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Research Article

# $\mathcal{P}, \mathcal{T}$, and $\mathcal{PT}-$symmetries of impulsive Dirac systems

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## Abstract

This article is concerned with locations of bound states and spectral singularities of an impulsive Dirac system. By using a transfer matrix, we obtain some spectral properties of this impulsive system. We also examine some special cases, where the impulsive condition at the origin has $\mathcal{P}, \mathcal{T}$, and $\mathcal{PT}-$symmetry.

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## 1. Introduction

Impulsive operators, that is, the operators involving impulsive effects appear in many different fields, including several world problems. Especially, Schrödinger differential equations subject to the general point interaction describe observed evolution phenomena. For instance, many chemical, physical phenomena, and pharmacokinetics do exhibit point interaction effects [8]. The spectral analysis of Schrödinger equations with general point interaction has been investigated in detail in [11, 12]. In literature, point interactions are called with various names; like impulsive conditions, jump conditions, interface conditions, transmission conditions, etc. In particular, spectral analysis of regular and singular impulsive Sturm–Liouville operators has been studied in [13, 14, 20, 21]. To be more precise, we should note that these equations with impulsive conditions have bound states, i.e., eigenvalues with square-integrable eigenfunctions and spectral singularities.

It is well known that the bound states of quantum mechanical system correspond to the energy. Also a physical interpretation for the spectral singularities that identifies with the energies of scattering states having infinite reflection and transmission coefficients. So spectral singularities correspond to the resonance states having a real energy. On the other hand, in spectral theory, it is a fact that spectral singularities are the poles of the kernel of the resolvent. Also, they belong to the continuous spectrum but they are not the eigenvalues. They are the spectral points that spoil the completeness of the eigenfunctions of certain non-Hermitian operators.

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The study of spectral singularities of the Sturm–Liouville operators with general boundary conditions has a long story. Many important and interesting results on this topic have been reported, see [10, 15–19]. Later, similar investigations have been done about Schrödinger, Klein-Gordon, and Dirac operators [1–4, 7]. Consequently, the spectral analysis of operators with spectral singularities is important to study in spectral theory and quantum mechanics. In elementary courses on quantum mechanics, it was learned that the condition that observables are Hermitian operators ensures the reality of their spectrum [22]. But, during the past ten years, there has been a renewed interest in quantum mechanics in the study of a special class of non-Hermitian operators that possess real spectrum. Surprisingly, it was observed that it is not an indispensable requirement that the operator must be Hermitian to get a real spectrum. The best known examples are $\mathcal{PT}$–operators. For this reason, a large class of non-Hermitian $\mathcal{PT}$–operators has become a very important direction in the theory of impulsive equations. A kind of such operators was studied by Bender, Guseinov, and Mostafazadeh in the recent years [5, 6, 11]. In 2011, Mostafazadeh investigated spectral singularities and bound states of a general point interaction of Schrödinger equation at a single point and examined the special cases where the point interaction is $\mathcal{P}, \mathcal{T}$, and $\mathcal{PT}$–symmetric [11]. In this context, we propose to discuss the analogous problem of locating spectral singularities of a general point interaction at a single point for Dirac system in this paper.

Let $\mathcal{L}$ denote the Dirac operator generated in $L_2(\mathbb{R}, \mathbb{C}^2)$ by the equation

$$i\sigma_2 \frac{d\psi}{dx} + m\sigma_3\psi = \lambda\psi, \qquad \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \qquad x \in \mathbb{R}\setminus\{0\} \tag{1.1}$$

with the point interaction

$$\begin{pmatrix} \psi_1(0^+) \\ \psi_2(0^+) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_1(0^-) \\ \psi_2(0^-) \end{pmatrix}, \qquad a, b, c, d \in \mathbb{C}, \tag{1.2}$$

where $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $m$ is the mass of particle and $\lambda$ is a complex spectral parameter. (1.2) is an impulsive condition for (1.1) at the point $x = 0$. Also, $\psi(x^-)$ and $\psi(x^+)$ are respectively the restrictions of $\psi$ to the sets of negative and positive real numbers, i.e.

$$\begin{cases} \psi(x^-) := \psi(x), & x \in \mathbb{R}^- \\ \psi(x^+) := \psi(x), & x \in \mathbb{R}^+ \end{cases}$$

and

$$\begin{cases} \psi(0^-) := \lim\limits_{\epsilon \to 0^-} \psi(\epsilon) \\ \psi(0^+) := \lim\limits_{\epsilon \to 0^+} \psi(\epsilon). \end{cases}$$

If we introduce the two-component wavefunction

$$\begin{cases} \Psi(x^-) := \begin{pmatrix} \psi_1(x^-) \\ \psi_2(x^-) \end{pmatrix}, & x \leqslant 0 \\[2ex] \Psi(x^+) := \begin{pmatrix} \psi_1(x^+) \\ \psi_2(x^+) \end{pmatrix}, & x \geqslant 0 \end{cases}, \tag{1.3}$$

we can express the point interaction (1.2) by imposing the matching condition

$$\Psi(0^+) = B\Psi(0^-), \qquad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{C}. \tag{1.4}$$

Depending on the choice of the coupling constants $a, b, c, d$, this interaction may display $\mathcal{P}, \mathcal{T}$, or $\mathcal{PT}-$symmetry. In this paper, we will give the definitions of $\mathcal{P}, \mathcal{T}$, and

$\mathcal{PT}-$symmetry on the point interaction (1.4) for the Dirac system and consider the problem of locating bound states and spectral singularities of this interaction.

## 2. Spectral singularities and bound states

Now, let us take into account our point interaction for (1.4):

$$\varphi_1(x) = \begin{pmatrix} \dfrac{\lambda+m}{k} ie^{ikx} \\ e^{ikx} \end{pmatrix}, \quad x \in \mathbb{R} \setminus \{0\} \tag{2.1}$$

and

$$\varphi_2(x) = \begin{pmatrix} -\dfrac{\lambda+m}{k} ie^{-ikx} \\ e^{-ikx} \end{pmatrix}, \quad x \in \mathbb{R} \setminus \{0\} \tag{2.2}$$

are the linearly independent solutions of (1.1) for $\lambda \in \mathbb{C}\backslash\{\pm m\}$, where $k := \sqrt{\lambda^2 - m^2}$. Hence we can express the general solution of (1.1) by

$$\begin{array}{ll} \psi(x^+) = A_+\varphi_1(x) + B_+\varphi_2(x), & x > 0 \\ \psi(x^-) = A_-\varphi_1(x) + B_-\varphi_2(x), & x < 0 \end{array}.$$

Using the point interaction given by (1.4), we obtain

$$\begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = \mathbf{M} \begin{pmatrix} A_- \\ B_- \end{pmatrix}, \tag{2.3}$$

such that

$$\mathbf{M} := N^{-1}BN, \qquad N := \begin{pmatrix} \dfrac{\lambda+m}{k} i & -\dfrac{\lambda+m}{k} i \\ 1 & 1 \end{pmatrix}, \tag{2.4}$$

where $\mathbf{M} = (M_{ij}); i,j = 1,2,\ldots$.

On the other hand, (1.1) admits a pair of solutions $\psi_{\lambda_\pm}(x) = \begin{pmatrix} \psi^{(1)}_{\lambda\pm}(x) \\ \psi^{(2)}_{\lambda\pm}(x) \end{pmatrix}$ fulfilling the asymptotic boundary conditions

$$\lim_{x\to-\infty} \frac{\psi_{\lambda_-}(x)}{\varphi_2(x)} = 1, \quad z \in \overline{\mathbb{C}}_+ \quad \text{and} \quad \lim_{x\to\infty} \frac{\psi_{\lambda_+}(x)}{\varphi_1(x)} = 1, \quad z \in \overline{\mathbb{C}}_+,$$

where $\overline{\mathbb{C}}_+ := \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda \geq 0\}$. These are called Jost solutions of (1.1). For the operator acting in $L_2(\mathbb{R}, \mathbb{C}^2)$, this is the Wronskian [9]

$$W[\psi_{\lambda_+}, \psi_{\lambda_-}] := \begin{vmatrix} \psi^{(1)}_{\lambda+}(x) & \psi^{(1)}_{\lambda-}(x) \\ \psi^{(2)}_{\lambda+}(x) & \psi^{(2)}_{\lambda-}(x) \end{vmatrix} = \psi^{(1)}_{\lambda+}(0)\psi^{(2)}_{\lambda-}(0) - \psi^{(2)}_{\lambda+}(0)\psi^{(1)}_{\lambda-}(0) \tag{2.5}$$

of the Jost solutions $\psi_{\lambda\pm}$ of the eigenvalue equation (1.1). A spectral singularity of $\mathcal{L}$ is a point $\lambda$ of the continuous spectrum of $\mathcal{L}$ such that $\psi_{\lambda_+}$ and $\psi_{\lambda_-}$ are linearly dependent, that is, they have a vanishing Wronskian.

Now, consider the left- and right-going scattering solutions of $\mathcal{L}\psi = \lambda\psi$ that we denote by $\psi^l_\lambda$ and $\psi^r_\lambda$, respectively. They are expressed as

$$\psi^l_\lambda(x) = \begin{cases} A^+_+\varphi_1(x) + B^+_+\varphi_2(x), & x \to +\infty \\ A^+_-\varphi_1(x) + B^+_-\varphi_2(x), & x \to -\infty \end{cases} \tag{2.6}$$

and

$$\psi^r_\lambda(x) = \begin{cases} A^-_+\varphi_1(x) + B^-_+\varphi_2(x), & x \to +\infty \\ A^-_-\varphi_1(x) + B^-_-\varphi_2(x), & x \to -\infty, \end{cases} \tag{2.7}$$

where $A^\pm_\pm$ and $B^\pm_\pm$ are probably $\lambda$-dependent complex coefficients. Note that, denoting the coefficients $A_\pm$ and $B_\pm$ for the Jost solutions $\psi_{\lambda_\pm}$ by $A^\pm_\pm$ and $B^\pm_\pm$, we obtain $\psi^l_\lambda$ and $\psi^r_\lambda$.

Thus, comparing left- and right-going scattering solutions with Jost solutions, we see that $\psi_\lambda^l$ and $\psi_\lambda^r$ are, respectively, proportional to the Jost solutions $\psi_+$ and $\psi_{\lambda_-}$. Therefore, at a spectral singularity $\lambda$, the scattering solutions $\psi_\lambda^l$ and $\psi_\lambda^r$ become linearly dependent.

**Theorem 2.1.** *The following equations hold.*

$$W[\psi_\lambda^l, \psi_\lambda^r] = \frac{2i\,(\lambda+m)\,M_{22}}{k}, \quad x \to +\infty, \tag{2.8}$$

$$W[\psi_\lambda^l, \psi_\lambda^r] = \frac{2i\,(\lambda+m)\,M_{22}}{k\det M}, \quad x \to -\infty. \tag{2.9}$$

***Proof.*** The left- and right-going scattering solutions are defined in terms of their asymptotic behaviors

$$\psi_\lambda^l(x) \to \varphi_1(x), \quad x \to +\infty$$

and

$$\psi_\lambda^r(x) \to \varphi_2(x), \quad x \to -\infty$$

respectively. Therefore, we obtain uniquely

$$A_+^+ = B_-^- = 1, \qquad A_-^- = B_+^+ = 0. \tag{2.10}$$

Next, for the left-going scattering solution, we write

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A_-^+ \\ B_-^+ \end{pmatrix}$$

using the expression (2.3) and (2.10). This implies that

$$A_-^+ = \frac{M_{22}}{\det M}, \qquad B_-^+ = -\frac{M_{21}}{\det M}\ . \tag{2.11}$$

Similarly, for the right-going scattering solution, we get

$$A_+^- = M_{12}, \qquad B_+^- = M_{22}. \tag{2.12}$$

Now, we can express the left- and right-going scattering solutions (2.6) and (2.7) such as

$$\begin{aligned} \psi_\lambda^l(x) &= \begin{cases} \varphi_1(x), & x \to +\infty \\ \dfrac{M_{22}}{\det M}\varphi_1(x) - \dfrac{M_{21}}{\det M}\varphi_2(x), & x \to -\infty \end{cases} \\ \psi_\lambda^r(x) &= \begin{cases} M_{12}\varphi_1(x) + M_{22}\varphi_2(x), & x \to +\infty \\ \varphi_2(x), & x \to -\infty. \end{cases} \end{aligned} \tag{2.13}$$

Because the Wronskian of solutions is independent of $x$, the equation (2.13) can be used to compute the Wronskian of the Jost solutions for $x \to +\infty$ and for $x \to -\infty$.

($i$) For $x \to +\infty$, we get

$$\begin{aligned} W[\psi_\lambda^l, \psi_\lambda^r] &= \left\{ \frac{\lambda+m}{k} i e^{ikx} \left( M_{12}e^{ikx} + M_{22}e^{-ikx} \right) \right. \\ &\quad \left. - \frac{\lambda+m}{k} i e^{ikx} \left( M_{12}e^{ikx} - M_{22}e^{-ikx} \right) \right\}_{x=0} \\ &= \frac{\lambda+m}{k} i \left( M_{12} + M_{22} \right) - \frac{\lambda+m}{k} i \left( M_{12} - M_{22} \right) \\ &= \frac{2i\,(\lambda+m)\,M_{22}}{k}. \end{aligned}$$

$(ii)$ For $x \to -\infty$, we see

$$\begin{aligned} W[\psi_\lambda^l, \psi_\lambda^r] &= \left\{ \frac{\lambda+m}{k \det M} i \left( M_{22} e^{ikx} + M_{21} e^{-ikx} \right) e^{-ikx} \right. \\ &\quad \left. + \frac{\lambda+m}{k \det M} i \left( M_{22} e^{ikx} - M_{21} e^{-ikx} \right) e^{-ikx} \right\}_{x=0} \\ &= \frac{\lambda+m}{k \det M} i \left( M_{22} + M_{21} \right) + \frac{\lambda+m}{k \det M} i \left( M_{22} - M_{21} \right) \\ &= \frac{2i\,(\lambda+m)\, M_{22}}{k \det M}. \end{aligned}$$

□

A direct consequence of (2.8) and (2.9) is

$$\det \mathbf{M} = \det B = ad - bc = 1. \tag{2.14}$$

We will call the point interactions violating this condition *anomalous point interactions* [11]. Using Theorem 2.1 and (2.14), we have the following.

**Corollary 2.2.** *A necessary and sufficient condition to investigate the bound states and spectral singularities of the Dirac operator $\mathcal{L}$ is to investigate the zeros of the function $M_{22}$.*

Combining (1.4)-(2.3), we find

$$\mathbf{M} = \frac{1}{2\tau} \begin{pmatrix} c\tau^2 + (a+d)\tau + b & -c\tau^2 - (a-d)\tau + b \\ c\tau^2 - (a-d)\tau - b & -c\tau^2 + (a+d)\tau - b \end{pmatrix}, \tag{2.15}$$

where

$$\tau := \frac{\lambda+m}{k} i.$$

Since the spectral singularities and bound states of $\mathcal{L}$ are given by zeros of $M_{22}$, according to (2.15), they correspond to $\lambda$ values for which $M_{22}(\tau) = 0$, i.e.

$$c\tau^2 - (a+d)\tau + b = 0. \tag{2.16}$$

$\sigma_d(\mathcal{L})$ and $\sigma_{ss}(\mathcal{L})$ will denote the bound states and spectral singularities of $\mathcal{L}$, respectively. Therefore, by the definitions of bound states and spectral singularities of an operator, we can write [10, 16],

$$\sigma_d(\mathcal{L}) = \left\{ \lambda \colon \lambda \in \mathbb{C} \backslash \mathbb{R}^*,\ \lambda = \left(1 - \frac{2}{\tau^2+1}\right) m,\ M_{22}(\tau) = 0 \right\}, \tag{2.17}$$

$$\sigma_{ss}(\mathcal{L}) = \left\{ \lambda \colon \lambda \in \mathbb{R}^*,\ \lambda = \left(1 - \frac{2}{\tau^2+1}\right) m,\ M_{22}(\tau) = 0 \right\}, \tag{2.18}$$

where $\mathbb{R}^* := (-\infty, -m) \cup (m, +\infty)$.

In order to examine the zeros of (2.16), we consider the following cases.

**Case 2.3.** $c \neq 0:$ In this case, (2.16) gives

$$\tau_{1,2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4bc}}{2c},$$

i.e.,

$$\tau_{1,2} = \frac{a+d}{2c} \pm \sqrt{\left(\frac{a+d}{2c}\right)^2 - \frac{b}{c}}.$$

Then, we can write

$$\tau_{1,2} = \mu \pm \sqrt{\mu^2 - \nu},$$

where

$$\mu := \frac{a+d}{2c}, \qquad \nu := \frac{b}{c}$$

and obtain

$$\lambda = \left(1 - \frac{2}{\left(\mu \pm \sqrt{\mu^2 - \nu}\right)^2 + 1}\right) m.$$

Therefore, a spectral singularity appears whenever

$$\operatorname{Re}\left(\mu \pm \sqrt{\mu^2 - \nu}\right) = 0, \qquad \operatorname{Im}\left(\mu \pm \sqrt{\mu^2 - \nu}\right) \neq \pm 1, \tag{2.19}$$

and a bound state exists if

$$\operatorname{Re}\left(\mu \pm \sqrt{\mu^2 - \nu}\right) \neq 0. \tag{2.20}$$

**Case 2.4.** $c = 0$ and $a + d = TrB \neq 0$: In this case, (2.16) gives

$$\tau = \frac{b}{a+d}$$

and thus, we have

$$\lambda = \left(1 - \frac{2}{\left(\frac{b}{a+d}\right)^2 + 1}\right) m.$$

Therefore, a spectral singularity appears whenever

$$\operatorname{Re}\left(\frac{b}{a+d}\right) = 0, \qquad \operatorname{Im}\left(\frac{b}{a+d}\right) \neq \pm 1,$$

and a bound state exists if

$$\operatorname{Re}\left(\frac{b}{a+d}\right) \neq 0.$$

**Case 2.5.** $c = 0$ and $a + d = TrB = 0$ : Then the condition of the existence of a spectral singularity or a bound state, namely $M_{22} = 0$ implies that $b = 0$. In this case $B = a\sigma_3$ and $M = -a\sigma_1$, where

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In particular, $\mathbf{M}$ is independent of $\lambda$, $M_{22}$ vanishes identically, and the interaction is anomalous for $a \neq \pm i$.

Now, we can give the following theorem.

**Theorem 2.6.** *For the point interaction* (1.4), *suppose that* $c \neq 0$, *we can summarize the conditions for the existence of spectral singularities and bound states as follows:*

(i) *There is a spectral singularity located at* $\lambda = \left(1 - \frac{2}{\tau^2 + 1}\right) m$, *for pure imaginary* $\tau \neq \pm i$.

(ii) *There is a bound state located at* $\lambda = \left(1 - \frac{2}{\tau^2 + 1}\right) m$, *where* $\operatorname{Re} \tau \neq 0$.

## 3. $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{PT}-$symmetries

In this section, we examine the consequences of imposing $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{PT}-$symmetries on the point interaction (1.4) and their spectral singularities and bound states.

### 3.1. $\mathcal{P}-$symmetry

**Definition 3.1.** Let $\mathcal{P}$ be the parity (reflection) operator acting in the space of all differentiable complex vector-valued functions $\varphi(x) = \begin{pmatrix}\varphi^{(1)}(x)\\ \varphi^{(2)}(x)\end{pmatrix}$, $\varphi : \mathbb{R} \to \mathbb{C}^2$. Then for all $x \in \mathbb{R}$, we have

$$(\mathcal{P}\varphi)(x) := \varphi(-x).$$

**Definition 3.2.** The point interaction (1.4) is $\mathcal{P}$ invariant (or has $\mathcal{P}-$symmetry) if

$$(\mathcal{P}\Psi)(0^+) = B(\mathcal{P}\Psi)(0^-), \tag{3.1}$$

where the action of $\mathcal{P}$ on a two-component wave function $\Psi$ is defined componentwise.

**Theorem 3.3.** *The point interaction* (1.4) *has $\mathcal{P}-$symmetry if and only if*

$$B = B^{-1}. \tag{3.2}$$

***Proof.*** Using (1.4) and (3.2), we can write

$$\begin{aligned}(\mathcal{P}\Psi)(0^+) = B(\mathcal{P}\Psi)(0^-) \quad &\Longleftrightarrow \quad \Psi(0^-) = B\Psi(0^+)\\ &\Longleftrightarrow \quad \Psi(0^+) = B^{-1}\Psi(0^-)\\ &\Longleftrightarrow \quad B\Psi(0^-) = B^{-1}\Psi(0^-)\end{aligned}$$

and this completes the proof. □

**Theorem 3.4.** *If the point interaction* (1.4) *has $\mathcal{P}-$symmetry, then the operator does not have any spectral singularity and bound state.*

***Proof.*** Suppose that, (1.4) has $\mathcal{P}-$symmetry. Using (3.2), we obtain

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}$$

and since $\det B = 1$, we find

$$a = d = 1,\ b = c = 0.$$

Therefore, we find

$$\tau = i\frac{\lambda+m}{k} = 0$$

and this means there is no bound state and spectral singularity. This completes the proof. □

### 3.2. $\mathcal{T}-$symmetry

**Definition 3.5.** Let $\mathcal{T}$ be the time-reversal operator acting on complex vector-valued functions $\varphi : \mathbb{R} \to \mathbb{C}^2$ according to

$$(\mathcal{T}\varphi)(x) := \varphi^*(x).$$

**Definition 3.6.** The point interaction (1.4) is the time-reversal invariant (or has $\mathcal{T}-$symmetry) if

$$(\mathcal{T}\Psi)(0^+) = B(\mathcal{T}\Psi)(0^-), \tag{3.3}$$

where the action of $\mathcal{T}$ on a two-component wave function $\Psi$ is defined componentwise.

**Theorem 3.7.** *The point interaction* (1.2) *has* $\mathcal{T}-$*symmetry if and only if* $B$ *is real matrix.*

***Proof.*** If (3.3) holds for the point interaction (1.4), it is easy to see that this relation is equivalent to the requirement that $B$ is a real matrix, i.e. $a, b, c, d$ must be real. $\square$

**Theorem 3.8.** *If the point interaction* (1.4) *has* $\mathcal{T}-$*symmetry, then we can summarize the conditions for the existence of spectral singularities and bound states as follows:*

(i) *If* $c \neq 0$*, then there exist bound states when* $a+d \neq 0$*, also there exist spectral singularities when* $a+d=0$*,* $b \neq c$ *and* $bc > 0$*.*

(ii) *If* $c = 0$ *and* $a+d = Tr\,(B) \neq 0$*, then there exist only bound states when* $b \neq 0$*.*

(iii) *If* $c = 0$ *and* $a+d = Tr\,(B) = 0$*, then the interaction is anomalous.*

***Proof.*** (i) Suppose that $c \neq 0$. Since $B = B^*$, we obtain $\mu, \nu \in \mathbb{R}$. Using (2.19) and (2.20), we can examine two special cases.

$$\mu^2 < \nu \implies \mathrm{Re}\left(\mu \pm \sqrt{\mu^2-\nu}\right) = \mu, \ \mathrm{Im}\left(\mu \pm \sqrt{\mu^2-\nu}\right) = \sqrt{\nu-\mu^2}$$

and thus, there exist spectral singularities if $\mu = 0$, $\nu > 0$ and $\nu \neq 1$, that is, $a+d=0$, $b \neq c$ and $bc > 0$. Also, there exist bound states if $a+d \neq 0$.

$$\mu^2 > \nu \implies \mathrm{Re}\left(\mu \pm \sqrt{\mu^2-\nu}\right) = \mu \pm \sqrt{\mu^2-\nu}, \ \mathrm{Im}\left(\mu \pm \sqrt{\mu^2-\nu}\right) = 0$$

and thus, there exist spectral singularities if $\nu = 0$, that is, $b = 0$. Also, there exist bound states if $b \neq 0$.

(ii) Suppose that $c = 0$ and $a+d = Tr\,(B) \neq 0$. Since $B = B^*$, we easily find $\tau = \dfrac{b}{a+d} \in \mathbb{R}$. There exist spectral singularities if and only if $b = 0$, that is, $\lambda = -m$. But it contradicts that $\lambda$ can not be $\pm m$. Thus, the operator does not have any spectral singularity, it only has bound states when $b \neq 0$.

(iii) Suppose that $c = 0$ and $a+d = Tr\,(B) = 0$. $b$ must be zero that the equation (2.16) is supplied. So,

$$b = c = 0, \qquad d = -a \implies \det \mathbf{M} = -a^2$$

and since $\det \mathbf{M} \neq 1$, the interaction is anomalous.

$\square$

## 3.3. $\mathcal{PT}-$symmetry

**Definition 3.9.** The point interaction (1.4) is $\mathcal{PT}$ invariant (or has $\mathcal{PT}-$symmetry) if

$$(\mathcal{PT}\Psi)(0^+) = B(\mathcal{PT}\Psi)(0^-). \tag{3.4}$$

**Theorem 3.10.** *The point interaction* (1.4) *has* $\mathcal{PT}-$*symmetry if and only if*

$$B^* = B^{-1} \tag{3.5}$$

*holds.*

***Proof.***

$$\begin{aligned}
\Psi(0^+) = B\Psi(0^-) \ &\Leftrightarrow \ \Psi^*(0^+) = B^*\Psi^*(0^-) \\
&\Leftrightarrow \ (\mathcal{T}\Psi)(0^+) = B^*(\mathcal{T}\Psi)(0^-) \\
&\Leftrightarrow \ (\mathcal{P}(\mathcal{T}\Psi))(0^-) = B^*(\mathcal{P}(\mathcal{T}\Psi))(0^+) \\
&\Leftrightarrow \ (\mathcal{P}(\mathcal{T}\Psi))(0^+) = (B^*)^{-1}(\mathcal{P}(\mathcal{T}\Psi))(0^-) \\
&\Leftrightarrow \ (B^*)^{-1} = B \\
&\Leftrightarrow \ B^{-1} = B^*.
\end{aligned}$$

$\square$

**Remark 3.11.** If the point interaction (1.4) has $\mathcal{PT}-$symmetry, we obtain

$$\begin{aligned} B^* = B^{-1} \quad &\Leftrightarrow \quad \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &\Leftrightarrow \quad a^* = d, \qquad b^* = -b, \qquad c^* = -c, \qquad ad - bc = 1 \\ &\Leftrightarrow \quad a^* = d, \qquad \operatorname{Re} b = 0, \qquad \operatorname{Re} c = 0, \end{aligned}$$

and so, in terms of the entries of $B$, (3.5) is equivalent to

$$a + d = 2\operatorname{Re} a, \qquad \operatorname{Re} b = 0, \qquad \operatorname{Re} c = 0.$$

**Theorem 3.12.** *If the point interaction* (1.4) *has* $\mathcal{PT}-$*symmetry, then we can summarize the conditions for the existence of spectral singularities and bound states as follows:*

(i) *If* $c \neq 0$*, then there exist spectral singularities when* $\left(\frac{\operatorname{Re} a}{\operatorname{Im} c}\right)^2 + \frac{b}{c} \geq 0$ *and there exist bound states when* $\left(\frac{\operatorname{Re} a}{\operatorname{Im} c}\right)^2 + \frac{b}{c} < 0$.

(ii) *If* $c = 0$ *and* $a + d = Tr\,(B) \neq 0$*, there exist spectral singularities if* $\frac{\operatorname{Im} b}{2\operatorname{Re} a} \neq \pm 1$.

(iii) *If* $c = 0$ *and* $a + d = Tr\,(B) = 0$*, then the interaction is anomalous when* $a \neq \pm i$.

***Proof.*** (i) Suppose that $c \neq 0$. It implies that

$$\tau_{1,2} = \mu \pm \sqrt{\mu^2 - \nu}$$

and gives

$$\tau = -i\left\{\frac{\operatorname{Re} a}{\operatorname{Im} c} \pm \sqrt{\left(\frac{\operatorname{Re} a}{\operatorname{Im} c}\right)^2 + \frac{b}{c}}\right\}.$$

Hence, by (2.19) and (2.20), there exist spectral singularities if $\left(\frac{\operatorname{Re} a}{\operatorname{Im} c}\right)^2 + \frac{b}{c} \geq 0$, also there exist bound states if $\left(\frac{\operatorname{Re} a}{\operatorname{Im} c}\right)^2 + \frac{b}{c} < 0$.

(ii) Suppose that $c = 0$ and $a + d = Tr\,(B) \neq 0$. Since $\tau = \frac{b}{a+d} = \frac{\operatorname{Im} b}{2\operatorname{Re} a} i$, there exist spectral singularities if $\frac{\operatorname{Im} b}{2\operatorname{Re} a} \neq \pm 1$.

(iii) Suppose that $c = 0$ and $a + d = Tr\,(B) = 0$. $b$ must be zero that the equation (2.16) is supplied. Then we get

$$b = c = 0, \qquad d = -a,$$

which yields $\det \mathbf{M} = -a^2$. Thus, we conclude that the interaction is anomalous for $a \neq \pm i$.

□

## 4. Conclusions

In this study, we investigated the bound states and spectral singularities of a Dirac operator with a point interaction at the origin. This paper emphasizes that if a discontinuity appears in a Dirac system, this may create some structural changes on the solutions of the system. Therefore, the locations of the spectral singularities and eigenvalues, so called bound states depend on the choice of the constants of given point interaction. In this paper, we followed a different way to examine some special cases by introducing a transfer matrix. The rest of the paper dealt with certain symmetries which are directly related with mathematical physics. We deduced some consequences about these symmetries in the last part of the paper.

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