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Araştırma Makalesi - Research Article

Jaynes-Cummings Modelinde Çiftlenim Sabitinin Kuantum Tahmini

Durgun DURAN[1*]

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ÖZ

Vazgeçilmez avantajlarıyla birlikte, kuantum Fisher bilişimi (QFI), bilinmeyen bir parametrenin değerini belirlemek ve çözünürlük hassasiyetini geliştirmek için anahtar özkaynaklardır. Bu çalışmada, biri Jaynes-Cummings kovuğunda diğeri ise tamamen izole edilmiş, uzaysal olarak ayrılmış iki atomun çiftlenim sabitine ilişkin olarak QFI dinamikleri incelenecek ve QFI'nın, en uygun tahmin için kuantum Cramér-Rao sınırı doyurulacak şekilde parametreler ayarlanarak maksimize edilebileceği gösterilecektir..

Anahtar Kelimeler*- *Kuantum Parametre Tahmini, Kuantum Fisher Bilişimi, Jaynes-Cummings Modeli

1* Corresponding author: durgun.duran@bozok.edu.tr (https://orcid.org/0000-0002-9458-3715)
Physics Department, Yozgat Bozok University, Atatürk Avenue, 7. Km, 66100, Yozgat, Turkey

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Quantum estimation of coupling constant in Jaynes-Cummings model

ABSTRACT

With its indispensable advantages, quantum Fisher information (QFI) is the key resources to determine the value of an unknown parameter and to enhance the precision of resolution. In the present paper, the QFI dynamics with respect to coupling constant of two spatial-separated atoms, one of them in Jaynes-Cummings cavity and the other isolated completely are investigated and it will be shown that the QFI is maximized by adjusting parameters so that quantum Cramér-Rao bound is saturated for the optimal estimation.

Keywords- Quantum Parameter Estimation, Quantum Fisher Information, Jaynes-Cummings Model

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I.INTRODUCTION

Quantum mechanics is a probabilistic theory and the intrinsically stochastic nature of measurements ultimately limits the achievable precision. When considering classical probes independently sensing a physical parameter, such as phase or frequency, the maximum attainable precision follows the standard limit (SL), say $1/\sqrt{N}$, where N is the number of probes [1-3]. In turn, it was shown that quantum entanglement allows one to achieve the Heisenberg limit in precision $(1/N)$ which gives a quadratic improvement compared to classical approaches. These precision limits apply to both single-shot protocols as well as protocols utilizing many repetitions. Still, it remains unclear to what extent such an improvement can be harnessed in practice under non-idealized conditions.

Characterization of the important features of quantum systems and many tasks that cannot be accomplished in a classical way are the aim of quantum information theory. It is necessary to determine the value of quantities that cannot be reached directly due to experimental impediments. This also applies to related quantities such as entanglement measure, phase, the coupling constants of interactions, temperature, which cannot correspond to any quantum observable.

Parameter estimation plays a crucial role in quantum information theory [1-3]. In this field, determining the value of an unknown parameter that encoded the quantum system is the main task and enhancing the resolution accuracy is the main goal. In this present study, we interest in the problem of estimating the coupling constant of the Jaynes-Cummings (JC) Hamiltonian which cannot classically accessible [4] and investigate the behavior of the quantum Fisher information (QFI) [5-8] of a system consisting of a two-qubit in which one of them interacts with a single-mode quantized radiation field. In particular, this system is suitable for the achievement of some quantum communication tasks since it includes a stationary qubit whose dimension is two and a qudit that quantifies a d-dimensional system [9-11]. In this context, the characterization and dynamics of the parameter estimation are provided by the Fisher information (FI) which represents an infinitesimal distance between different probability distributions and it gives the ultimate precision accessible by a quantum estimator called as unbiased via the Cramér-Rao bound (CRB) [5, 12-16]. Its quantum counterpart, QFI is a measure of the degree of distinguishability of a quantum state from its neighbors and it gives an ultimate bound to the precision on the estimation that is allowed by the laws of quantum mechanics. Particularly, QFI provides a good boundary to distinguish each member of the family of different probability distributions. In the quantum estimation theory, the optimal measurement can be found for problems in which the interested quantity is not directly accessible using the quantum mechanical tools in case of the quantum systems and then the quantum Cramér-Rao bound (QCRB) can be established to attain the lower bound imposed by QFI [14-15].

We first construct the probe state prepared in a Fock state or number state that defines any state of the Fock space with a well-defined number of particles in each state and a generic pure state with two 2-qubit states which are in a maximally entangled. We address the overall estimation properties by evaluating the QFI for the whole system undergone the JC evolution. We also focus on the two-qubit subsystem obtained as the partial trace over the cavity and evaluate the QFI of the corresponding reduced density matrix to identify how much quantum information about the parameter to be estimated is contained in the subsystem. Finally, we investigate the dynamics of QFI between each atom and the field.

II. QUANTUM FISHER INFORMATION

An essential goal of the quantum parameter estimation is to archive the best observable. For example, to estimate the true value of parameter θ if the quantum system is in one state of the family $\{\rho_\theta\}$, an observable $\tilde{\theta}$ is called to be the unbiased estimator. Therefore, the expectation value of the estimator should satisfy $Tr(\tilde{\theta}\rho_\theta) = \theta$ and the unbiased estimator $\tilde{\theta}$ is not unique in general. We can quantify how a quantum state can accurately measure an unknown parameter with the QFI associated with the QCRB. QFI is defined as [6, 14, 15]

$$F(\rho_\theta) = Tr(\rho_\theta L^2) \tag{1}$$

where ρ_θ is the density matrix of the system encoded by parameter θ, θ is the parameter to be estimated and L is the symmetric logarithmic derivation given by [14-17]

$$\partial_\theta \rho_\theta = \frac{\rho_\theta L + L\rho_\theta}{2} \tag{2}$$

The QCRB has been formulated in which the bound is asymptotically archived by the maximum likelihood estimator as well as the classical theory [14, 15]

$$\Delta^2\theta \geq \frac{1}{NF(\rho_\theta)} \tag{3}$$

where $\Delta^2\theta$ is the variance or error in the parameter θ and N is the number of independent measurements which repeated. The inequality (3) describes the principally smallest possible uncertainty in the estimation of value of parameter. Given the spectral decomposition of the density operator which is dependent on the parameter θ

$$\rho_\theta = \sum_{i=1}^{s} \lambda_i |\psi_i\rangle\langle\psi_i| \tag{4}$$

where λ_i and $|\psi_i\rangle$ are respectively the parameter-dependent eigenvalues and eigenstates of ρ_θ and s is the dimension of the support set of ρ_θ, i.e. $s = dim[supp(\rho_\theta)]$, QFI for density matrices with arbitrary ranks can be expressed by [18-22]

$$F(\rho_\theta) = \sum_{i=1}^{s} \frac{(\partial_\theta\lambda_i)^2}{\lambda_i} + \sum_{i=1}^{s} 4\lambda_i\langle\partial_\theta\psi_i|\partial_\theta\psi_i\rangle - \sum_{i\neq j=1}^{s} \frac{8\lambda_i\lambda_j}{\lambda_i+\lambda_j}\left|\langle\psi_i|\partial_\theta\psi_j\rangle\right|^2 \tag{5}$$

with $\lambda_i + \lambda_j \neq 0$. The first term in the right-hand side of Eq. (5) is the classical contribution of QFI whereas the second and third terms can be regarded as the pure quantum contribution because factor $\left|\langle\psi_i|\partial_\theta\psi_j\rangle\right|$ illustrates the quantum coherence between the eigenvectors of ρ_θ.

For a unitary parametrization process, the final state ρ_θ is expressed as $\rho_\theta = U_\theta \rho U_\theta^\dagger$ where ρ is the input state and independent of the parameter θ, $U = e^{-i\theta H}$ is unitary operator and H is some Hamiltonian. In this situation, the first term in the right-hand side of Eq. (5) vanishes since the spectrum of density matrix is unchanged. Moreover, it is zero for pure states. In the meantime, with some transformation, Eq. (5) can be rewritten as [22-25]

$$F(\rho_\theta) = \sum_{i=1}^{s} 4\lambda_i\langle\Delta^2\mathcal{H}\rangle_{\psi_i} - \sum_{i\neq j=1}^{s} \frac{8\lambda_i\lambda_j}{\lambda_i+\lambda_j}\left|\langle\psi_i|\mathcal{H}|\psi_j\rangle\right|^2 \tag{6}$$

where $\mathcal{H} \coloneqq i(\partial_\theta U^\dagger)U$ (denote $U = U_\theta$ for simplicity) is a Hermitian operator since $(\partial_\theta U^\dagger)U = -U^\dagger(\partial_\theta U)$. Besides

$$\langle\Delta^2\mathcal{H}\rangle_{\psi_i} = \langle\psi_i|\mathcal{H}^2|\psi_i\rangle - |\langle\psi_i|\mathcal{H}|\psi_i\rangle|^2 \tag{7}$$

is the variance of $\mathcal{H}$ on the ith eigenstate of ρ.

III. MODEL AND DYNAMICS

The Jaynes-Cummings model (JCM) describes the interaction of a two-level atom with a single-mode quantized radiation field. The Hamiltonian of a quantum system which considered here can be written as ($\hbar = 1$) [9-11]

$$H = H_{AB} + H_F + H_I \tag{8}$$

with

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$$H_{AB} = \frac{1}{2}\omega_0(\sigma_z^A \otimes \mathbb{I}^{BC} + \mathbb{I}^A \otimes \sigma_z^B \otimes \mathbb{I}^C), \tag{9a}$$

$$H_F = \mathbb{I}^{AB} \otimes \omega a^\dagger a, \tag{9b}$$

$$H_I = g(\sigma_-^A \otimes \mathbb{I}^B \otimes a^\dagger + \sigma_+^A \otimes \mathbb{I}^B \otimes a), \tag{9c}$$

where ω_0 and ω are respectively the frequencies of atoms and field, g represents the coupling constant of interaction between the cavity and first atom A, σ_z^j (j = A, B) is the atomic inversion or Pauli z-operator of jth atom, $\sigma_\pm$ are the atomic raising and lowering operators and $a^\dagger(a)$ is the creation (annihilation) operator of the cavity field. Atom B interacts neither with atom A nor with cavity C. The eigenstates of this Hamiltonian can be constructed as products of the state of second atom B and the dressed quantum state of the well-known JC system.

We start by assuming that in time $t = 0$ the probe state is prepared in an entangled pure state and no initial correlations between the two-qubit and the bosonic field in the Fock state $|n\rangle$ whose action is defined as $|n\rangle = (1/\sqrt{n!})(a^\dagger)^n|0\rangle$. The initial probe state of the total system constructed with tensor product of states of two atoms and field can be written as

$$|\Psi_0\rangle = (\cos\phi\,|eg\rangle + \sin\phi\,|ge\rangle) \otimes |n\rangle \tag{10}$$

where $|e\rangle(|g\rangle)$ denotes the excited (ground) state of atom A and B. There are possibly $n - 1, n$ or $n + 1$ photons in the cavity. Then, the solution of the model in terms of the standard basis $\{|gg\rangle, |ge\rangle, |eg\rangle, |ee\rangle\}$ in time t can be written as follows

$$|\Psi(t)\rangle = U|\Psi_0\rangle = x_1(t)|egn\rangle + x_2(t)|gen\rangle + x_3(t)|gg(n+1)\rangle + x_4(t)|ee(n-1)\rangle. \tag{11}$$

Solving the Schrödinger equation the coefficients follow the equations

$$i\dot{x}_1(t) = x_3(t)g\sqrt{n+1} + x_1(t)n\omega, \tag{12a}$$

$$i\dot{x}_2(t) = x_4(t)g\sqrt{n} + x_2(t)n\omega, \tag{12b}$$

$$i\dot{x}_3(t) = x_1(t)g\sqrt{n+1} + x_3(t)[(n+1)\omega - \omega_0], \tag{12c}$$

$$i\dot{x}_4(t) = x_2(t)g\sqrt{n} + x_4(t)[(n-1)\omega + \omega_0], \tag{12d}$$

where the coefficients stand for the following time-dependent formulae in case of zero detuning or resonance $\Delta = \omega - \omega_0 = 0$;

$$x_1(t) = e^{-in\omega_0 t}\cos\left(\sqrt{n+1}gt\right)\cos\phi, \tag{13a}$$

$$x_2(t) = e^{-in\omega_0 t}\cos\left(\sqrt{n}gt\right)\sin\phi, \tag{13b}$$

$$x_3(t) = -ie^{-in\omega_0 t}\sin\left(\sqrt{n+1}gt\right)\cos\phi, \tag{13c}$$

$$x_4(t) = -ie^{-in\omega_0 t}\sin\left(\sqrt{n}gt\right)\sin\phi. \tag{13d}$$

A. The QFI Between Two Atoms

Information about two atoms is contained in the reduced density matrix ρ^{AB} for the two atoms which can be obtained from Eq. (11) by tracing out the cavity of the total state, $\rho^{AB} = Tr_C(\rho^{ABC})$ where $\rho^{ABC} = |\Psi(t)\rangle\langle\Psi(t)|$. The explicit 4×4 matrix written on the basis $\{|gg\rangle, |ge\rangle, |eg\rangle, |ee\rangle\}$ is given by

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$$\rho^{AB} = \begin{pmatrix} |x_4(t)|^2 & 0 & 0 & 0 \\ 0 & |x_1(t)|^2 & x_1(t)x_2^*(t) & 0 \\ 0 & x_2(t)x_1^*(t) & |x_2(t)|^2 & 0 \\ 0 & 0 & 0 & |x_3(t)|^2 \end{pmatrix}. \tag{14}$$

The spectrum and corresponding unnormalized eigenvectors of ρ^{AB} are evaluated as for abbreviation $x_i(t) = x_i$,

$$spec(\rho^{AB}) = \{0, 1 - |x_2|^2 - |x_3|^2, |x_2|^2, |x_3|^2\}, \tag{15}$$

$$\{|\Phi_i\rangle\}_{i=1}^4 = \left\{\left(0, -\frac{x_2}{x_1}, 1, 0\right)^T, \left(0, \frac{x_1}{x_2}, 1, 0\right)^T, (1,0,0,0)^T, (0,0,0,1)^T\right\}, \tag{16}$$

where T denotes the transposition operation. The parameter $\theta = gt$ is the estimated parameter, encoded with a unitary adjoint action $ad_U(\cdot) = U(\cdot)U^\dagger$ to the initial probe state and we will focus on its estimation properties by evaluating the QFI. Hereafter, we denote the output density matrix ρ^{AB} as ρ_θ^{AB}.

Now, we can calculate the QFI by the help of Eq. (6) as follows

$$F(\rho_\theta^{AB}) = 4\cot^2\phi \sec^2(\sqrt{n}\theta)\left[1 + \sin^2\phi \cos(2\sqrt{n}\theta) + \cos^2\phi \cos(2\sqrt{n+1}\theta)\right]$$
$$\times \left[\sqrt{n+1}\sin^2(\sqrt{n+1}\theta) - \sqrt{n}\cos^2(\sqrt{n+1}\theta)\tan^2(\sqrt{n}\theta)\right]^2. \tag{17}$$

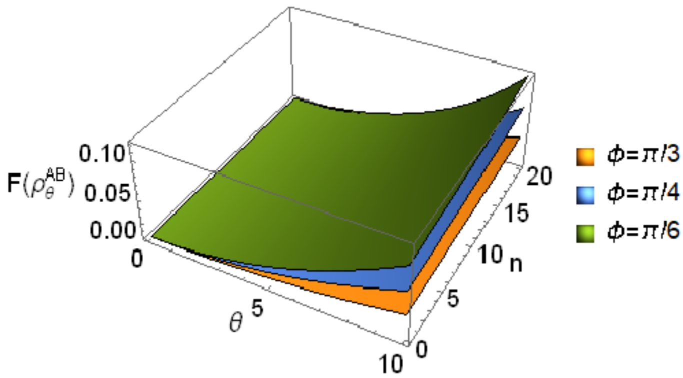


Figure 1. Plot of QFI given by Eq. (17) for the output ρ_θ^{AB} concerning to parameters θ and n.

The behavior of the QFI of the output state ρ_θ^{AB} under the action of Jaynes-Cummings Hamiltonian given by Eq. (8) to input state $\rho_0 = |\Psi_0\rangle\langle\Psi_0|$ is plotted in Figure 1. It is clear that QFI increases with the increasing values of parameter to be estimated θ and photon number n. On the other hand, it takes place its maximum value for small values of the initial parameter ϕ except for $\phi = 0$. Evidently, the QFI is maximized by adjusting parameters θ, n and ϕ so that Eq. (3) is saturated for the optimal estimation.

B. The QFI Between Atoms and Cavity

The reduced density matrices between atom A (B) and cavity C become 6×6 matrix that defines a qubit-qutrit system for $n \neq 0$ and $\rho_\theta^{AC}(\rho_\theta^{BC})$ are explicitly written as a 6×6 matrix in the basis $\{|e(n+1)\rangle, |en\rangle, |e(n-1)\rangle, |g(n+1)\rangle, |gn\rangle, |g(n-1)\rangle\}$

$$\rho_\theta^{AC} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & |x_1(t)|^2 & 0 & x_1(t)x_3^*(t) & 0 & 0 \\ 0 & 0 & |x_4(t)|^2 & 0 & x_4(t)x_2^*(t) & 0 \\ 0 & x_3(t)x_1^*(t) & 0 & |x_3(t)|^2 & 0 & 0 \\ 0 & 0 & x_2(t)x_4^*(t) & 0 & |x_2(t)|^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{18}$$

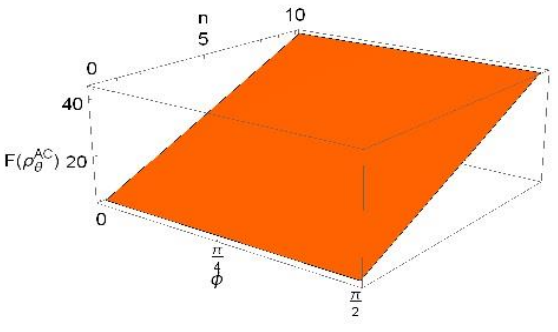
$$\rho_\theta^{BC} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & |x_2(t)|^2 & 0 & x_2(t)x_3^*(t) & 0 & 0 \\ 0 & 0 & |x_4(t)|^2 & 0 & x_4(t)x_1^*(t) & 0 \\ 0 & x_3(t)x_2^*(t) & 0 & |x_3(t)|^2 & 0 & 0 \\ 0 & 0 & x_1(t)x_4^*(t) & 0 & |x_1(t)|^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

Similar to the previous case, the spectra and corresponding eigenvectors of the atom-cavity output states ρ_θ^{AC} and ρ_θ^{BC} can be calculated and are not reported here. Now, QFI between atom A and cavity C whose output state is given by Eq. (18) can be obtained according to Eq. (6) as follows

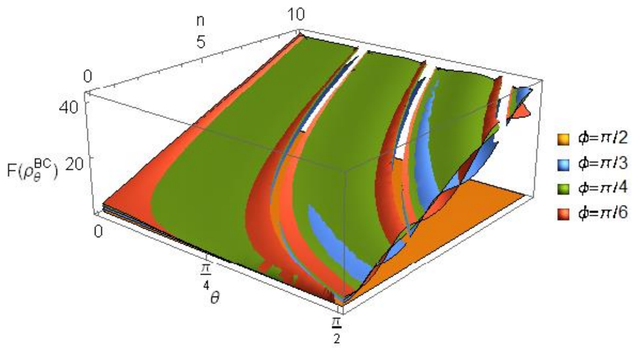
$$F(\rho_\theta^{AC}) = 4(n + \cos^2\phi). \tag{20}$$

We note that the Eq. (20) is independent of the parameter θ and contribution to QFI comes from only quantum part, since the eigenvalues of the output (18) is independent of the parameter θ. On the other hand, QFI between atom B and cavity C has more complicated form since some eigenvalues of the output (19) are dependent to the parameter θ. QFI for the output given by Eq. (19) can be explicitly written as follows

$$F(\rho_\theta^{BC}) = \sin^2 2\phi \left\{ \frac{\left[\sqrt{1+n}\cos(\sqrt{n}\theta)\cot(\sqrt{1+n}\theta) + \sqrt{n}\sin(\sqrt{n}\theta)\right]^2}{\cos^2\phi + \sin^2\phi\cos^2(\sqrt{n}\theta)\csc^2(\sqrt{1+n}\theta)} + \frac{\left[\sqrt{1+n}\sin(\sqrt{n}\theta)\tan(\sqrt{1+n}\theta) + \sqrt{n}\cos(\sqrt{n}\theta)\right]^2}{\cos^2\phi + \sin^2\phi\sin^2(\sqrt{n}\theta)\sec^2(\sqrt{1+n}\theta)} \right\}. \tag{21}$$



(a)



(b)

Figure 2. The evolution of QFIs given by (a) Eq. (20) with respect to parameters ϕ and n, (b) Eq. (21) with respect to parameters n and $\theta = gt$ for different values of ϕ.

Figure 2 displays an interesting behavior of the QFI. It has the same behavior for both output states denoted by ρ_θ^{AC} and ρ_θ^{BC}. We can see that QFI between atom A and cavity C monotonically increases with the increasing value of the photon number n for all ϕ from Fig. 2 (a). Since it is independent of the parameter to be estimated θ we can say that when the atom is in the cavity its information content is lost or the cavity destroys the information content of the atom about the parameter θ. Evidently, quantum information flows from the system to the environment. On the other hand, from Fig. 2 (b) similar to the Fig. 2 (a) QFI increases with increasing values of n for all values of the estimated parameter θ. Moreover, QFI is strictly dependent on the choice of initial state parameter ϕ that characterize the entanglement contents of the initial state of the atom A and B. It is clearly said that from Eq. (21) and Fig. (2) QFI vanishes for the values of $\phi = 0$ and $\phi = k\pi/2, (k = 0,1,2, \dots)$ that correspond to initial separable atom state, namely, product state. Generally, it attains its maximum value for $\phi = \pi/4$ where the initial atomic state is in the maximally entangled state.

IV. CONCLUDING REMARKS

In this present paper, we have studied the behavior of QFI quantify the information content of a quantum state under the Jaynes-Cummings model consisting of a two 2-level atom in which one of them interacts with a single-mode quantized radiation field. Our results clearly show that the actions of different input states have different effects on the QFI. Although there is an inevitable loss of information for the JC model, we have relatively observed enhancements of QFI choosing a large number of photon n for two-qubit entangled input states. Evidently, we should note that further improvements in QFI are possible with the choice of the parameters. This is all to say, it has great importance to choose the convenient parameters to increase the accuracy of quantum parameter estimation. Besides all this, it can be worth to study the behavior of the QFI for the different physical models such as spin-boson interaction, Heisenberg spin system and so on, and it may be interesting to investigate the multiparameter estimation under the actions of these models.

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