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Analysis of The Dynamics of The Classical Epidemic Model with Beta Distributed Random Components

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ABSTRACT: In this study, the classical epidemic model of Kermack and McKendrick is analyzed with beta distributed random components. A random analysis is done for the deterministic epidemic model by transforming the parameters and initial values of the system to random variables with beta distribution. The approximations for the expectations of the model variables are compared with the deterministic results to comment on the randomness of the cases with random parameters and random initial values. Results for some numerical characteristics of these two cases are also given to investigate the accuracy of the approximations for the expected values.

Keywords: SIR Model, Random Effect, Beta Distribution, Moment, Random Differential Equation.

Klasik Salgın Hastalik Modeli Dinamiklerinin Beta Dağılımına Sahip Rastgele Bileşenlerle İncelenmesi

ÖZET: Bu çalışmada Kermack ve McKendrick'in klasik salgın hastalık modeli beta dağılımına sahip rastgele bileşenlerle incelenmektedir. Deterministik model için sistemin parametreleri ve başlangıç koşulları beta dağılımına sahip rastgele değişkenlere dönüştürülerek bir rastgele inceleme yapılmaktadır. Model değişkenlerinin beklenen değerleri için elde edilen yaklaşımlar deterministik sonuçlarla karşılaştırılarak rastgele başlangıç koşulları ve rastgele parametre içeren durumların rastgele yapıları hakkında yorum yapılmaktadır. Beklenen değerlerin yaklaşımlarının doğruluğunun incelenmesi için iki durumun bazı sayısal karakteristiklerinin sonuçları da verilmektedir.

Anahtar Kelimeler: SIR Modeli, Rastgele Etki, Beta Dağılımı, Moment, Rastgele Diferansiyel Denklem.

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INTRODUCTION

The SIR model, also known as the classical epidemic model, is a milestone in mathematical epidemiology studies. It enables the investigation of disease dynamics through the analysis of the compartments of any population under consideration. The total population is divided into 3 compartments (Susceptible – Infected – Recovered) and the changes in these compartments over time are analyzed to monitor the course of the disease. SIR model was introduced by Kermack and McKendrick in early 1900s and takes its name from the capital letters of the compartments (Kermack and McKendrick, 1927). Various other models have been formed from the classical model by adding new compartments and parameters to analyze many other diseases and events in medicine, biology etc. (Khan et al., 2015; Merdan et al., 2017). SIS (Susceptible – Infected – Susceptible), SEIR (Susceptible – Exposed – Infected – Recovered) and MSEIR (Passively immune – Susceptible – Exposed – Infected – Recovered) are some of the derivatives of the classical epidemic model (Hethcote, 2000).

A vast majority of the literature on mathematical modeling studies in various fields of science are made on a deterministic level. That is, the randomness of the real life course of events is ignored in these studies. The recent works are focused on a fractional and/or numerical perspective for modeling studies (Araz and Durur, 2018; Yokus and Yavuz, 2018; Dokuyucu et al., 2018; Singh et al., 2018; Prakasha et al., 2019; Dokuyucu, 2019; Durur et al., 2019; Rasool et al., 2019; Yokus, 2019; Durur, 2019; Yokus, 2020). However, it is known that some components of modeling studies in epidemiology, biochemistry and etc. are determined through statistical investigations and hence possess a, non-negligibly, random nature. In this study, we will be transforming some components of the deterministic classical epidemic model to random variables with beta distribution to represent the real life randomness of some events in the mathematical models. The motivation of such an analysis is the previous studies of the authors which contain similar random modeling and analyses of influenza and bacterial resistance (Merdan and Khaniyev, 2008; Merdan et al., 2017). The transformation of the parameters and initial values to random variables enables the modeling of a random scenario for disease transmission, where the disease spreads in the population randomly or where the population is structured of randomly distributed susceptible and infected individuals. Beta distribution has previously been used to describe the dispersion of data about sunshine data and HIV transmission probability (Wiley et al., 1989; Suleiman et al., 1999). Since Beta distribution has been used for the distributions of such random events in various fields and because of the fact that we can arrange the parameters of this distribution to obtain a symmetrical continuous distribution in a limited support with a bell shaped density for certain parameters, Beta distribution will be used to for the random parameters and the random initial values.

The outline of the paper is as follows: The classical epidemic model will be introduced in the next section on a deterministic level. In the following sections, the random models with random parameters and random initial data will be given, respectively. In the last three parts, results for the expectations of these two random models will be compared and a conclusion will be given, along with the results of the simulations.

MATERIALS AND METHODS

The Deterministic Classical Epidemic Model

The deterministic model in (Hethcote, 2000) is used in this study. Here, s, i, r are the population compartments denoting the susceptible, infected and recovered humans, respectively. $n(t)$ denotes the total population:

$$\begin{aligned}\frac{ds}{dt} &= -\beta i\frac{s}{n}, \qquad s(0) = s_0 \geq 0,\\ \frac{di}{dt} &= \beta i\frac{s}{n} - \gamma i, \qquad i(0) = i_0 \geq 0,\\ \frac{dr}{dt} &= \gamma i, \qquad r(0) = r_0 \geq 0.\end{aligned} \tag{1}$$

We use the general "fixed population" assumption of compartmental models:

$$s(t) + i(t) + r(t) \rightarrow \text{fixed}.$$

Dividing all population groups by $n(t)$, we form a new set of variables: $S(t), I(t), R(t)$. Since $R(t) = 1 - S(t) - I(t)$, the non-dimensionalized model is obtained as:

$$\begin{aligned}\frac{dS}{dt} &= -\beta IS,\\ \frac{dI}{dt} &= \beta IS - \gamma I.\end{aligned} \tag{2}$$

The parameters of the model are given as follows: β denotes the contact rate and has the deteministic value 1 while γ denotes the recovery rate and has the value $1/3$. These parameter values are used along with initial conditions: $S(0) = 0.99, I(0) = 0.01$ for the numerical simulations. The values of the parameters have been taken from (Hethcote, 2000). It should also be noted that the time t, is given in days.

SIR Model with Beta Distributed Random Parameters

The general beta distribution is a continuous distribution with left and right shape parameters a and b, a location parameters c and a scale parameter d. The random variable X has the general beta distribution with parameters (a, b) in (c, d), if it has the probability density function

$$f(x) = \frac{1}{B(a,b)d^{a+b-1}}(x-c)^{a-1}(c+d-x)^{b-1}, x \in (c, c+d), a > 0, b > 0.$$

We will be using a symmetrical general Beta distribution by choosing identical left and right shape parameters ($a = b$), since we will be assuming equal dispersion of the data around the deterministic value of the model parameters and initial values.

The parameters of system (2) will be transformed into random variables with symmetrical general Beta distribution to model the randomness of the values of recovery and contact rates in real life. The expected value of a general Beta distributed random variable X^* with parameters (a, b, c, d) is given as (Bekiryazici et al., 2016)

$$E(X^*) = c + d\frac{a}{a+b}.$$

The parameters of system (2), β and γ are transformed into random variables β^*, γ^* such that

$$\begin{aligned}\beta^* &= c_1 + d_1 Z_1,\\ \gamma^* &= c_2 + d_2 Z_2,\end{aligned}$$

where Z_1, Z_2 are independent random variables with standard Beta distribution having the parameters $(a_i, b_i), i = 1{,}2$ and $c_i, d_i, i = 1{,}2$ are the lower bounds and lengths of their supports, respectively. Since the deterministic values are $\beta = 1$ and $\gamma = 1/3$, we determine the parameters of the general Beta distributions $(a_i, b_i, c_i, d_i), i = 1{,}2$ such that $E(\beta^*) = 1$ and $E(\gamma^*) = 1/3$. If the new parameters β^*, γ^* are written in the system (2), we get the random model with random parameters:

$$\frac{dS}{dt} = -\beta^* IS, \qquad \frac{dI}{dt} = \beta^* IS - \gamma^* I, \tag{3}$$

along with the initial values $S(0) = 0.99, I(0) = 0.01$. Since $S(t)$ and $I(t)$ are determined through the random equations in (3), they become random variables as well. If we denote a generally Beta distributed random variable $X^* \sim gBeta(c, d, a, b)$, for $\gamma^* \sim gBeta\left(\frac{19}{60}, \frac{1}{30}, 4,4\right)$ and $\beta^* \sim gBeta(0.99,0.02,3,3)$; (where $gBeta$ denotes a generally distributed random variable)

$$E(\gamma^*) = c_1 + d_1 \frac{a_1}{a_1 + b_1} = \frac{19}{60} + \frac{1}{30}\frac{4}{4+4} = \frac{1}{3},$$
$$E(\beta^*) = c_2 + d_2 \frac{a_2}{a_2 + b_2} = 0.99 + 0.02\frac{3}{3+3} = 1.$$

If we place the random variables $\gamma^* = c_1 + d_1 Z_1 = \frac{19}{60} + \frac{1}{30} Z_1, \beta^* = 0.99 + 0.02 Z_2$ in the system (3), where $Z_1 \sim Beta(4,4)$ and $Z_2 \sim Beta(3,3)$ are independent, we get

$$\frac{dS}{dt} = -(0.99 + 0.02Z_2)IS, \qquad \frac{dI}{dt} = (0.99 + 0.02Z_2)IS - \left(\frac{19}{60} + \frac{1}{30}Z_1\right)I, \tag{4}$$

along with the initial values $S(0) = 0.99, I(0) = 0.01$. System (4) can be simulated in MATLAB by using the command '`betarnd(A,B)`'. The "`betarnd(A,B)`" command in MATLAB generates standard Beta distributed random variables with parameters A, B.

RESULTS AND DISCUSSION

SIR Model with Beta Distributed Random Initial Values

For the second random model, the initial values of (2) will be transformed into random variables with symmetrical general Beta distribution. The initial values

$$S(0) = S_0 = 0.99, \qquad I(0) = I_0 = 0.01,$$

will be similarly randomized. By selecting the new random initial values S_0^* and I_0^* as

$$S_0^* \sim gBeta(0,98,0.02,3,3), \qquad I_0^* \sim gBeta(0,0.02,4,4).$$

Since S_0 and I_0 represent the initial fractions of susceptible and infected humans, they have to be in the interval $(0,1)$. The supports of the new random initial values were determined accordingly:

$$S_0^* \in (0.98,1), \qquad I_0^* \in (0,0.02).$$

Using the new initial values, we obtain the second model:

$$\frac{dS}{dt} = -\beta IS, \quad S_0^* \sim gBeta(0,98,0.02,3,3), \qquad \frac{dI}{dt} = \beta IS - \gamma I, \quad I_0^* \sim gBeta(0,0.02,4,4), \tag{5}$$

where $\beta = 1$ and $\gamma = 1/3$. Once again, we will simulate system (5) in MATLAB by using the command '`betarnd(A,B)`'.

Comparison of Random Characteristics

Differential Transformation Method (DTM) is a popular scheme for analyzing the approximate solutions of differential equations (Pukhov, 1982; Zhou, 1986). The mean square convergence of random DTM has been previously given by Villafuerte and Cortés (Villafuerte and Cortés, 2013). Using

Differential Transformation Method, we can obtain approximate analytical solution of system (2) and then introduce the random parameters and initial values that are general Beta distributed to investigate the random characteristics (Khudair, Haddad and Khalaf, 2016). Using $n = 2$ iterations, the approximate analytical solution of $S(t)$ in the random model with random parameters is obtained as (we will use $S^*(t), I^*(t)$ for the model with random parameters and $S^{**}(t), I^{**}(t)$ for the model with random initial values to distinguish these two cases):

$$S^*(t) = 0.99 - 0.0099\beta t + (-0.004851\beta^2 + 0.00495\beta\gamma)t^2,$$
$$I^*(t) = 0.01 + (0.0099\beta - 0.01\gamma)t + (0.004851\beta^2 - 0.0099\beta\gamma + 0.005\gamma^2)t^2.$$

Similarly for the random model with random initial conditions, (using $n = 2$ iterations), we get:

$$S^{**}(t) = S_0 - S_0 I_0 t + \left(-\frac{1}{2}S_0^2 I_0 + \frac{1}{6}S_0 I_0 + \frac{1}{2}S_0 I_0^2\right)t^2,$$
$$I^{**}(t) = I_0 + \left(S_0 I_0 - \frac{1}{3}I_0\right)t + \left(\frac{1}{2}S_0^2 I_0 - \frac{1}{3}S_0 I_0 - \frac{1}{2}S_0 I_0^2 + \frac{1}{18}I_0\right)t^2.$$

Hence, by the expected values of $S(t)$ with random parameters and initial values can be compared as below. The expectation of $S(t)$ with random parameters is:

$$E\left(S^*(t)\right) = E(0.99 - 0.0099\beta t + (-0.004851\beta^2 + 0.00495\beta\gamma)t^2)$$
$$= 0.99 - 0.0099E(\beta)t - \left(0.004851E(\beta^2) + 0.00495E(\beta)E(\gamma)\right)t^2,$$
$$E\left(S^{**}(t)\right) = E\left(S_0 - S_0 I_0 t + \left(-\frac{1}{2}S_0^2 I_0 + \frac{1}{6}S_0 I_0 + \frac{1}{2}S_0 I_0^2\right)t^2\right)$$
$$= E(S_0) - E(S_0)E(I_0)t + \left(-\frac{1}{2}E(S_0^2)E(I_0) + \frac{1}{6}E(S_0)E(I_0) + \frac{1}{2}E(S_0)E(I_0^2)\right)t^2.$$

Higher moments of a random variable $X^* \sim gBeta(c, d, a, b)$ can be obtained by using the equality,

$$X^* = c + dZ, Z \sim Beta(a, b) \Rightarrow E((X^*)^k) = E((c + dZ)^k),$$

and the higher moment formulas of a standard Beta random variable $Z \sim Beta(a, b)$ (Feller, 1968):

$$E(Z^k) = \frac{a(a+1)\ldots(a+(k-1))}{(a+b)(a+b+1)\ldots(a+b+(k-1))}.$$

Using this equality and the fact that $E((X^*)^k) = E((c + dZ)^k)$, we obtain the first three moments of the general beta distributed random variables β^*, γ^*, S_0^* and I_0^* as:

$$\beta^* \sim gBeta(0.99, 0.02, 3, 3)$$
$$\Rightarrow E(\beta^*) = 1, E\left(\beta^{*2}\right) \simeq 0.9999, E\left(\beta^{*3}\right) \simeq 1, E\left(\beta^{*4}\right) \simeq 0.9606.$$
$$\gamma^* \sim gBeta(19/60, 1/30, 4, 4)$$
$$\Rightarrow E(\gamma^*) \simeq 0.3333, E\left(\gamma^{*2}\right) \simeq 0.1108, E\left(\gamma^{*3}\right) \simeq 0.0371, \left(\gamma^{*4}\right) \simeq 0.0101.$$
$$S_0^* \sim gBeta(0, 98, 0.02, 3, 3)$$
$$\Rightarrow E(S_0^*) = 0.99, E\left(S_0^{*2}\right) = 0.98, E\left(S_0^{*3}\right) \simeq 0.9703, E\left(S_0^{*4}\right) \simeq 0.9224.$$
$$I_0^* \sim gBeta(0, 0.02, 4, 4)$$
$$\Rightarrow E(I_0^*) = 0.01, E\left(I_0^{*2}\right) = 0, E\left(I_0^{*3}\right) \simeq 1.3334 \times 10^{-6}, E\left(I_0^{*3}\right) \simeq 1.6970 \times 10^{-8}.$$

The results have been obtained for the approximations of $S(t)$ and $I(t)$ in the deterministic model and the random models with random parameters and initial values (Tables 1, 2).

The results show that for the first 36 hours of the disease ($t \in (0, 1.5)$), the results for the random and deterministic models are almost the same. Hence, DTM has provided accurate approximation formulas for the early expectations of the model components. Although the formulas for the expectations of the model components $S(t)$ and $I(t)$ may continue to provide accurate results for $t > 0$, the nonlinearity and the fact that this model consists of coupled differential equations may increase the error in the approximate results as t gets larger.

Table 1. Results for $S(t)$ in the models (2), (4) and (5) for $t \in (0,1.5)$.

t	$S(t)$	$E(S^*(t))$	$E(S^{**}(t))$
0	0.9900	0.9900	0.9900
0.1	0.9890	0.9890	0.9890
0.2	0.9879	0.9879	0.9879
0.3	0.9867	0.9867	0.9867
0.4	0.9855	0.9855	0.9855
0.5	0.9842	0.9842	0.9841
0.6	0.9828	0.9828	0.9827
0.7	0.9813	0.9813	0.9812
0.8	0.9797	0.9797	0.9796
0.9	0.9780	0.9780	0.9779
1.0	0.9762	0.9762	0.9761
1.1	0.9744	0.9744	0.9742
1.2	0.9724	0.9724	0.9722
1.3	0.9703	0.9703	0.9701
1.4	0.9680	0.9680	0.9678
1.5	0.9657	0.9657	0.9654

Table 2. Results for $I(t)$ in the models (2), (4) and (5) for $t \in (0,1.5)$.

t	$I(t)$	$E(I^*(t))$	$E(I^{**}(t))$
0	0.0100	0.0100	0.0100
0.1	0.0107	0.0107	0.0107
0.2	0.0114	0.0114	0.0114
0.3	0.0122	0.0122	0.0122
0.4	0.0130	0.0130	0.0130
0.5	0.0139	0.0139	0.0139
0.6	0.0148	0.0148	0.0148
0.7	0.0158	0.0158	0.0158
0.8	0.0168	0.0168	0.0169
0.9	0.0179	0.0179	0.0180
1.0	0.0191	0.0191	0.0192
1.1	0.0203	0.0203	0.0205
1.2	0.0217	0.0216	0.0218
1.3	0.0230	0.0230	0.0232
1.4	0.0245	0.0245	0.0247
1.5	0.0260	0.0260	0.0263

It can also be seen that the difference between the deterministic results and the random results are higher for the case with random initial values. This is due to the width of the supports of the randomized model components, β^*, γ^*, S_0^* and I_0^*. The supports are $\gamma^* \in \left(\frac{19}{60}, \frac{21}{60}\right), \beta^* \in (0.99,1.01), S_0^* \in (0.98,1)$ and $I_0^* \in (0,0.02)$. While the random effects for β^*, γ^* and S_0^* are similar

and around 5% of the deterministic value, I_0^* has a much larger random effect of 100%. This amount of randomness in the random initial data has caused a difference in the results for models (4) and (5).

Simulations

The deterministic solutions for the deterministic model (2) can be found in the referred study (Hethcote, 2000). The comparison of the expectations for the models with random initial data and random parameters can be seen in Figure 1.

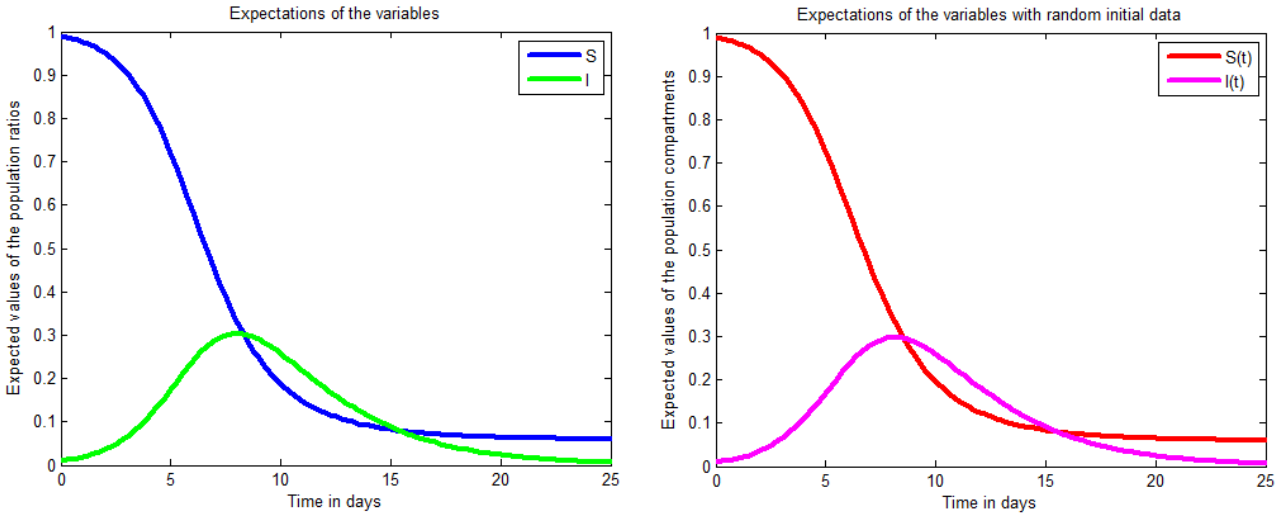

Figure 1. Expectations of the variables with random parameters (left) and random initial values (right).

Figure 2 shows the effects of a higher random effect in model (5) through the confidence intervals. Once again, the effects of the larger support for the random initial data can be seen in the graphs for confidence intervals. Although the expected values are similar for the cases with random parameters and random initial values, the 99% confidence intervals are remarkably larger in the second case.

The effects of a greater randomness in the case with random initial data can also be seen by analyzing the coefficients of variation in both cases. In the model with random parameters, the maximum standard deviation for $S(t)$ is obtained at $t = 7.5$ with a value of 0.009086 and since $E\big(S(t)\big) = 0.3865$ at $t = 7.5$ we see that the coefficient of variation can go up to about 2.35%; whereas for the model with random initial data, the maximum standard deviation for $S(t)$ is obtained at $t = 6.5$ with a value of 0.08224 and considering $E\big(S(t)\big) = 0.529$ for $t = 6.5$, it is seen that the coefficient of variation is around 15.55%. Similarly for $I(t)$ in the model with random parameters, maximum standard deviation is 0.006685 and the expectation is $E\big(I(t)\big) = 0.2878$ at $t = 7$ which corresponds to a coefficient of variation around 2.32%, whereas in the model with random initial data, maximum standard deviation is 0.04127 and the expectation is $E\big(I(t)\big) = 0.2013$ at $t = 5.5$ which corresponds to a coefficient of variation around 20.50%. The results are given in Table 3.

Table 3. Coefficient of variations for $S(t)$ and $I(t)$ in both cases, calculated at the instants of their largest standard deviations.

	$S(t)$	$I(t)$
Random Parameters	0.0235	0.0232
Random Initial Data	0.1555	0.2050

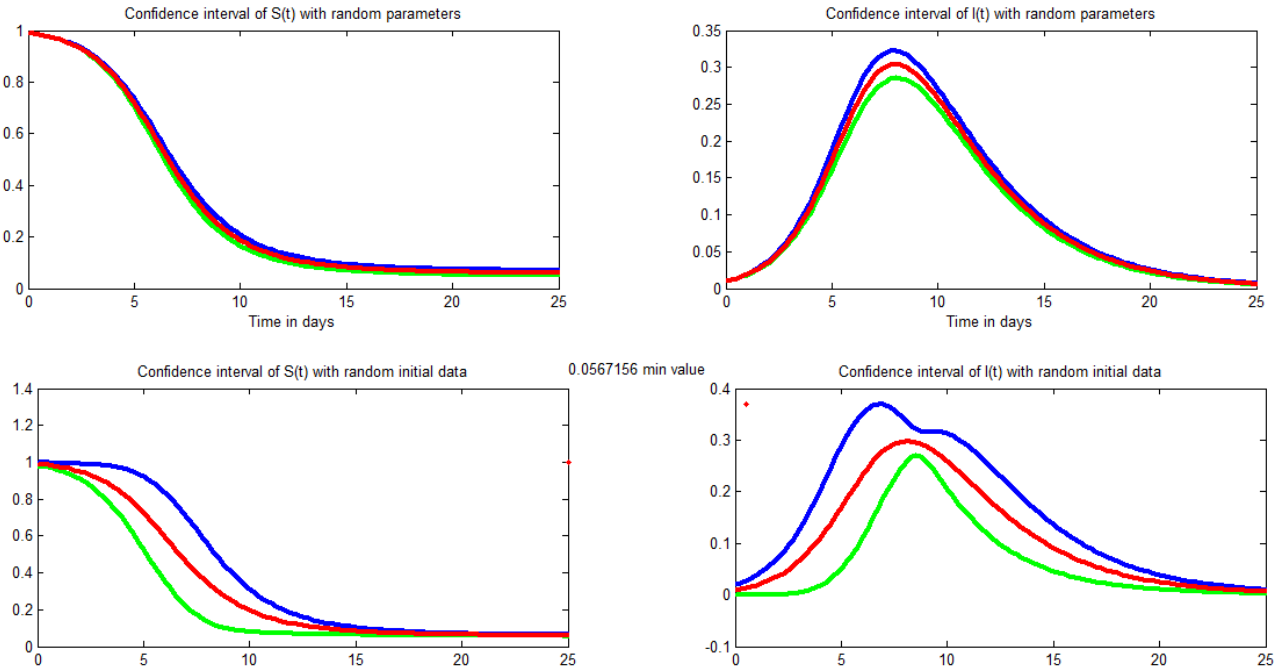

Figure 2. Confidence intervals of $S(t)$ and $I(t)$ with random parameters (upper graphs) and random initial data (lower graphs) are given. The three curves in each graphs denote the upper bound of the confidence interval, expected value and the lower bound of the confidence interval, from top to bottom, respectively.

CONCLUSION

In this study, we used the classical epidemic model of Kermack and McKendrick with the parameters and initial values of the referred study by H.W. Hethcote. We obtained two random models consisting of differential equations with random parameters and random initial values, respectively. We compared the results for the expectation formulas obtained by DTM in both random models. The coefficients of variation were also considered for the random models (4) and (5). It is seen that the approximations for the expected values in both random cases were accurate in the first 36 hours ($t \in (0,1)$). The random models have been simulated in MATLAB using $N > 100000$ simulations and the expectations for both models proved similar results to the deterministic results given by H.W. Hethcote. The confidence intervals for the random models and the variation coefficients show that there is a higher amount of randomness in the results for the case with random initial values. This has been a result of the amount of random effects that were added to the initial value I_0. When the results for the expectations obtained from the simulations and the approximations by DTM are considered, it can be concluded that the models with random parameters and random initial data produce similar results with the deterministic results. Hence, the random models are meaningful and may be used to model the real life randomness of various phenomena that can be modeled by using this epidemiological model.

The model with random parameters suggest that the infected fraction of the population is expected to assume its maximum value of 0.303 at $t = 7.75$, whereas the susceptible fraction at this time is expected to be 0.3584. Using the model assumption $S(t) + I(t) + R(t) = 1$, it can be found that $R(t) = 0.3584$ at $t = 7.75$. Thus the model with random parameters suggests that at most 30.3% of the population is expected to be infected, with a 0.64% deviation in this expectation since $\text{std}(I(t)) = 0.006364$ for $t = 7.75$. For the model with random initial data, $\max(I(t)) = 0.2981$ at $t = 8$ and since $S(t) = 0.3446$ at this time, $R(t) = 0.3573$ for $t = 8$. However, $\text{std}(I(t)) = 0.01444$ at $t = 8$ means there may be about 1.4% deviation in this expectation within the model with random initial data. It

should be noted that the numerical differences are a result of the amount of random effects used for the random parameters and random initial data. Similar analysis can be made for mathematical models of other diseases or events and different distributions can be used for the random effects to analyze or compare the random results under varying conditions.

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