Three Equivalent $n$-Norms on the Space of $p$-Summable Sequences

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Abstract

Given a normed space, one can define a new $n$-norm using a semi-inner product $g$ on the space, different from the $n$-norm defined by Gähler. In this paper, we are interested in the new $n$-norm which is defined using such a functional $g$ on the space $\ell^p$ of $p$-summable sequences, where $1 \leq p < \infty$. We prove particularly that the new $n$-norm is equivalent with the one defined previously by Gunawan on $\ell^p$.

1. Introduction

On a normed space $(X, \| \cdot \|)$, let $g : X^2 \to \mathbb{R}$ be the functional defined by the formula

$$g(x,y) := \frac{1}{2} \|x\|^2 \left[ \tau_+(x,y) + \tau_-(x,y) \right],$$

with

$$\tau_\pm(x,y) := \lim_{t \to 0^\pm} \frac{\|x+ty\| - \|x\|}{t}.$$ 

Then, one may check that $g$ satisfies the following properties:

1. $g(x,x) = \|x\|^2$ for every $x \in X$;
2. $g(\alpha x, \beta y) = \alpha \beta g(x,y)$ for every $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$;
3. $g(x,x+y) = \|x\|^2 + g(x,y)$ for every $x, y \in X$;
4. $|g(x,y)| \leq \|x\| \|y\|$ for every $x, y \in X$.

Assuming that the $g$-functional is linear in the second argument then $[y,x] = g(x,y)$ is a semi-inner product on $X$. Note that all vector spaces in text are assumed to be over $\mathbb{R}$. For example, one may observe that the functional

$$g(x,y) := \|x\|^{2-p} \sum_{k} |x_k|^{p-1} \text{sgn}(x_k) y_k, \quad x := (x_k), y := (y_k) \in \ell^p$$

is a semi-inner product on $\ell^p$, $1 \leq p < \infty$ [1].

Remark 1.1. Note that not all vector spaces have the property that the $g$-functional is linear in the second argument. If the normed space is smooth, then the $g$-functional is linear in the second argument. A normed space with the property that the $g$-functional is linear in the second argument is referred to as normed spaces of $(G)$-type [2].
By using a semi-inner product $g$, Miličić [3] introduced the following orthogonality relation on $X$: $x$ is said to be $g$-orthogonal to $y$, denoted by $x \perp_g y$, provided that $g(x, y) = 0$. For more recent works, see in [4, 5].

Recently, Nur and Gunawan in [6] defined a 2-norm on $X$ by

$$
\|x_1, x_2\|_g := \sup_{|y_1| \leq 1, j = 1, 2} \left| \frac{g(y_1, x_1)}{g(y_1, x_2)} \right|.
$$

Similarly, we can define an $n$-norm (with $n \geq 2$) using the semi-inner product $g$ on $X$. An $n$-norm on $X$ is a mapping $\|\cdot, \cdot, \cdot, \cdot\| : X \times \cdots \times X \to \mathbb{R}$ which satisfies the following four properties:

1. $\|x_1, \ldots, x_n\| = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent;
2. $\|x_1, \ldots, x_n\|$ is invariant under permutation;
3. $\|\alpha x_1, \ldots, x_n\| = |\alpha| \|x_1, \ldots, x_n\|$ for every $x_1, \ldots, x_n \in X$ and for every $\alpha \in \mathbb{R}$;
4. $\|x_1, \ldots, x_{n-1}, y + z\| \leq \|x_1, \ldots, x_{n-1}, y\| + \|x_1, \ldots, x_{n-1}, z\|$ for every $x, y, z \in X$.

The pair $(X, \|\cdot, \cdot, \cdot, \cdot\|)$ is called an $n$-normed space.

The theory of 2-normed spaces was initially introduced by Gähler [7] in the 1960's. Meanwhile, the theory of $n$-normed spaces for $n \geq 2$ was developed in [8]-[10]. See [11]-[15] for recent results on this subject.

On the space $\ell^p$ of $p$-summable sequences, where $1 \leq p < \infty$, the following $n$-norm

$$
\|x_1, \ldots, x_n\|_p := \left(\frac{1}{n} \sum_{k_1} \cdots \sum_{k_n} \left| \begin{array}{cccc}
| x_1k_1 | & \cdots & | x_1k_n |
\end{array} \right| \right)^{\frac{1}{p}},
$$

is defined by Gunawan in [16]. As shown in [17, 18], this $n$-norm is equivalent with the one formulated by Gähler in [8]-[10], namely

$$
\|x_1, \ldots, x_n\|_p := \sup_{|y_1|^p \leq 1, j = 1, \ldots, n} \left| \frac{\sum x_1k y_{1k}}{\sum x_1k y_{nk}} \right|.
$$

where $p'$ denotes the dual exponent of $p$. Precisely, we have the following theorem.

**Theorem 1.2.** [19] For every $x_1, \ldots, x_n \in \ell^p$ ($1 \leq p < \infty$), we have

$$
(n!)^{\frac{1}{p}} \|x_1, \ldots, x_n\| \leq \|x_1, \ldots, x_n\|_p \leq (n!)^{\frac{1}{p'}} \|x_1, \ldots, x_n\|.
$$

In this article, we shall first prove that, on $\ell^p$ ($1 \leq p < \infty$), the new 2-norm $\|\cdot, \cdot\|_g$ is equivalent with the 2-norm $\|\cdot, \cdot\|_p$ which is defined in (1.1). Using this result, we can prove that the 2-normed space $(\ell^p, \|\cdot, \cdot\|_g)$ is complete. We then extend the result for all $n \geq 2$.

### 2. Main results

Before we discuss the equivalence between the two 2-norms on $\ell^p$ ($1 \leq p < \infty$), we need some definitions. Let $(X, \|\cdot\|)$ be a normed space. We define the $g$-orthogonal projection of a vector $y$ on a subspace $S$ of $X$ as follows.

**Definition 2.1.** [20] Let $y \in X$ and $S = \text{span}\{x_1, \ldots, x_m\}$ be a subspace of $X$ with $\Gamma(x_1, \ldots, x_m) = \text{det}[g(x_i, x_j)] \neq 0$. The $g$-orthogonal projection of $y$ on $S$, which we denote by $y_S$, is defined by

$$
y_S := \frac{1}{\Gamma(x_1, \ldots, x_m)} \left| \begin{array}{cccc}
0 & x_1 & \cdots & x_m \\
g(x_1, y) & g(x_1, x_1) & \cdots & g(x_1, x_m) \\
\vdots & \vdots & \ddots & \vdots \\
g(x_m, y) & g(x_m, x_1) & \cdots & g(x_m, x_m)
\end{array} \right|,
$$

and its $g$-orthogonal complement $y - y_S$ is given by

$$
y - y_S = \frac{1}{\Gamma(x_1, \ldots, x_m)} \left| \begin{array}{cccc}
y & x_1 & \cdots & x_m \\
g(x_1, y) & g(x_1, x_1) & \cdots & g(x_1, x_m) \\
\vdots & \vdots & \ddots & \vdots \\
g(x_m, y) & g(x_m, x_1) & \cdots & g(x_m, x_m)
\end{array} \right|.
$$

Observe here that $x_i \perp g y - y_S$ for every $i = 1, \ldots, m$. Note that, if $S = \text{span}\{x\}$, then

$$
y_S = \frac{g(x, y)}{\|x\|^2} x,
$$

and $y - y_S$ is the $g$-orthogonal complement $y$ on $S$. It is clear here that $x \perp g y - y_S$. 

Next, let \( x_1, \ldots, x_n \in X \) be a set of \( n \) linearly independent vectors. We may construct a left \( g \)-orthogonal sequence \( x'_1, \ldots, x'_n \) with \( x'_1 := x_1 \), and

\[
x'_i := x_i - \langle x_i, s \rangle s_i,
\]

where \( S_{i-1} = \text{span} \{ x'_1, \ldots, x'_{i-1} \} \) for \( i = 2, \ldots, n \). Observe here that \( \langle x'_i, x'_j \rangle = 0 \) for \( i < j \) (see \([15, 20] \)).

For \( X = \ell^p \) (\( 1 \leq p < \infty \)), we have relation for the \( n \)-norm \( \| x_1, \ldots, x_n \|_p \) and the ‘volume’ of the \( n \)-dimensional parallelepiped spanned by \( \{ x_1, \ldots, x_n \} \) in \( \ell^p \), namely \( V(x_1, \ldots, x_n) = \prod_{i=1}^n \| x'_i \|_p \), as follows.

**Theorem 2.2.** [19] Let \( \{ x_1, \ldots, x_n \} \) be a set of linearly independent vectors in \( \ell^p \) (\( 1 \leq p < \infty \)). Then we have

\[
(n!)^{\frac{1}{p} - 1} \| x_1, \ldots, x_n \|_p \leq V(x_1, \ldots, x_n) \leq (n!)^{\frac{1}{p}} \| x_1, \ldots, x_n \|_p
\]

for any permutation \( (i_1, \ldots, i_n) \) of \( (1, \ldots, n) \).

Note that the value of \( V(x_1, \ldots, x_n) \) may not be invariant under permutation of \( (x_1, \ldots, x_n) \) because \( g(\cdot, \cdot) \) may not be symmetry. The above theorem states that all possible values of \( V(x_1, \ldots, x_n) \) lie between two multiples of \( \| x_1, \ldots, x_n \|_p \), independent of the permutation.

### 2.1. The equivalence between two \( p \)-norms

Let us consider Gunawan’s definition and Gähler’s definition of 2-norm on \( \ell^p \) (\( 1 \leq p < \infty \)), namely:

\[
\| x_1, x_2 \|_p = \left[ \sum_{k_1} \sum_{k_2} \left( \| x_{k_1} \|_{x_{k_2}}^p \right)^{\frac{1}{p}} \right]^{\frac{1}{2}}
\]

and

\[
\| x_1, x_2 \|' := \sup_{\| y \|_{x} \leq 1, j = 1, 2} \left| \sum_{k_1} \sum_{k_2} x_{k_1} y_{k_2} \right|
\]

Meanwhile, Nur and Gunawan’s 2-norm is given by

\[
\| x_1, x_2 \|_{p, \rho} = \sup_{\| y \|_{x} \leq 1, j = 1, 2} \left| \sum_{k_1} \sum_{k_2} y_{k_1} \| y_{k_1} \|_{y_{k_2}}^p \right| \sum_{k_1} \sum_{k_2} \left| y_{k_1} \| y_{k_1} \|_{y_{k_2}}^p \right| \sum_{k_1} \sum_{k_2} \left| y_{k_1} \| y_{k_1} \|_{y_{k_2}}^p \right| \sum_{k_1} \sum_{k_2} \left| y_{k_1} \| y_{k_1} \|_{y_{k_2}}^p \right| \sum_{k_1} \sum_{k_2} \left| y_{k_1} \| y_{k_1} \|_{y_{k_2}}^p \right| \sum_{k_1} \sum_{k_2} \left| y_{k_1} \| y_{k_1} \|_{y_{k_2}}^p \right|
\]

**Remark 2.3.** Using properties of determinants, the above 2-norm may be rewritten as

\[
\| x_1, x_2 \|_{p, \rho} = \sup_{\| y_j \|_{x_j} \leq 1, j = 1, 2} \frac{1}{2} \prod_{j=1}^2 \left( \sum_{k_1} \sum_{k_2} y_{k_1} \| y_{k_1} \|_{y_{k_2}}^p \right) \sum_{k_1} \sum_{k_2} \left| y_{k_1} \| y_{k_1} \|_{y_{k_2}}^p \right| \sum_{k_1} \sum_{k_2} \left| y_{k_1} \| y_{k_1} \|_{y_{k_2}}^p \right| \sum_{k_1} \sum_{k_2} \left| y_{k_1} \| y_{k_1} \|_{y_{k_2}}^p \right| \sum_{k_1} \sum_{k_2} \left| y_{k_1} \| y_{k_1} \|_{y_{k_2}}^p \right| \sum_{k_1} \sum_{k_2} \left| y_{k_1} \| y_{k_1} \|_{y_{k_2}}^p \right|
\]

For \( p = 2 \), we observe that

\[
\| x_1, x_2 \|_{2, \rho} = \sup_{\| y_j \|_{x_j} \leq 1, j = 1, 2} \frac{1}{2} \prod_{j=1}^2 \left( \sum_{k_1} \sum_{k_2} y_{k_1} \| y_{k_1} \|_{y_{k_2}}^p \right) \sum_{k_1} \sum_{k_2} \left| y_{k_1} \| y_{k_1} \|_{y_{k_2}}^p \right| \sum_{k_1} \sum_{k_2} \left| y_{k_1} \| y_{k_1} \|_{y_{k_2}}^p \right| \sum_{k_1} \sum_{k_2} \left| y_{k_1} \| y_{k_1} \|_{y_{k_2}}^p \right| \sum_{k_1} \sum_{k_2} \left| y_{k_1} \| y_{k_1} \|_{y_{k_2}}^p \right| \sum_{k_1} \sum_{k_2} \left| y_{k_1} \| y_{k_1} \|_{y_{k_2}}^p \right|
\]

One may then verify that the three 2-norms \( \| \cdot \|_2, \| \cdot \|' \) and \( \| \cdot \|_{2, \rho} \) are identical (see \([6, 12] \)).

For other values of \( p \), we have the following theorem.

**Theorem 2.4.** For every \( x_1, x_2 \in \ell^p \) (\( 1 \leq p < \infty \)), we have

\[
2^{\frac{1}{p} - 1} \| x_1, x_2 \|_p \leq \| x_1, x_2 \|_{p, \rho} \leq \| x_1, x_2 \|' \leq 2^{\frac{1}{p}} \| x_1, x_2 \|_p.
\]

**Proof.** For \( j = 1, 2 \), let \( y_j \in \ell^p \) with \( \| y_j \|_p \leq 1 \). Take \( u_j = (u_{j,k}) \) with \( u_{j,k} = \| y_j \|_p \) \( y_{j,k} \| y_{j,k} \|_{y_{j,k}}^{-1} \). We observe that \( u_j \in \ell^p \) with \( \| u_j \|_{\rho'} = \| y_j \|_p \). As a consequence, we have \( \| x_1, x_2 \|_{p, \rho} \leq \| u_1, u_2 \|_{p, \rho} \). By using Theorem 1.2, we obtain

\[
\| x_1, x_2 \|_{p, \rho} \leq \| u_1, u_2 \|_{p, \rho} \leq 2^{\frac{1}{p}} \| x_1, x_2 \|_p.
\]

Next, assume that \( \{ x_1, x_2 \} \) is linearly independent. Using the process in (2.1), we obtain the left \( g \)-orthogonal set \( \{ x'_1, x'_2 \} \). Then, by Theorem 2.2, we have

\[
2^{\frac{1}{p} - 1} \| x_1, x_2 \|_p \leq V(x_1, x_2) = \| x'_1 \|_p \| x'_2 \|_p.
\]
For $j = 1, 2$, let $y_j = \frac{x_j}{\|x_j\|}$, so that $\|y_j\|_p = 1$. It follows from the properties of semi-inner product $g$ and matrix determinants that
\[
\begin{vmatrix}
g(y_1, x_1) & g(y_2, x_1) \\
g(y_1, x_2) & g(y_2, x_2)
\end{vmatrix}
= \frac{1}{\|x_1\|_p} g(x_1^*, x_1) \frac{1}{\|x_2\|_p} g(x_2^*, x_2)
= \|x_1^*\|_p \|x_2^*\|_p = V(x_1, x_2)
\geq 2^{\frac{1}{p} - 1} \|x_1, x_2\|_p.
\]
By the definition of $\|\cdot\|_{p, p}$, we conclude that $\|x_1, x_2\|_{p, p} \geq 2^{\frac{1}{p} - 1} \|x_1, x_2\|_p$. Combining with the previous inequalities, we have
\[
2^{\frac{1}{p} - 1} \|x_1, x_2\|_p \leq \|x_1, x_2\|_{p, p} \leq \|x_1, x_2\|_p \leq 2^{\frac{1}{p}} \|x_1, x_2\|_p.
\]
Note that if $\{x_1, x_2\}$ is a linearly dependent set, then all the 2-norms are equal 0, and so we have the equalities.

**Corollary 2.5.** For $1 \leq p < \infty$, the three 2-norms $\|\cdot\|_{p, p}$, $\|\cdot\|_{p', p'}$ and $\|\cdot\|_p$ are pairwise equivalent.

Since $(\ell^p, \|\cdot\|_{p, p})$ is a 2-Banach space [1], we obtain the following corollary.

**Corollary 2.6.** For $1 \leq p < \infty$, the 2-normed space $(\ell^p, \|\cdot\|_{p, p})$ is a 2-Banach space.

### 2.2. The equivalence between two $n$-norms

All results above subsection can be extended to $n$-normed spaces for any $n \geq 2$. Suppose that $g$ is a semi-inner product on $(X, \|\cdot\|)$. Consider the following mapping $\|\cdot, \ldots, \cdot\|_g$ on $X \times \cdots \times X$:
\[
\|x_1, \ldots, x_n\|_g = \sup_{\|y\|_g \leq 1, j = 1, \ldots, n} \begin{vmatrix}
g(y_1, x_1) & \cdots & g(y_n, x_1) \\
\vdots & \ddots & \vdots \\
g(y_1, x_n) & \cdots & g(y_n, x_n)
\end{vmatrix} = \sup_{\|y\|_g \leq 1, j = 1, \ldots, n} \det(g(y_j, x_i)).
\]
(2.2)
If $\|y\|_g \leq 1$ for $j = 1, \ldots, n$, then $\det(g(y_j, x_i)) \leq n! \prod_{i=1}^{n} |x_i|$. Note that the factor $n!$ comes from the number of terms in the expansion of $\det(g(y_j, x_i))$. The following fact tells us that $\|\ldots, \|_g$ is a finite number.

**Fact 2.7.** The inequality
\[
\|x_1, \ldots, x_n\|_g \leq n! \prod_{i=1}^{n} |x_i|
\]
holds whenever $x_1, \ldots, x_n \in X$.

Moreover, we have the following result.

**Proposition 2.8.** The mapping (2.2) defines an $n$-norm on $X$.

**Proof.** It is obvious that if $\{x_1, \ldots, x_n\}$ is linearly dependent, then we have $\|x_1, \ldots, x_n\|_g = 0$. Conversely, if $\|x_1, \ldots, x_n\|_g = 0$, then the rows of the matrix $[g(y_j, x_i)]$ are linearly dependent for every $y_1, \ldots, y_n \in X$ with $\|y_j\|_g \leq 1, j = 1, \ldots, n$. This happens only if $x_1, \ldots, x_n$ are linearly dependent.

Next, by using the properties of supremum and matrix determinants, we obtain the invariance of $\|x_1, \ldots, x_2\|_g$ under permutation. Furthermore, we have $\|\alpha x_1, \ldots, x_n\|_g = |\alpha| \|x_1, \ldots, x_n\|_g$ for $\alpha \in \mathbb{R}$.

Finally, for arbitrary $x_0, x_1, \ldots, x_n \in X$, we obtain
\[
\|x_0 + x_1, \ldots, x_n\|_g = \sup_{\|y\|_g \leq 1, j = 1, \ldots, n} \begin{vmatrix}
g(y_1, x_0 + x_1) & \cdots & g(y_n, x_0 + x_1) \\
\vdots & \ddots & \vdots \\
g(y_1, x_n) & \cdots & g(y_n, x_n)
\end{vmatrix}
\leq \sup_{\|y\|_g \leq 1, j = 1, \ldots, n} \begin{vmatrix}
g(y_1, x_0) & \cdots & g(y_n, x_0) \\
\vdots & \ddots & \vdots \\
g(y_1, x_n) & \cdots & g(y_n, x_n)
\end{vmatrix}
+ \sup_{\|y\|_g \leq 1, j = 1, \ldots, n} \begin{vmatrix}
g(y_1, x_1) & \cdots & g(y_n, x_1) \\
\vdots & \ddots & \vdots \\
g(y_1, x_n) & \cdots & g(y_n, x_n)
\end{vmatrix}
= \|x_0, \ldots, x_n\|_g + \|x_1, \ldots, x_n\|_g.
\]
This completes the proof. □

The following theorem holds for an inner product space $(X, \langle \cdot, \cdot \rangle)$. 
Theorem 2.9. If \((X, \langle \cdot, \cdot \rangle)\) is a real inner product space, then the two \(n\)-norms \(\| \cdot \|_g\) in (2.2) and \(\| \cdot \|_s\) given by
\[
\| x_1, \ldots, x_n \|_g := \left| \begin{array}{c}
\langle x_1, x_1 \rangle \\
\vdots \\
\langle x_n, x_n \rangle
\end{array} \right|^{\frac{1}{n}}
\]
are identical.

Proof. On the inner product space \(X\), the functional \(g(\cdot, \cdot)\) is identical with the inner product \(\langle \cdot, \cdot \rangle\). Therefore,
\[
\| x_1, \ldots, x_n \|_g = \sup_{\| y \| \leq 1, y_1, \ldots, y_n} \left| \begin{array}{c}
\langle y_1, x_1 \rangle \\
\vdots \\
\langle y_n, x_n \rangle
\end{array} \right|^{\frac{1}{n}}
\]
Now, applying the generalized Cauchy-Schwarz inequality \([21]\) and Hadamard’s inequality \([22]\), we get
\[
\| x_1, \ldots, x_n \|_g \leq \sup_{\| y \| \leq 1, y_1, \ldots, y_n} \| x_1, y_1, \ldots, x_n, y_n \|_s \leq \| x_1, \ldots, x_n \|_s.
\]
Conversely, suppose that \(\{x_1, \ldots, x_n\}\) is linearly independent. Using the Gram-Schmidt process, we get the orthogonal set \(\{x_1', \ldots, x_n'\}\). Because the determinant of the Gram matrix of a linearly independent set being equal to the Gram matrix of the associated orthogonal set (obtained using Gram-Schmidt process), we have \(\| x_1, \ldots, x_n \|_g = \| x_1', \ldots, x_n' \|_s\). For \(j = 1, \ldots, n\), let \(y_j = \frac{x_j}{\| x_j \|}\), so that \(\| y_j \| = 1\). Then, by the properties of the inner product and matrix determinants, we obtain
\[
\| x_1, \ldots, x_n \|_g = \| x_1, \ldots, x_n \|_s \leq \| x_1 \| \cdots \| x_n \|,
\]
which is better than that in Fact ??.

Remark 2.10. Note that, in an inner product space, we have the well-known Hadamard’s inequality \([22]\)
\[
\| x_1, \ldots, x_n \|_g = \| x_1, \ldots, x_n \|_s \leq \| x_1 \| \cdots \| x_n \|,
\]
For \(X = l^p, 1 \leq p < \infty\), we rewrite the formula in (2.2) as
\[
\| x_1, \ldots, x_n \|_{g,p} = \sup_{\| y \| \leq 1, y_1, \ldots, y_n} \left| \begin{array}{c}
g(y_1, x_1) \\
\vdots \\
g(y_n, x_n)
\end{array} \right|^{\frac{1}{n}}
\]
Substituting \(g(y_j, x_k) = \| y_j \|^{2-p} \sum_k |y_{jk}|^{p-1} \text{sgn}(y_{jk})x_{jk}\) and using the properties of determinants, we have
\[
\| x_1, \ldots, x_n \|_{g,p} = \sup_{\| y \| \leq 1, y_1, \ldots, y_n} \left| \begin{array}{c}
\| y_1 \|^{2-p} \sum_k |y_{1k}|^{p-1} \text{sgn}(y_{1k})x_{1k} \\
\vdots \\
\| y_n \|^{2-p} \sum_k |y_{nk}|^{p-1} \text{sgn}(y_{nk})x_{nk}
\end{array} \right|^{\frac{1}{n}}
\]
\[
= \sup_{\| y \| \leq 1, y_1, \ldots, y_n} \prod_{k=1}^n \left| \sum_{j=1}^n \prod_{l=1}^n |y_{jk}|^{p-1} \text{sgn}(y_{jk}) x_{jk} \right|^{\frac{1}{n}}
\]
\[
= \prod_{k=1}^n \left| \sum_{j=1}^n \prod_{l=1}^n |y_{jk}|^{p-1} \text{sgn}(y_{jk}) x_{jk} \right|^{\frac{1}{n}}
\]
\[
= \prod_{k=1}^n \left| \sum_{j=1}^n \prod_{l=1}^n |y_{jk}|^{p-1} \text{sgn}(y_{jk}) x_{jk} \right|^{\frac{1}{n}}
\]
\[
= \prod_{k=1}^n \left| \sum_{j=1}^n \prod_{l=1}^n |y_{jk}|^{p-1} \text{sgn}(y_{jk}) x_{jk} \right|^{\frac{1}{n}}
\]
Corollary 2.11. For \(p = 2\), the three \(n\)-norms \(\| \cdot \|_{g,2}, \| \cdot \|_{s,2}\) in (1.1), \(\| \cdot \|_2, \| \cdot \|_g, \| \cdot \|_s, \| \cdot \|_s, \| \cdot \|_{g,2}\) in (2.3) are identical.

For \(p \neq 2\), we have the following generalization of Theorem 2.4.

Theorem 2.12. For every \(x_1, \ldots, x_n \in l^p, 1 \leq p < \infty\), we have
\[
(\alpha_1) \| x_1, \ldots, x_n \|_p \leq \| x_1, \ldots, x_n \|_{g,p} \leq \| x_1, \ldots, x_n \|_g \leq (\alpha_1) \| x_1, \ldots, x_n \|_p.
\]
Proof. For each \( j = 1, \ldots, n \), let \( y_j \in \ell^p \) with \( \|y_j\|_p \leq 1 \). Then take \( u_j = (u_{jk}) \) with \( u_{jk} = |y_{jk}|^{2-p} |y_{jk}|^{p-1} \text{sgn}(y_{jk}) \). We observe that \( u_j \in \ell^p \) with \( \|u_j\|_p = \|y_j\|_p \leq 1 \). As a consequence, we have

\[
\|x_1, \ldots, x_n\|_{g, p} \leq \|x_1, \ldots, x_n\|_p.
\]

By using Theorem 1.2, we obtain

\[
\|x_1, \ldots, x_n\|_{g, p} \leq \|x_1, \ldots, x_n\|_p^p \leq (n!)^{\frac{1}{p}} \|x_1, \ldots, x_n\|_p.
\]

Conversely, suppose that \( \{x_1, \ldots, x_n\} \) is a linearly independent set. Using \( x_1 = x_1 \) and so forth as in (2.1), we obtain the left \( g \)-orthogonal set \( \{x_1^*, \ldots, x_n^*\} \). Then, it follows from Theorem 2.2 that

\[
(n!)^{\frac{1}{p}-1} \|x_1, \ldots, x_n\|_p \leq V(x_1, \ldots, x_n) = \|x_1^*\|_p \cdots \|x_n^*\|_p.
\]

For \( j = 1, \ldots, n \), let \( y_j = \frac{x_j^*}{\|x_j^*\|_p} \), so that \( \|y_j\|_p = 1 \). Next, using the properties of matrix determinants and the semi-inner product \( g \), we have

\[
\begin{vmatrix}
g(y_1, x_1) & \cdots & g(y_n, x_1) \\
\vdots & \ddots & \vdots \\
g(y_1, x_n) & \cdots & g(y_n, x_n)
\end{vmatrix}
= \begin{vmatrix}
\frac{1}{\|x_1^*\|_p} g(x_1^*, x_1^*) & \cdots & \frac{1}{\|x_n^*\|_p} g(x_n^*, x_1^*) \\
\vdots & \ddots & \vdots \\
\frac{1}{\|x_1^*\|_p} g(x_1^*, x_n^*) & \cdots & \frac{1}{\|x_n^*\|_p} g(x_n^*, x_n^*)
\end{vmatrix}
= \|x_1^*\|_p \cdots \|x_n^*\|_p = V(x_1, \ldots, x_n)
\geq (n!)^{\frac{1}{p}-1} \|x_1, \ldots, x_n\|_p,
\]

whence \( \|x_1, \ldots, x_n\|_{g, p} \geq (n!)^{\frac{1}{p}-1} \|x_1, \ldots, x_n\|_p \). Combining with the previous inequalities, we obtain

\[
(n!)^{\frac{1}{p}-1} \|x_1, \ldots, x_n\|_p \leq |x_1, \ldots, x_n|_{g, p} \leq \|x_1, \ldots, x_n\|_p \leq (n!)^{\frac{1}{p}} \|x_1, \ldots, x_n\|_p.
\]

If \( \{x_1, \ldots, x_n\} \) is linearly dependent, then all the \( n \)-norms vanish and so we have the equalities.

\[
\Box
\]

Corollary 2.13. For \( 1 \leq p < \infty \), the three \( n \)-norms \( \|\cdot, \ldots, \|_{g, p}, \|\cdot, \ldots, \|_p \) and \( \|\cdot, \ldots, \|_p \) are equivalent.

Knowing that the space \((\ell^p, \|\cdot, \ldots, \|_p)\) is an \( n \)-Banach space in [16], we have a generalization of Corollary 2.6 as follows.

Corollary 2.14. For \( 1 \leq p < \infty \), the space \((\ell^p, \|\cdot, \ldots, \|_{g, p})\) is an \( n \)-Banach space.

3. Concluding remarks

In this paper, a new \( n \)-norm is defined using a semi-inner product \( g \) on \( \ell^p \) for \( 1 \leq p < \infty \). Accordingly, on the space \( \ell^p \) \( (1 \leq p < \infty) \), we have three different \( n \)-norms, namely Gähler’s \( n \)-norm \( \|\cdot, \ldots, \|_p \) defined in [8]-[10], Gunawan’s \( n \)-norm \( \|\cdot, \ldots, \|_p \) defined in [16], and \( \|\cdot, \ldots, \|_{g, p} \) defined here in (2.3). In Corollary 2.13, we have just seen that the three \( n \)-norms on \( \ell^p \) are equivalent. As expected, the case where \( p = 2 \) is special. Here, the three \( n \)-norms on \( \ell^2 \) are identical.

In addition to the above three \( n \)-norms, we also have a formula for another \( n \)-norm using the semi-inner product \( g \) on \( \ell^p \) \( (1 \leq p < \infty) \), namely

\[
\|x_1, \ldots, x_n\|_{g, p}^p = \sup_{y_1, \ldots, y_n \leq 1} g(y_1, x_1) \cdots g(y_n, x_1)
= \begin{Vmatrix}
g(y_1, x_1) & \cdots & g(y_n, x_1) \\
\vdots & \ddots & \vdots \\
g(y_1, x_n) & \cdots & g(y_n, x_n)
\end{Vmatrix}.
\]

Since \( g(y, x) = \|y\|_p^p \sum_k |y_k|^{p-1} \text{sgn}(y_k)x_k \), we obtain

\[
\|x_1, \ldots, x_n\|_{g, p} = \left[ \sup_{y_1, \ldots, y_n \leq 1} \frac{1}{n!} \prod_{k=1}^n |y_k|^{2-p} \times \sum_{k_1} \cdots \sum_{k_n} |y_{ik_1}|^{p-1} \text{sgn}(y_{ik_1}) \cdots |y_{ik_n}|^{p-1} \text{sgn}(y_{ik_n}) \right]^{\frac{1}{p}} \begin{Vmatrix} x_{ik_1} \cdots x_{ik_n} \end{Vmatrix}.
\]

Note that, for \( p = 2 \), we have \( \|x_1, \ldots, x_n\|_{g, 2} = \|x_1, \ldots, x_n\|_{g, 2}^2 \). For other values of \( p \), we can show that

\[
\|x_1, \ldots, x_n\|_{g, p} \leq (n!)^{\frac{1}{2} - \frac{1}{p}} \|x_1, \ldots, x_n\|_{g, p}.
\]
Indeed, assuming that $x_1, \ldots, x_n$ are linearly independent, let $x_1', \ldots, x_n'$ be the vectors obtained from $x_1, \ldots, x_n$ through the process in (2.1). By taking $y_j = \frac{x_j}{\|x_j\|_p}$ ($j = 1, \ldots, n$), we obtain $\|y_1, \ldots, y_n\|_p = 1$. Next, using the properties of matrix determinants and the semi-inner product $g$, we have

$$
\begin{vmatrix}
g(y_1, x_1) & \cdots & g(y_n, x_1) \\
\vdots & \ddots & \vdots \\
g(y_1, x_n) & \cdots & g(y_n, x_n)
\end{vmatrix}
= \begin{vmatrix}
\frac{1}{\|x_1\|_p} g(x_1', x_1') & \cdots & \frac{1}{\|x_n\|_p} g(x_n', x_n') \\
\vdots & \ddots & \vdots \\
\frac{1}{\|x_1\|_p} g(x_1', x_n') & \cdots & \frac{1}{\|x_n\|_p} g(x_n', x_n')
\end{vmatrix}
= \frac{1}{\|x_1\|_p^2 \cdots \|x_n\|_p^2} \|x_1, \ldots, x_n\|_p^2.
$$

Since $\|x_1, \ldots, x_n\|_p \leq (\alpha_n)^{\frac{1}{\alpha_n}} \|x_1\|_p^{\alpha_n} \cdots \|x_n\|_p^{\alpha_n}$ by Theorem 2.2 and $\|x_1', \ldots, x_n'\|_p = \|x_1, \ldots, x_n\|_p$, we obtain

$$
\|x_1, \ldots, x_n\|_{p, p} \geq (\alpha_n)^{\frac{1}{\alpha_n}} \|x_1\|_p^{\alpha_n} \cdots \|x_n\|_p^{\alpha_n}.
$$

Moreover, using Theorem 2.12, we have

$$
\|x_1, \ldots, x_n\|_{p, p} \leq (\alpha_n)^{\frac{1}{\alpha_n}} \|x_1\|_p^{\alpha_n} \cdots \|x_n\|_p^{\alpha_n}.
$$

It follows from this inequality that the convergence of a sequence in $\|\cdot, \cdot, \ldots\|_{p, p}$ implies the convergence in $\|\cdot, \cdot, \cdot\|_{p, p}$, and hence also in $\|\cdot, \cdot, \cdot\|_p$. Unfortunately, up to now, we do not know if the converse is true.

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**References**


