\|CU(t)z_0\|_Y^2 \leq -2|g(U(t)z_0)|^2\|U(t)z_0\|^2. \quad (2.11)$$

Integrating (2.11) from nT and $(n+1)T$, $(n \in \mathbb{N})$, we obtain

$$\|CU(nT)z_0\|_Y^2 - \|CU((n+1)T)z_0\|_Y^2 \geq 2\int_{nT}^{(n+1)T} |g(U(s)z_0)|^2\|U(s)z_0\|^2 ds.$$

Using (2.10), (2.11) and the fact that $\|CU(t)z_0\|_Y$ decreases, it follows

$$\left(1+2\left(\frac{\gamma}{M}\right)^2\right)\|CU((n+1)T)z_0\|_Y^2 \leq \|CU(nT)z_0\|_Y^2.$$

Then

$$\|CU((n+1)T)z_0\|_Y \leq \beta\|CU(nT)z_0\|_Y,$$

where $\beta = \frac{1}{\left(1+2\left(\frac{\gamma}{M}\right)^2\right)^{\frac{1}{2}}}$. By recurrence, we show that $\|CU(nT)z_0\|_Y \leq \beta^n\|Cz_0\|_Y$. Taking $n = E(\frac{t}{T})$ the integer part of $\frac{t}{T}$, we deduce that

$$\|CU(t)z_0\|_Y \leq Re^{-\sigma t}\|z_0\|,$$

where $R = \alpha\left(1+2\left(\frac{\gamma}{M}\right)^2\right)^{\frac{1}{2}}$, with $\alpha > 0$ and $\sigma = \frac{\ln\left(1+2\left(\frac{\gamma}{M}\right)^2\right)}{2T} > 0$, which achieves the proof. □

2.2. **Strong stabilization.** The following result will be used to prove strong stabilization of the output (1.5).

Theorem 2.2. *Let A generate a semigroup $(S(t))_{t\geq 0}$ of contractions on H and $B : H \to H$ is a bounded linear operator. If the conditions:*
1. $\mathcal{R}e\big(\langle C^*CA\psi,\psi\rangle\big) \leq 0,\ \forall\psi \in D(A)$,
2. $\mathcal{R}e\big(\langle C^*CB\psi,\psi\rangle\langle B\psi,\psi\rangle\big) \geq 0,\ \forall\psi \in H$,
hold, then control

$$v(t) = -\frac{\langle C^*CBz(t),z(t)\rangle}{1+|\langle C^*CBz(t),z(t)\rangle|}, \tag{2.12}$$

allows the estimate

$$\left(\int_0^T |\langle C^*CBS(s)z(t),S(s)z(t)\rangle|ds\right)^2 = \Theta\left(\int_t^{t+T}\frac{|\langle C^*CBz(s),z(s)\rangle|^2}{1+|\langle C^*CBz(s),z(s)\rangle|}ds\right), \quad \text{as } t\to+\infty. \tag{2.13}$$

Proof. We have

$$\frac{1}{2}\frac{d}{dt}\langle Cz(t),Cz(t)\rangle_Y = \mathcal{R}e\big(\langle CAz(t),Cz(t)\rangle_Y\big) + \mathcal{R}e\big(v(t)\langle CBz(t),Cz(t)\rangle_Y\big).$$

Then

$$\frac{1}{2}\frac{d}{dt}\langle Cz(t),Cz(t)\rangle_Y = \frac{1}{2}\frac{d}{dt}\|Cz(t)\|_Y^2 = \mathcal{R}e\big(\langle C^*CAz(t),z(t)\rangle\big)+\mathcal{R}e\big(v(t)\langle C^*CBz(t),z(t)\rangle\big).$$

From hypothesis 1 of Theorem 2.2, we have

$$\frac{1}{2}\frac{d}{dt}\|Cz(t)\|_Y^2 \leq \mathcal{R}e\big(v(t)\langle C^*CBz(t),z(t)\rangle\big).$$

In order to make the function $\frac{1}{2}\|Cz(t)\|_Y^2$ nonincreasing, we consider the control

$$v(t) = -\frac{\langle C^*CBz(t),z(t)\rangle}{1+|\langle C^*CBz(t),z(t)\rangle|},$$

so that the resulting closed-loop system is

$$\dot{z}(t) = Az(t) + f(z(t)),\ z(0) = z_0, \tag{2.14}$$

where $f(z) = -\frac{\big\langle C^*CBz,z\big\rangle Bz}{1+|\big\langle C^*CBz,z\big\rangle|}$, $\forall\ z \in H$.
Since f is locally Lipschitz, then system (2.14) has a unique mild solution $z(t)$ (see Theorem 1.4, pp 185 in [8]) defined on a maximal interval $[0, t_{max}]$ by

$$z(t) = S(t)z_0 + \int_0^t S(t-s)f(z(s))ds. \tag{2.15}$$

Because of the contractions of the semigroup (i.e $\mathcal{R}e\big(\big\langle A\psi, \psi\big\rangle\big) \leq 0,\ \forall\ \psi \in D(A)$), we have

$$\frac{d}{dt}\|z(t)\|^2 \leq -2\frac{\big\langle C^*CBz(t), z(t)\big\rangle\langle Bz(t), z(t)\rangle}{1+|\big\langle C^*CBz(t), z(t)\big\rangle|}.$$

Integrating this inequality over the interval $[0, t]$, we deduce

$$\|z(t)\|^2 - \|z(0)\|^2 \leq -2\int_0^t \frac{\big\langle C^*CBz(s), z(s)\big\rangle\langle Bz(s), z(s)\rangle}{1+|\big\langle C^*CBz(s), z(s)\big\rangle|}ds.$$

Using condition 2 of Theorem 2.2, it follows that

$$\|z(t)\| \leq \|z_0\|. \tag{2.16}$$

From hypothesis 1 of Theorem 2.2, we have

$$\frac{d}{dt}\|Cz(t)\|_Y^2 \leq -2\frac{|\big\langle C^*CBz(t), z(t)\big\rangle|^2}{1+|\big\langle C^*CBz(t), z(t)\big\rangle|}.$$

Integrating this inequality, we deduce

$$\|Cz(t)\|_Y^2 - \|Cz(0)\|_Y^2 \leq -2\int_0^t \frac{|\big\langle C^*CBz(s), z(s)\big\rangle|^2}{1+|\big\langle C^*CBz(s), z(s)\big\rangle|}ds. \tag{2.17}$$

While from (2.15) and using Schwartz inequality, we obtain

$$\|z(t) - S(t)z_0\| \leq \|B\|\|z_0\|\left(T\int_0^t \frac{|\big\langle C^*CBz(s), z(s)\big\rangle|^2}{1+|\big\langle C^*CBz(s), z(s)\big\rangle|}ds\right)^{\frac{1}{2}},\ \forall t \in [0, T]. \tag{2.18}$$

Since B is bounded and C continuous, we have

$$|\langle C^*CBS(s)z_0, S(s)z_0\rangle| \leq 2K\|B\|\|z(s) - S(s)z_0\|\|z_0\| + |\langle C^*CBz(s), z(s)\rangle|, \tag{2.19}$$

where K is a positive constant. Replacing z_0 by $z(t)$ in (2.18) and (2.19), we deduce

$$\begin{aligned} |\langle C^*CBS(s)z(t), S(s)z(t)\rangle| \ &\leq\ 2K\|B\|^2\|z_0\|^2\left(T\int_t^{t+T} \frac{|\big\langle C^*CBz(s), z(s)\big\rangle|^2}{1+|\big\langle C^*CBz(s), z(s)\big\rangle|}ds\right)^{\frac{1}{2}} \\ &+\ |\langle C^*CBz(t+s), z(t+s)\rangle|, \quad \forall t \geq s \geq 0. \end{aligned}$$

Integrating this relation over $[0, T]$ and using Cauchy-Schwartz, we obtain

$$\begin{aligned} \int_0^T |\langle C^*CBS(s)z(t), S(s)z(t)\rangle|ds \ &\leq\ \left(2K\|B\|^2T^{\frac{3}{2}} + T\left(1 + K\|B\|\|z_0\|^2\right)\right) \\ &\times\ \left(\int_t^{t+T} \frac{|\big\langle C^*CBz(s), z(s)\big\rangle|^2}{1+|\big\langle C^*CBz(s), z(s)\big\rangle|}ds\right)^{\frac{1}{2}}, \end{aligned}$$

which achieves the proof. □

The following result gives sufficient conditions for strong stabilization of the output (1.5).

Theorem 2.3. *Let A generate a semigroup $(S(t))_{t\geq 0}$ of contractions on H and B is a bounded linear operator. If the assumptions 1, 2 of Theorem 2.2 and*

$$\int_0^T |\langle C^*CBS(t)\psi, S(t)\psi\rangle| dt \geq \gamma \|C\psi\|_Y^2, \ \ \forall \ \psi \in H, \ (for\ some\ T, \gamma > 0), \quad (2.20)$$

hold, then control (2.12) strongly stabilizes the output (1.5) with decay estimate

$$\|Cz(t)\|_Y = \Theta\left(\frac{1}{\sqrt{t}}\right), \ \ as \ \ t \longrightarrow +\infty. \quad (2.21)$$

Proof. Using (2.17), we deduce

$$\|Cz(kT)\|_Y^2 - \|Cz((k+1)T)\|_Y^2 \geq 2\int_{kT}^{k(T+1)} \frac{|\big\langle C^*CBz(t), z(t)\big\rangle|^2}{1 + |\big\langle C^*CBz(t), z(t)\big\rangle|} dt, \ k \geq 0.$$

From (2.13) and (2.20), we have

$$\|Cz(kT)\|_Y^2 - \|Cz((k+1)T)\|_Y^2 \geq \beta \|Cz(kT)\|_Y^4, \quad (2.22)$$

where $\beta = \frac{\gamma^2}{2\left(2K\|B\|^2T^{\frac{3}{2}} + T\left(1+K\|B\|\|z_0\|^2\right)\right)^2}$. Taking $s_k = \|Cz(kT)\|_Y^2$, the inequality (2.22) can be written as

$$\beta s_k^2 + s_{k+1} \leq s_k, \ \ \forall k \geq 0.$$

Since $s_{k+1} \leq s_k$, we obtain

$$\beta s_{k+1}^2 + s_{k+1} \leq s_k, \ \ \forall k \geq 0.$$

Taking $p(s) = \beta s^2$ and $q(s) = s - (I+p)^{-1}(s)$ in Lemma 3.3, page 531 in [5], we deduce

$$s_k \leq x(k), \ \ k \geq 0,$$

where $x(t)$ is the solution of equation $x'(t) + q(x(t)) = 0, \ \ x(0) = s_0$.
Since $x(k) \geq s_k$ and $x(t)$ decreases give $x(t) \geq 0$, $\forall t \geq 0$. Furthermore, it is easy to see that $q(s)$ is an increasing function such that

$$0 \leq q(s) \leq p(s), \forall s \geq 0.$$

We obtain $-\beta x(t)^2 \leq x'(t) \leq 0$, which implies that

$$x(t) = \Theta(t^{-1}), \text{ as } t \to +\infty.$$

Finally the inequality $s_k \leq x(k)$, together with the fact that $\|Cz(t)\|_Y$ decreases, we deduce the estimate

$$\|Cz(t)\|_Y = \Theta\left(\frac{1}{\sqrt{t}}\right), \ \text{ as } \ t \longrightarrow +\infty.$$

□

2.3. **Weak stabilization.** The following result provides sufficient conditions for weak stabilization of the output (1.5).

Theorem 2.4. *Let A generate a semigroup $(S(t))_{t\geq 0}$ of contractions on H and B is a compact operator. If the conditions:*
1. $\mathcal{R}e\big(\big\langle C^*CA\psi, \psi\big\rangle\big) \leq 0$, $\forall \psi \in D(A)$,
2. $\mathcal{R}e\big(\big\langle C^*CB\psi, \psi\big\rangle\big\langle B\psi, \psi\big\rangle\big) \geq 0$, $\forall \psi \in H$,
3. $\big\langle C^*CBS(t)\psi, S(t)\psi\big\rangle = 0, \ \ \forall t \geq 0 \Longrightarrow C\psi = 0$,
hold, then control (2.12) weakly stabilizes the output (1.5).

Proof. Let us consider the nonlinear semigroup $\Gamma(t)z_0 := z(t)$ and let (t_n) be a sequence of real numbers such that $t_n \longrightarrow +\infty$ as $n \longrightarrow +\infty$.
From (2.16), $\Gamma(t_n)z_0$ is bounded in H, then there exists a subsequence $(t_{\phi(n)})$ of (t_n) such that

$$\Gamma(t_{\phi(n)})z_0 \rightharpoonup \psi, \text{ as } n \to \infty.$$

Since B is compact and C continuous, we have

$$\lim_{n\to+\infty} \langle C^*CBS(t)\Gamma(t_{\phi(n)})z_0, S(t)\Gamma(t_{\phi(n)})z_0\rangle = \langle C^*CBS(t)\psi, S(t)\psi\rangle.$$

For all $n \geq$, we set

$$\Lambda_n(t) := \int_{\phi(n)}^{\phi(n)+t} \frac{|\langle C^*CB\Gamma(s)z_0, \Gamma(s)z_0\rangle|^2}{1+|\langle C^*CB\Gamma(s)z_0, \Gamma(s)z_0\rangle|} ds.$$

It follows that $\forall t \geq 0$, $\Lambda_n(t) \to 0$ as $n \to +\infty$.
Using (2.13), we deduce

$$\lim_{n\to+\infty} \int_0^t |\langle C^*CBS(s)\Gamma(t_{\phi(n)})z_0, S(s)\Gamma(t_{\phi(n)})z_0\rangle| ds = 0.$$

Hence, by the dominated convergence Theorem, we have

$$\int_0^t |\langle C^*CBS(s)\psi, S(s)\psi\rangle| ds = 0.$$

We conclude that

$$\langle C^*CBS(s)\psi, S(s)\psi\rangle = 0, \ \ \forall s \in [0,t].$$

Using condition 3 of Theorem 2.4, we deduce that

$$C\Gamma(t_{\phi(n)})z_0 \rightharpoonup 0, \ \text{ as } \ n \longrightarrow +\infty. \tag{2.23}$$

On the other hand, it is clear that (2.23) holds for each subsequence $(t_{\phi(n)})$ of (t_n) such that $C\Gamma(t_{\phi(n)})z_0$ weakly converges in Y. This implies that $\forall \varphi \in Y$, we have $\langle C\Gamma(t_n)z_0, \varphi\rangle \to 0$ as $n \longrightarrow +\infty$ and hence

$$C\Gamma(t)z_0 \rightharpoonup 0, \ \text{ as } \ t \longrightarrow +\infty.$$

□

3. Examples

Example 3.1. *Let Ω denote a bounded open subset of $\mathbb{R}^n$, and consider the following wave equation*

$$\begin{cases} \dfrac{\partial^2 z(x,t)}{\partial t^2} - \Delta z(x,t) = v(t)\dfrac{\partial z(x,t)}{\partial t} & \Omega\times]0,+\infty[\\ z(x,t) = 0 & \partial\Omega\times]0,+\infty[\\ z(x,0) = z_0(x), \ \dfrac{\partial z(x,0)}{\partial t} = z_1(x) & \Omega. \end{cases} \tag{3.1}$$

This system has the form of equation (1.1) if we set $H = H_0^1(\Omega) \times L^2(\Omega)$ with $\langle (y_1,z_1),(y_2,z_2)\rangle = \langle y_1,y_2\rangle_{H^1(\Omega)} + \langle z_1,z_2\rangle_{L^2(\Omega)}$, $A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$.

We consider the output operator $C = I$, we have A is skew-adjoint on H and the assumption (2.20) holds (see [2]). Then the control

$$v(t) = -\frac{\|\frac{\partial z(.,t)}{\partial t}\|^2_{L^2(0,1)}}{1+\|\frac{\partial z(.,t)}{\partial t}\|^2_{L^2(0,1)}}, \tag{3.2}$$

strongly stabilises system (3.1) with the decay estimate

$$\|(z(.,t), \frac{\partial z(.,t)}{\partial t})\|_H = \Theta(\frac{1}{\sqrt{t}}), \quad as\ t \longrightarrow +\infty.$$

Example 3.2. *Let us consider a system defined on $\Omega =]0,1[$ by*

$$\begin{cases} \dfrac{\partial z(x,t)}{\partial t} = Az(x,t) + v(t)a(x)z(x,t) & \Omega\times]0,+\infty[\\ z(x,0) = z_0(x) & \Omega, \end{cases} \tag{3.3}$$

where $H = L^2(\Omega)$, $Az = -z$, and $a \in L^\infty(]0,1[)$ such that $a(x) \geq 0$ a.e on $]0,1[$ and $a(x) \geq c > 0$ on subregion ω of Ω and $v(.) \in L^\infty(0,+\infty)$ the control function. System (3.3) is augmented with the output

$$w(t) = \chi_\omega z(t), \tag{3.4}$$

where $\chi_\omega : L^2(\Omega) \longrightarrow L^2(\omega)$, the restriction operator to ω and χ^*_ω is the adjoint operator of χ_ω. The operator A generates a semigroup of contractions on $L^2(\Omega)$ given by $S(t)z_0 = e^{-t}z_0$. For all $z_0 \in L^2(\Omega)$ and $T = 2$, we obtain

$$\begin{aligned} \int_0^2 \langle \chi^*_\omega \chi_\omega BS(t)z_0, S(t)z_0\rangle dt &= \int_0^2 e^{-2t}dt \int_\omega a(x)|z_0|^2 dx \\ &\geq \beta \|\chi_\omega z_0\|^2_{L^2(\omega)}, \end{aligned}$$

with $\beta = c\displaystyle\int_0^2 e^{-2t}dt > 0$.
Then the control

$$v(t) = -\frac{\displaystyle\int_\omega a(x)|z(x,t)|^2 dx}{1+\displaystyle\int_\omega a(x)|z(x,t)|^2 dx},$$

strongly stabilizes the output (3.4) with decay estimate

$$\|\chi_\omega z(t)\|_{L^2(\omega)} = \Theta\left(\frac{1}{\sqrt{t}}\right), \quad as\ t \longrightarrow +\infty.$$

Example 3.3. *Consider a system defined in $\Omega =]0,+\infty[$, and described by*

$$\begin{cases} \dfrac{\partial z(x,t)}{\partial t} = -\dfrac{\partial z(x,t)}{\partial x} + v(t)Bz(x,t) & \Omega\times]0,+\infty[\\ z(x,0) = z_0(x) & \Omega, \end{cases} \tag{3.5}$$

where $Az = -\dfrac{\partial z}{\partial x}$ *with domain* $D(A) = \{z \in H^1(\Omega) \mid z(0) = 0,\ z(x) \to 0$ *as* $x \to +\infty\}$ *and* $Bz = \displaystyle\int_0^1 z(x)dx$. *The operator* A *generates a semigroup of contractions*

$$(S(t)z_0)(x) = \begin{cases} z_0(x-t) & if\, x \geq t \\ 0 & if\, x < t. \end{cases}$$

Let $\omega =]0,1[$ *be a subregion of* Ω *and system (3.5) is augmented with the output*

$$w(t) = \chi_\omega z(t). \tag{3.6}$$

We have

$$\begin{aligned} \mathcal{R}e(\langle \chi_\omega^* \chi_\omega Az, z \rangle) &= -\mathcal{R}e(\int_0^1 z'(x)z(x)dx) \\ &= -\frac{z^2(1)}{2} \leq 0, \end{aligned}$$

so, the assumption 1 of Theorem 2.4 holds. The operator B *is compact and verifies*

$$\langle \chi_\omega^* \chi_\omega BS(t)z_0, S(t)z_0 \rangle = \left(\int_0^{1-t} z_0(x)dx \right)^2, \quad 0 \leq t \leq 1.$$

Thus

$$\langle \chi_\omega^* \chi_\omega BS(t)z_0, S(t)z_0 \rangle = 0, \ \ \forall t \geq 0 \implies z_0(x) = 0, \ a.e \ \ on\, \omega.$$

Then, the control

$$v(t) = -\frac{\left(\displaystyle\int_0^1 z(x,t)dx \right)^2}{1 + \left(\displaystyle\int_0^1 z(x,t)dx \right)^2}, \tag{3.7}$$

weakly stabilizes the output (3.6).

4. Simulations

Consider system (3.5) with $z(x,0) = \sin(\pi x)$, and augmented with the output (3.6).

• For $\omega =]0,2[$, figure 1 shows that the state is stabilized on ω with error equals 3.4×10^{-4}, and the evolution of control function is given by figure 2.

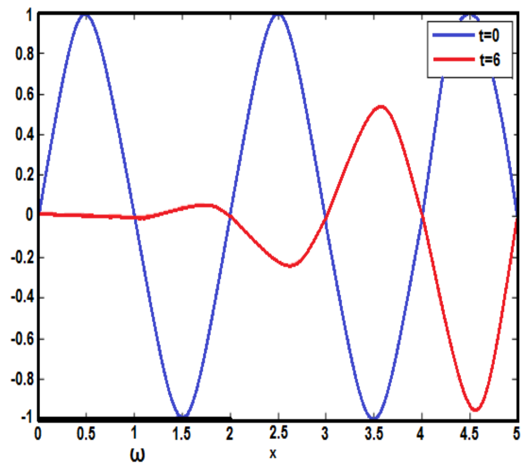

Figure 1. The stabilization of the state on $\omega =]0, 2[$.

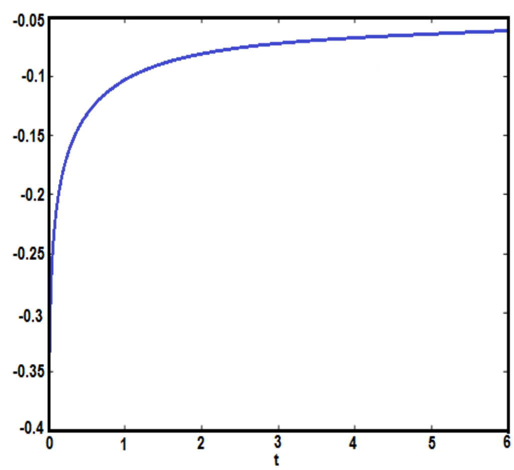

Figure 2. Evolution of control function.

• For $\omega =]0, 3[$, figure 3 shows that the state is stabilized on ω with error equals 7.8×10^{-4} and the evolution of control is given by figure 4.

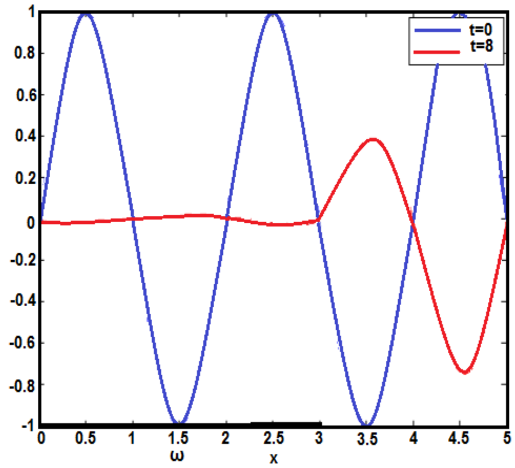

FIGURE 3. The stabilization of the state on $\omega =]0, 3[$.

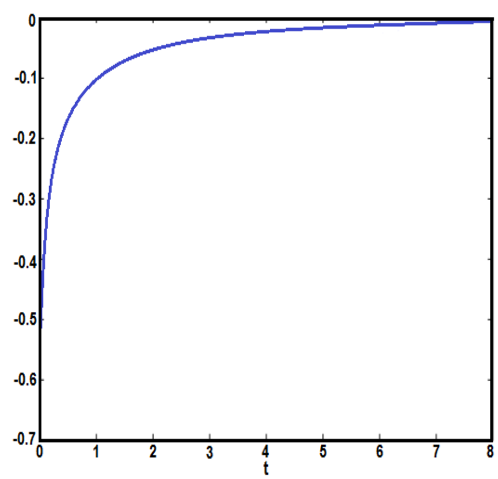

FIGURE 4. Evolution of control function.

5. Conclusion

The output stabilization of bilinear systems is discussed. Under sufficient conditions, we give bounded controls depending on the output operator that exponentially, strongly and weakly stabilizes the output of such systems. Numerical simulations illustrate the efficiency theoretical results. This work gives an opening to others questions, this is the case of output stabilization of semilinear systems with bounded controls.

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