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Research Article

A lexicographical order induced by Schauder bases

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Abstract

In this paper, we show that every Banach space with a Schauder basis can be seen as a totally ordered vector space. Indeed, this order can be considered as a lexicographical order since it is a generalization of lexicographical order in $\mathbb{R}^n$. We also provide order structural properties of the order by approaching geometrical (cone) sense.

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1. Introduction

The ordered vector spaces have been studying since at the beginning of the last century and it has efficient applications the other disciplines, see [1, 2, 4, 5]. Since the significant properties of the optimization problems in a vector space are frequently based on an order-structure, the optimality concept has been started to approached by the properties of the cone which is a geometric way to understand order structures in the vector spaces. Some of these studies have a wide range of applications such as equilibrium theory and well-posedness problems, see [7, 9–11, 16].

In this study, we show that we can obtain a totally order by using projections of a Schauder basis of a Banach space that gives us a lexicographical-like order structure. In fact, this cone can be considered as a "generalization" of the lexicographical cone in $\mathbb{R}^n$. We also show that the equivalent Schauder basis generates order-isomorphic vector lattices. By associating our findings with some well-known results in Banach space theory, such as every infinite-dimensional Banach space has a subspace that has a Schauder basis [6], we can immediately get the following conclusions: Every separable Banach space has a totally ordered subspace, every infinite-dimensional Banach space has an infinite-dimensional quotient space which can be considered as a totally ordered vector lattice. Among the other things mentioned above, we obtain a generalization of the main results of the papers [9, 10].

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2. Preliminaries

Let us recall some of the notions of the ordered vector spaces. In this section, all definitions and aligned properties can be found in [3, 4, 8, 12–14, 19]. Throughout of this section, let E be a real vector space and θ be zero vector in E. A subset $\mathcal{K}$ of E is called a *cone* if: $\mathcal{K} + \mathcal{K} \subset \mathcal{K}$, $\alpha.\mathcal{K} \subset \mathcal{K}$ for all $\alpha \in \mathbb{R}^{\geq 0}$ and $-\mathcal{K} \cap \mathcal{K} = \{\theta\}$. A subset $\mathcal{W}$ of E is called a *wedge* if it satisfies all cone axioms except the axiom $-\mathcal{K} \cap \mathcal{K} = \{\theta\}$. The *Minkowski sum* of $A, B \subseteq E$ is defined by $A + B = \{a + b : a \in A \text{ and } b \in B\}$ and the scalar multiplication is defined by $\alpha.A = \{\alpha.a : a \in A\}$. It is well known that if $\mathcal{K}$ is a cone in a vector space E then :

$$a \leq b \text{ if and only if } b - a \in \mathcal{K}$$

is a partial order in E. So that, the vector space E with a cone structure $\mathcal{K}$ can be seen as a pair $(E, \mathcal{K})$ which is ordered vector space. Two ordered vector spaces $(E, \mathcal{K})$ and $(M, \mathcal{L})$ is called *order isomorphic* if there is a linear bijection, $T : E \to M$ such that $T(\mathcal{K}) = \mathcal{L}$. If a pair $(E, \mathcal{K})$ has *lattice* property (i.e., $\sup\{x, y\}$ or $\inf\{x, y\}$ exists for every pair of $x, y \in E$) then the pair $(E, \mathcal{K})$ is called an *ordered vector lattice* or a *Riesz space*. A sequence $\{x_n : n \in \mathbb{N}\}$ in the ordered vector lattice E is called *order convergent* to an element $x \in E$ if there exits a monotone decreasing sequence $\{q_n : n \in \mathbb{N}\}$ in E with $\inf\{q_n\} = \theta$ such that $\sup\{(x_n - x), (x - x_n)\} < q_n$ for all $n \in \mathbb{N}$.

If a cone $\mathcal{K}$ has additional property that $-\mathcal{K} \cup \mathcal{K} = E$, then the cone $\mathcal{K}$ is called a *totally ordering cone*. In this case, the order relation which is induced by the cone structure is called a *totally order*. Let us introduce some of subspaces of an ordered vector lattice which provide useful information about whole vector lattice. Let I be a subspace of E. If I has lattice property then I is called a *vector sub-lattice* of E. If a vector sub-lattice I has *solid* property (i.e, if a triple of $x, y, y - x \in \mathcal{K}$ with $y \in I$ implies that $x \in I$) then I is called an *order ideal* of E. The Minkowski sum of order ideals and intersection of order ideals is an order ideal as well. An order ideal I is called a *maximal order ideal* if it is the only proper ideal which is contained by itself. Let $(E, \mathcal{K})$ be an ordered vector lattice, $(E, \mathcal{K})$ is called an *Archimedean* vector lattice if for all $n \in \mathbb{N}$ and for any $x \in E$, $y, y - nx \in \mathcal{K}$ implies that $x = \theta$ or $x \in -\mathcal{K}$. It is well known that the unique totally ordered Achimedean vector lattice is $\mathbb{R}$ with the cone $[0, \infty)$, up to vector lattice isomorphism. The *lexicographical order* is a non-Archimedean totally order which is defined on $\mathbb{R}^{n \geq 2}$, with the following relation: $(x_1, x_2, ..., x_n) < (y_1, y_2, ..., y_n) \iff x_i < y_i$ for the smallest i for which $x_i \neq y_i$. A sequence (b_n) in a Banach space X is called a *Schauder basis* of X if for every $x \in X$ there is an unique sequence of scalars (α_n) so that $x = \sum_{n=1}^{\infty} \alpha_n.b_n$. We should emphasise that for a Schauder basis, there is not only countability, but a specific ordering of base elements. Let E and L be two Banach spaces with Schauder basis (b_n) and (c_n), respectively. Basis (b_n) and (c_n) are called *equivalent base* if any convergence of $\sum_{n=1}^{\infty} \alpha_n.b_n$ or $\sum_{n=1}^{\infty} \alpha_n.c_n$ implies each other.

3. Totally ordering cones with Schauder basis

Let E be an infinite dimensional Banach space with Schauder basis (b_n). Each of element $x \in E$ correspond to unique scalar sequence (α_n) where $x = \sum_{n=1}^{\infty} \alpha_n.b_n$, in the sense of norm convergence. The linear mappings $P_n : E \to E$, defined by

$$P_n(x) = \sum_{k=1}^{n} \alpha_k.b_k.$$

Let $b_n^* : E \to \mathbb{R}$ denote the functional, where b_n^* assigns to every vector x in E the coordinate α_n of x in the above expansion. Each b_n^* is a bounded linear functional on E.

Let us define the sequence of sets

$$B_1 = \{x \in E : b_1^*(x) > 0\},$$
$$B_2 = \{x \in E : b_1^*(x) = 0 \text{ and } b_2^*(x) > 0\}, ...$$
$$B_n = \{x \in E : (b_i^*(x) = 0 \text{ for all } i < n)\text{and } b_n^*(x) > 0\}, ...$$

If $\mathcal{K} = \bigcup_{n=1}^{\infty} B_n \cup \{\theta\}$ then $\mathcal{K}$ is cone in E that produces totally order for the elements of E.

Theorem 3.1. *$(E, \mathcal{K})$ is a totally ordered vector lattice.*

Proof. We will show that $\mathcal{K} = \bigcup_{n=1}^{\infty} B_n \cup \{\theta\}$ is a totally ordered cone. Let us first show that $\mathcal{K} + \mathcal{K} \subset \mathcal{K}$. If at least one of $x, y \in \mathcal{K}$ is zero vector then $x + y \in \mathcal{K}$. If $x \neq \theta$ and $y \neq \theta$ then $b_i^*(x) > 0$, $b_j^*(y) > 0$ for some $i, j \in \mathbb{N}$, and $b_n^*(x) = b_n^*(y) = 0$ for all $n < \min\{i, j\}$ Since b_k^* is a linear functional for all $k \in \mathbb{N}$, $x + y \in B_{\min\{i,j\}}$, and so $\mathcal{K} + \mathcal{K} \subset \mathcal{K}$. The linearity of b_k^*'s implies that $\alpha\mathcal{K} \subset \mathcal{K}$ for all $\alpha \geq 0$. Thus $\mathcal{K}$ is a wedge. The linearity of b_k^*'s also implies that $-\mathcal{K} \cap \mathcal{K} = \{\theta\}$. Now let $\theta \neq x \in E$ then let us define $k := \min\{i \in \mathbb{N} : b_i^*(x) \neq 0\}$. Thus $x \in -B_k \cup B_k \subset -\mathcal{K} \cup \mathcal{K}$. Therefore, the cone $\mathcal{K}$ is a totally ordering cone in E. □

The rest of the paper, the notation "$(E, \mathcal{K})$" means that totally ordered vector lattice with the cone $\mathcal{K}$ which is induced by Schauder basis of the vector space E.

Proposition 3.2. *Let E and L be two Banach spaces with equivalent basis (b_n) and (c_n), respectively. If $\mathcal{B}$ and $\mathcal{C}$ are totally ordering cones induced by (b_n) and (c_n), respectively, then $(E, \mathcal{B})$ and $(L, \mathcal{C})$ are order isomorphic vector lattices.*

Proof. From Closed Graph Theorem, b_n and c_n are equivalent basis if and only if there is an isomorphism $T : E \to L$ such that $T(b_n) = c_n$ for all $n \in \mathbb{N}$. It is easy to see that for each $x \in E$, we have $b_n^*(T(x)) = c_n^*(x)$ for all $n \in \mathbb{N}$. Therefore the equality $T(\mathcal{B}) = \mathcal{C}$ holds and so, T is an order isomorphism. □

The following corollary is immediately obtained from Proposition 3.2 by considering the case $E = L$.

Corollary 3.3. *Let $\mathcal{B}$ and $\mathcal{C}$ be totally ordering cones in a Banach space E which are induced by equivalent basis b_n and c_n, respectively. Then $(E, \mathcal{B})$ and $(E, \mathcal{C})$ are order isomorphic vector lattices.*

Proposition 3.4. *The subset $I_1 = \{x \in E : b_1^*(x) = 0\}$ of E is a maximal order ideal in $(E, \mathcal{K})$.*

Proof. Firstly, let us show that I_1 is an order ideal in $(E, \mathcal{K})$. It is not hard to see that I_1 is a vector sub-lattice of E. To show I_1 has the solid property, let $x, y, y - x \in \mathcal{K}$ with $y \in I_1$. Since $b_1^*(y) = 0$ and $b_1^*(y - x) = b_1^*(y) - b_1^*(x)$, then $b_1^*(x)$ is zero or a negative real number. But the case being negative contradicts with being $x \in K$. Therefore $x \in I_1$ and so that I_1 is an order ideal in E.

Now let us show that it is a maximal order ideal. Suppose L is an order ideal in E such that $I_1 \subsetneq L$. If $x \in L \setminus I_1$ then $b_1^*(x) \neq 0$. We will show that $L = E$. Let us assume that there exists $e \in E \setminus L$, then it is easily to see that $b_1^*(e) \neq 0$. We can assume that both of $b_1^*(e)$ and $b_1^*(x)$ are positive otherwise we can rearrange $-x$ or $-e$ as the positive values. Now, since real numbers are Archimedean there exists $\alpha \in \mathbb{R}$ such that $\alpha b_1^*(x) > b_1^*(e)$. The solid property of L implies $e \in L$. Therefore $L = E$ and I_1 is a maximal order ideal in $(E, \mathcal{K})$. □

Indeed, it is not hard to see that $I_n = \{x \in E : b_i^*(x) = 0 \text{ for all } i \leq n\}$ is an order ideal for each $n \geq 1$. Let $I(E)$ be the family of all order ideals in E. It is well known that $I(E)$ has a lattice structure if one consider Minkowski sum and intersection as the lattice operations.

Proposition 3.5. *$I(E)$ has countable cardinality.*

Proof. We will show that all order ideals of E, except itself and $\{\theta\}$, are one of the $I_n = \{x \in E : b_i^*(x) = 0 \text{ for all } i \leq n\}$ for some $n \in \mathbb{N}$. Suppose that a proper order ideal $M \neq I_n$ for all $n \geq 1$. Then from maximality of I_1, it is easy to see that $M \subset I_1$. Otherwise, by following second part of proof of Proposition 3.4, M must contain all elements of E. Indeed, M should be also a subset of I_2. If it is between I_1 and I_2 then again by following second part of proof of Proposition 3.4, M should be equal I_1. Now, one can get the desired result by induction over $n \geq 1$. Therefore all order ideals of E must be equal one of $\{I_n\}$, $\{\theta\}$ or E. □

Corollary 3.6. *The lattice $I(E)$ is totally ordered.*

It is well known that if I is an order ideal in a vector lattice E, then the quotient vector space E/I is a vector lattice with the following order : $\phi(x) > 0$ if $x + y > \theta$ for all $y \in I$, where ϕ is the canonical map from E to E/I. If I is a maximal order ideal in a vector lattice E then the quotient vector lattice E/I is order isomorphic to the real numbers, see [17]. So the following corollary is obtained immediately from the proof of Proposition 3.5, since I_1 is the unique maximal ideal of E we have the following corollary.

Corollary 3.7. *E/I_1 is lattice isomorphic to $\mathbb{R}$.*

The cone $\mathcal{K}$ is not Archimedean (A totally ordered cone is closed if and only if it has at most 1 dimension, see [4]), nevertheless, we have following relationship between order convergence and base projections.

Lemma 3.8. *If a sequence $\{x_n\}$ of E is order convergent to $x \in E$, then the real sequence $\{b_k^*(x_n - x)\}$ converges to zero for each $k \in \mathbb{N}$.*

Proof. First of all, let us show that if $q_n \downarrow \theta$ in E, then $b_k^*(q_n) \downarrow 0$ for each $k \in \mathbb{N}$. Let us assume that $q_n \downarrow \theta$ in E but $r := \inf_{n\in\mathbb{N}} b_{k_0}^*(q_n) \neq 0$ for a $k_0 \in \mathbb{N}$. We can assure that this infimum exits because of that the sequence $b_{k_0}^*(q_n)$ is bounded below from zero. Let y be chosen such that $0 < b_{k_0}^*(y) < r$ and $b_k^*(y) = 0$ for all $k < k_0$. Then obviously $y \neq \theta$ and $q_n > y$ for all $n \in \mathbb{N}$ which contradicts with being $q_n \downarrow \theta$.

Now, let x_n be order convergent to $x \in E$. Then there exits a sequence $q_n \downarrow \theta$ such that $|x_n - x| < q_n$ for each $k \in \mathbb{N}$. From the inequality $b_k^*(|x_n - x|) < b_k^*(q_n)$ and with the previous observation, we obtain that the sequence $b_k^*(|x_n - x|)$ converges to zero for each $k \in \mathbb{N}$. By using linearity of b_k^*, we can easily get the desired result. □

Example 3.9. The norm convergence does not imply the order convergence and vice versa. Let us consider the Banach space c_0 with sup norm. Now consider sequence of $x_n = (\frac{1}{n}, 0, 0, ...)$ for $n \in \mathbb{N}$. It is easy to see that the sequence $\{x_n : n \in \mathbb{N}\}$ converges to zero with sup norm. But it does not order converge to zero . To see this, it is enough to observe that $\inf_{n\in\mathbb{N}}\{x_n\} > (0, 1, 0, 0, ...) > \theta$.

In order to see that order convergence does not imply norm convergence, let us consider the Schauder basis $(e_n)_{n=1}^{\infty}$ of c_0 which is not a Cauchy sequence with respect to sup norm, but it is order convergent to zero vector. It is clear that zero vector is a lower bound for the sequence $\{e_n\}$ and let us assume that $e \in E$ is another lower bound for $\{e_n\}$ such that $e > \theta$. Since $e > \theta$ then there exits an integer n_0 such that n_0th term of the sequence e is a positive real number. But in this case we obtain $e_{n_0+1} < e$ and this contradict with property of e that being lower bound of $\{e_n\}$. Therefore θ is greatest lower bound of $\{e_n\}$, so that it is order convergent to zero vector.

It is well known that Hamel base of the finite dimensional Banach spaces can be seen as a Schauder basis and they are all equivalent to Hamel base of $\mathbb{R}^n$. Since there is only one totally ordering cone in $\mathbb{R}^n$ and by Proposition 3.2, we can re-state the following well-known corollary.

Corollary 3.10. *Every finite dimensional totally ordered vector lattice is order isomorphic to* $(\mathbb{R}^n, <_{lex})$.

Indeed, it is well known that in a Hilbert space, all orthonormal basis are equivalent. Since every orthogonal base in a separable Hilbert space can be seen as a Schauder basis, Proposition 3.2 gives us the following corollary which is the main result of [10].

Corollary 3.11. *Every separable Hilbert space has totally ordering cone.*

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