Suzuki - $F(\psi - \phi) - \alpha$ type fixed point theorem on quasi metric spaces

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Abstract
In this paper, we obtain a $\alpha$ - Suzuki fixed point theorem by using $C$ - class function on quasi metric spaces. Also we give an example which supports our main theorem.

Keywords: Quasi - metric space, Suzuki type contraction, $C$ - Class function, $\alpha$ - admissible mapping.

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1. Introduction

In this paper $\mathbb{N}$ and $\mathbb{R}$ denote the sets of positive integers, respectively the set of real numbers, while $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_0^+ := [0, \infty)$. In 2008, the generalization theorem of Banach contraction principle [2], which was introduced by T.Suzuki [7], later this theorem is also referred as Suzuki type contraction. In 2014, Ansari [1] introduced the concept of $C$- class functions and proved the unique fixed point theorems for certain contractive mappings with respect to the $C$ - class functions.

The aim of this paper is to prove a $\alpha$-Suzuki type fixed point theorem by using $(C)$- class functions in quasi metric spaces.

2. Preliminaries

The aim of Suzuki [7] is to extend the well-known Edelstein’s Theorem by using the notion of C-condition. Popescu [5] re-considered this approach to extend Bogin’s fixed point theorem:
Theorem 2.1. Let a self-mapping $T$ on a complete metric space $(X, d)$ satisfies the following condition:

$$\frac{1}{2}d(x, Tx) \leq d(x, y)$$  \hspace{1cm} (1)

implies

$$d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)]$$  \hspace{1cm} (2)

where $a \geq 0$, $b > 0$, $c > 0$ and $a + 2b + 2c = 1$. Then $T$ has a unique fixed point.

First we recall some basic definitions which play crucial role in the theory of quasi metric spaces.

Definition 2.2. Let $X$ be a non-empty set $X$ and $q : X \times X \to \mathbb{R}^+$ be a function which satisfies: such that

$q(1)$ $q(x, y) = 0$ if and only if $x = y$;

$q(2)$ $q(x, y) \leq q(x, z) + q(z, y)$.

The pair $(X, q)$ is called a quasi-metric space.

Example 2.3. Let $X = l_1$ be defined by

$$l_1 = \{\{x_n\}_{n \geq 1} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n| < \infty\}.$$ 

Consider $d : X \times X \to [0, \infty)$ such that

$$q(x, y) = \begin{cases} 0 & \text{if } x \preceq y, \\ \sum_{n=1}^{\infty} |x_n| & \text{if } x \succeq y. \end{cases}$$

$q$ is a quasi-metric. Mention that $x \succeq y$ if $x_n \geq y_n$ for all $n$, where $x = \{x_n\}$ and $y = \{y_n\}$ are in $X$.

Definition 2.4. Let $(X, q)$ be a quasi-metric space.

$q(i)$ A sequence $\{x_n\}$ in $X$ is said to be convergent to $x$ if $\lim_{n \to \infty} q(x_n, x) = \lim_{n \to \infty} q(x, x_n) = 0$.

$q(ii)$ A sequence $\{x_n\}$ in $X$ is called left-Cauchy if for every $\epsilon > 0$ there exists a positive integer $N = N(\epsilon)$ such that $q(x_n, x_m) < \epsilon$ for all $n \geq m > N$.

$q(iii)$ A sequence $\{x_n\}$ in $X$ is called right-Cauchy if for every $\epsilon > 0$ there exists a positive integer $N = N(\epsilon)$ such that $q(x_n, x_m) < \epsilon$ for all $m \geq n > N$.

$q(iv)$ A sequence $\{x_n\}$ in $X$ is called Cauchy sequence if for every $\epsilon > 0$ there exists a positive integer $N = N(\epsilon)$ such that $q(x_n, x_m) < \epsilon$ for all $m, n > N$.

Remark: From definition it is obvious that a sequence $\{x_n\}$ in a quasi-metric space is Cauchy if and only if it is both left-Cauchy and right-Cauchy.

Ansari [1] introduced the concept of $C$-class functions as the following:

Definition 2.5. (See [1]) A mapping $F : [0, +\infty)^2 \to \mathbb{R}$ is called a $C$-class function if it is continuous and for all $s, t \in [0, +\infty)$,

(a) $F(t, s) \leq s$;

(b) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.

We denote $\mathcal{C}$ as the family of all $C$-class functions.
Example 2.6. (See [7]) The following functions $F : [0, +\infty)^2 \to R$ are elements in $C$.

1. $(F(s, t) = s - t \text{ for all } s, t \in [0, \infty);$ 
2. $(F(s, t) = ms \text{ for all } s, t \in [0, \infty) \text{ where } 0 < m < 1;$
3. $(F(s, t) = \left[\frac{s}{t+r}\right] \text{ for all } s, t \in [0, \infty) \text{ where } r \in (0, \infty);$ 
4. $(F(s, t) = (s + l)\frac{1}{1+s} - l \text{ for all } s, t \in [0, \infty) \text{ where } l > 1, r \in (0, \infty);$ 
5. $(F(s, t) = s\log_{t+a} a \text{ for all } s, t \in [0, \infty) \text{ where } a > 1;$ 
6. $(F(s, t) = s - \left(\frac{1}{2}\right) \left(\frac{t}{s}\right) \text{ for all } s, t \in [0, \infty);$ 
7. $(F(s, t) = s\beta(s) \text{ for all } s, t \in [0, \infty) \text{ where } \beta : [0, \infty) \to [0, 1) \text{ and is continuous;}$ 
8. $(F(s, t) = s - \varphi(s) \text{ for all } s, t \in [0, \infty) \text{ where } \varphi : [0, \infty) \to [0, \infty) \text{ is a continuous function such that}$ 
   $\varphi(t) = 0 \text{ if and only if } t = 0;$ 
9. $(F(s, t) = sh(s, t) \text{ for all } s, t \in [0, \infty) \text{ where } h : [0, \infty) \times [0, \infty) \to [0, \infty) \text{ is a continuous function such}$ 
   $\text{that } h(t, s) < 1 \text{ for all } s, t \in [0, \infty);$ 
10. $(F(s, t) = s - \left(\frac{2}{s+1}\right)t \text{ for all } s, t \in [0, \infty);$ 
11. $(F(s, t) = \sqrt{\ln(1 + s^n)} \text{ for all } s, t \in [0, \infty).$

Definition 2.7. (See [7]) A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

(a) $\psi$ is nondecreasing and continuous;
(b) $\psi(t) = 0 \text{ if and only if } t = 0.$

We denote $\Phi$ the family of all altering distance function.

Definition 2.8. (See [7]) A function $\varphi : [0, \infty) \to [0, \infty)$ is called an ultra altering distance function if the following properties are satisfied:

(a) $\varphi$ is continuous;
(b) $\varphi(t) > 0 \text{ for all } t > 0.$

We denote $\Psi$ the family of all altering distance function.

In 2012, Samet et al. [6] introduced $\alpha$-admissible mappings as the following:

Definition 2.9. (See. [1], [3]) A mapping $f : X \to X$ is called $\alpha$-admissible if for all $x, y \in X$ we have

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1,$$

where $\alpha : X \times X \to [0, \infty)$ is a given function.

Definition 2.10. [7] A mapping $f : X \to X$ is called a triangular $\alpha$-admissible if it is $\alpha$-admissible and satisfies

$$\left\{\begin{array}{c}
\alpha(x, y) \geq 1 \\
\alpha(y, z) \geq 1
\end{array}\right\} \Rightarrow \alpha(x, z) \geq 1,$$

where $x, y, z \in X$ and $\alpha : X \times X \to [0, \infty)$ is a given function.

Definition 2.11. [7] A mapping $f : X \to X$ is said to be weak triangular $\alpha$-admissible if it is $\alpha$-admissible and satisfies

$$\alpha(x, fx) \geq 1 \Rightarrow \alpha(x, f^2x) \geq 1,$$

where $\alpha : X \times X \to [0, \infty)$ is a given function.

Lemma 2.12. [7] Let $f : X \to X$ be a weak triangular $\alpha$-admissible mapping. Assume that there exists $x_0 \in x$ such that $\alpha(x_0, fx_0) \geq 1$. If $x_n = f^n x_0$, then $\alpha(x_m, fx_n) \geq 1$ for all $m, n \in N_0$ with $m < n.$

The following auxiliary result is going to be used in the proof of existence theorems.
Lemma 2.13. Let \( f : X \to X \) be a triangular \( \alpha \)-admissible mapping. Assume that there exists \( x_0 \in X \) such that \( \alpha(x_0, fx_0) \geq 1 \) and \( \alpha(fx_0, x_0) \geq 1 \). If \( x_n = f^n x_0 \), then \( \alpha(x_m, x_n) \geq 1 \) for all \( m, n \in \mathbb{N} \).

Definition 2.14. Let \((X, q)\) be a quasi metric space and let \( f : X \to X \) be a given mapping \( f \) is an \( F(\psi - \phi) - \alpha \)-Suzuki-type rational contractive if there exist two functions \( \alpha : X \times X \to [0, \infty) \) such that \( \alpha(x, y) \geq 1 \) and
\[
\frac{1}{2} \psi(q(x, fx)) \leq q(x, y)
\]
implies that
\[
\psi(q(fx, fy)) \leq F(\psi(M(x, y)), \varphi(M(x, y))),
\]
for all \( x, y \) in \( X \), where
\[
M(x, y) = \max \left\{ q(x, y), \frac{1+q(x, fx)q(y, fy)}{1+q(x, y)} \right\},
\]
\( \psi \in \Psi, \varphi \in \Phi \) and \( F \in \mathcal{C} \).

Now we prove our main result.

3. Main Results

Theorem 3.1. Let \((X, q)\) be a complete quasi metric space and \( f : X \to X \) be mappings such that \( f \) is \( F(\psi - \phi) - \alpha \)-Suzuki-type rational contractive suppose that

(i) \( f : X \to X \) is weak triangular \( \alpha \)-admissible mapping

(ii) there exists \( x_0 \in X \) such that \( \alpha(x_0, fx_0) \geq 1 \) and \( \alpha(fx_0, x_0) \geq 1 \)

(iii) \( f \) is continuous or If \( \{x_n\} \) is a sequence in \( X \) such that \( \alpha(x_n, x_{n+1}) \geq 1 \) and \( \alpha(x_{n+1}, x_n) \geq 1 \) for all \( n \) and as \( n \to \infty \), then there exists a subsequence \( \{x_{n(k)}\} \) of \( x_n \) such that \( \alpha(x_{n(k)}, x) \geq 1 \) and \( \alpha(x, x_{n(k)}) \geq 1 \) for all \( k \)

Then \( f \) has fixed point in \( X \).

Proof. By assumption (ii), there exists \( x_0 \in X \), such that \( \alpha(x_0, fx_0) \geq 1 \) and \( \alpha(fx_0, x_0) \geq 1 \). Define the sequence \( \{x_n\} \) in \( X \) as \( f x_n = x_{n+1} \), \( n = 1, 2, 3, \ldots \)
If \( x_{n_0} = x_{n_0+1} \) for some \( n_0 > 0 \), then \( x_{n_0} \) is a fixed point of \( f \) and the proof is done. Assume that \( x_n \neq x_{n+1} \) for all \( n \geq 0 \). Since \( f \) is \( \alpha \)-admissible,
\[
\alpha(x_0, fx_0) = \alpha(x_0, x_1) \geq 1 \Rightarrow \alpha(fx_0, f x_1) = \alpha(x_1, x_2) \geq 1
\]
and continuing we obtain
\[
\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}.
\]
Since
\[
\frac{1}{2} \psi(q(x, fx)) \leq q(x, x_{n+1}).
\]
From (3), we get
\[
\psi(q(fx, fy)) \leq F(\psi(M(x, x_{n+1})), \varphi(M(x, x_{n+1}))).
\]
\[
M(x_n, x_{n+1}) = \max \left\{ \frac{1+q(x_{n+1}, x_{n+2})q(x_{n+1}, x_{n+2})}{1+q(x_{n+1}, x_{n+2})} \right\} = \max \{ q(x_n, x_{n+1}), q(x_{n+1}, x_{n+2}) \}.\]
Hence,

\[ \psi(q(x_{n+1}, x_{n+2})) \leq F(\psi(\max\{q(x_n, x_{n+1}), q(x_{n+1}, x_{n+2})\}), \varphi(\max\{q(x_n, x_{n+1}), q(x_{n+1}, x_{n+2})\}) \].

If \( q(x_{n+1}, x_{n+2}) \) is maximum then we have

\[ \psi(q(x_{n+1}, x_{n+2})) \leq F(\psi(q(x_{n+1}, x_{n+2})), \varphi(q(x_{n+1}, x_{n+2}))) < \psi(q(x_{n+1}, x_{n+2})) \]

, which is a contradiction.

Hence \( q(x_n, x_{n+1}) \) is maximum. Thus

\[ \psi(q(x_{n+1}, x_{n+2})) \leq F(\psi(q(x_n, x_{n+1})), \varphi(q(x_n, x_{n+1})) \] (4)

Since \( \psi \) is increasing we have \( q(x_{n+1}, x_{n+2}) \leq q(x_n, x_{n+1}) \).

Thus \( \{q(x_n, x_{n+1})\} \) is a non-increasing sequence of non-negative real numbers and must converge to a real number, say, \( r \geq 0 \). Suppose \( r > 0 \).

Letting \( n \to \infty \) in (4), we get

\[ \psi(r) \leq F(\psi(r), \varphi(r)). \]

This implies that \( \psi(r) = 0 \) and \( \varphi(r) = 0 \) which yields

\[ \lim_{n \to \infty} q(x_n, x_{n+1}) = 0. \] (5)

Now we prove that \( \{x_n\} \) is a left-Cauchy sequence in \((X, q)\). On contrary suppose that \( \{x_n\} \) is not left-Cauchy.

Then there exist an \( \epsilon > 0 \) and monotone increasing sequences of natural numbers \( \{m_k\} \) and \( \{n_k\} \) such that \( n_k > m_k \),

\[ q(x_{m_k}, x_{n_k}) \geq \epsilon \] (6)

and

\[ q(x_{m_k}, x_{n_k-1}) < \epsilon. \] (7)

From (6) and (7), we obtain

\[ \epsilon \leq q(x_{m_k}, x_{n_k}) \]
\[ \leq q(x_{m_k}, x_{n_k-1}) + q(x_{n_k-1}, x_{n_k-1}) + q(x_{n_k-1}, x_{n_k}) \]
\[ < \epsilon + q(x_{n_k-1}, x_{n_k-1}) + q(x_{n_k-1}, x_{n_k}). \]

Letting \( k \to \infty \) and then using (6), we get

\[ \lim_{k \to \infty} q(x_{m_k}, x_{n_k}) = \epsilon. \] (8)

Letting \( k \to \infty \) and then using (5) and (8) in

\[ |q(x_{m_k-1}, x_{n_k}) - q(x_{m_k-1}, x_{m_k})| \leq q(x_{m_k}, x_{n_k}) \]

we obtain

\[ \lim_{k \to \infty} q(x_{m_k-1}, x_{n_k}) = \epsilon. \] (9)

Letting \( k \to \infty \) and then using (5) and (8) in

\[ |q(x_{m_k}, x_{n_k+1}) - q(x_{n_k}, x_{n_k+1})| \leq q(x_{m_k}, x_{n_k}) \]

we obtain

\[ \lim_{k \to \infty} q(x_{m_k}, x_{n_k+1}) = \epsilon. \] (10)

Hence, we get

Since \( f \) is weak triangular \( \alpha \)-admissible. Then, from Lemma 2.13 we have

\[ \alpha(x_{n_k}, x_{m_k}) \geq 1 \]
If \( \frac{1}{2}q(x_{m-1}, x_m) > q(x_{m-1}, u) \) then letting \( k \to \infty \), we get \( 0 \geq \epsilon \) from \([3]\) and \([6]\).

It is a contradiction. Hence

\( \frac{1}{2}q(x_{m-1}, x_m) \leq q(x_{m-1}, x_n) \).

From \([3]\), we have

\[
\psi(q(x_{m-1}, x_{m+1})) = \psi(q(fx_{m-1}, fx_n)) 
\leq F(\psi(M(x_{m-1}, x_n)), \varphi(M(x_{m-1}, x_n))) ,
\]

where

\[ M(x_{m-1}, x_n) = \max\left\{ q(x_{m-1}, x_n), \frac{1+q(x_{m-1}, x_m), q(x_n, x_{n+1})}{1+q(x_{m-1}, x_n)} \right\} . \]

Letting \( k \to \infty \) and then using \([10]\) and \([5]\) we have

\[
\psi(\epsilon) \leq F(\psi(\max\{\epsilon, 0\}), \varphi(\max\{\epsilon, 0\})) 
\leq F(\psi(\epsilon), \varphi(\epsilon)) .
\]

It follows that \( \psi(\epsilon) = 0 \) or \( \varphi(\epsilon) = 0 \). This implies that \( \epsilon = 0 \) which is a contradiction. Hence \( \{x_n\} \) is left - Cauchy in \((X, q)\). Similarly, \( \{x_n\} \) is right - Cauchy.

Thus \( \{x_n\} \) is a Cauchy sequence in \((X, q)\).

Hence,

\[
\lim_{n, m \to \infty} q(x_n, x_m) = 0 .
\] (11)

Since \( x_{n+1} = fx_n \), it follows \( \{x_n\} \) is a Cauchy sequence in the complete quasi - metric space \((X, q)\). Therefore, there exists \( u \in X \) such that

\[
\lim_{n \to \infty} q(x_n, u) = \lim_{n \to \infty} q(u, x_n) = 0 .
\] (12)

From continuity of \( f \) we get

\[
\lim_{n \to \infty} q(x_n, fu) = \lim_{n \to \infty} q(fx_{n-1}, fu) = 0 .
\] (13)

and

\[
\lim_{n \to \infty} q(fu, x_n) = \lim_{n \to \infty} q(fu, fx_{n-1}) = 0 .
\] (14)

Combining \([13]\) and \([14]\), we deduce

\[
\lim_{n \to \infty} q(x_n, fu) = \lim_{n \to \infty} q(fu, fx_n) = 0 .
\] (15)

From \([12]\) and \([13]\), due to the uniqueness of the limit, we conclude that \( u = fu \), that is , \( u \) is a fixed point of \( f \).

Now we claim that, for each \( n \geq 1 \), at least one of the following assertions holds.

\[
\frac{1}{2}q(x_{n-1}, x_n) \leq q(x_{n-1}, u) \quad \text{or} \quad \frac{1}{2}q(x_n, x_{n+1}) \leq q(x_n, u) .
\]

On the contrary suppose that

\[
\frac{1}{2}q(x_{n-1}, x_n) > q(x_{n-1}, u) \quad \text{and} \quad \frac{1}{2}q(x_n, x_{n+1}) > q(x_n, u)
\]

for some \( n \geq 1 \).

Then we have

\[
q(x_{n-1}, x_n) \leq q(x_{n-1}, u) + q(u, x_n) 
< \frac{1}{2}[q(x_{n-1}, x_n) + q(x_n, x_{n+1})]
\leq q(x_{n-1}, x_n) ,
\]
which is a contradiction and so the claim holds.

Suppose \( \frac{1}{2}q(x_n, x_{n+1}) \leq q(x_n, u) \).

Suppose \( fu \neq u \).

Since the sequence \( \{x_n\} \) converges to \( u \in X \), from (iii), there exists a subsequence \( \{x_n(k)\} \) of \( x_n \) such that \( \alpha(x_n(k), u) \geq 1 \) and \( \alpha(u, x_n(k)) \geq 1 \) for all \( k \).

We have

\[
\frac{1}{2}q(x_{n_k}, x_{n_k+1}) \leq \frac{1}{2}q(x_{n_k}, u)
\]

from (3), we have

\[
\psi (\psi (M(x_{n_k}, u)) \geq 1, \varphi (M(x_{n_k}, u)))
\]

where

\[
M(x_{n_k}, u) = \max \left\{ q(x_{n_k}, u), \frac{1+q(x_{n_k}, u)q(u, fu)}{1+q(u, u)} \right\}
\]

Letting \( n \to \infty \) and using 14 we get

\[
\psi (q(u, fu)) \leq F \left( \psi \left( \max \left\{ q(u, u), \frac{1+q(u, u)q(u, fu)}{1+q(u, u)} \right\} \right), \varphi \left( \max \left\{ q(u, u), \frac{1+q(u, u)q(u, fu)}{1+q(u, u)} \right\} \right)\right),
\]

which is a contradiction.

Thus, \( fu = u \).

Hence, \( u \) is a fixed point of \( f \).

\( (H) \) for all \( x, y \in \text{Fix}(f) \), we have \( \alpha(x, y) \geq 1 \), where \( \text{Fix}(f) \) denotes the set of fixed points of \( f \).

**Theorem 3.2.** Adding \( (H) \) to the hypotheses of Theorem [3.1], \( f \) has a unique fixed point.

**Proof.** Due to Theorem [3.1], we have \( u \) is a fixed point of \( f \). Let \( w \) be another fixed point of \( f \).

Suppose \( u \neq w \).

From (H), we have

\[
\alpha(u, w) \geq 1, \text{ for all } u, w \in \text{Fix}(f).
\]

Since \( \frac{1}{2}q(u, fu) \leq q(u, w) \), from (3), we obtain

\[
\psi (q(u, w)) = \psi (q(fu, fw)) \leq F (\psi (M(u, w)), \varphi (M(u, w))),
\]

where

\[
M(u, w) = \max \left\{ q(u, w), \frac{1+q(u, u)q(w, w)}{1+q(u, w)} \right\}
\]

Thus

\[
\psi (q(u, w)) \leq F (\psi (q(u, w)), \varphi (q(u, w)))
\]

It follows that \( \psi (q(u, w)) = 0 \) or \( \varphi (q(u, w)) = 0 \).

This implies that \( q(u, w) = 0 \) which is a contradiction.

Hence \( u = w \).

\( \square \)
Example 3.3. Let $X = [0, \infty)$ and $q$ be the quasi metric on $X$ given by

$$q(x, y) = \begin{cases} |x| & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

for all $x, y \in X$. It is obvious that $(X, q)$ be a complete quasi- metric space. Suppose that $f : X \to X$ is defined by

$$fx = \begin{cases} x^3 - 2x & \text{if } x > 2, \\ x & \text{if } x \in [0, 2]. \end{cases}$$

Now, define $\alpha : X \times X \to [0, \infty)$ as

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Let $F(s, t) = s - t$ for all $s, t \in [0, \infty)$. Let $\psi(t) = t$, $\varphi(t) = \frac{1}{2}t$.

Then

$$\frac{1}{2}q(x, fx) \leq x \leq q(x, y)$$

and

$$\psi(q(fx, fy)) = q(fx, fy) = fx = \frac{x}{2} = \frac{1}{2}q(x, y) \leq \frac{1}{2}M(x, y) = M(x, y) - \frac{1}{2}M(x, y) = F(\psi(M(x, y)), \varphi(M(x, y)))$$

Therefore, all of the conditions of Theorem 3.1 are satisfied and 0 is the fixed point of $f$.

If we let $\alpha(x, y) = 1$ for all $x \in X$, we get the following result.

Corollary 3.4. Let $(X, q)$ be a complete quasi metric space and let $f : X \to X$ be a given mapping $f$ is an $F(\psi - \varphi)$-Suzuki- type rational contraction condition. If there exist functions $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$ such that

$$\frac{1}{2}q(x, fx) \leq q(x, y)$$

implies that

$$\psi(q(fx, fy)) \leq F(\psi(M(x, y)), \varphi(M(x, y))),$$

where

$$M(x, y) = \max \left\{ q(x, y), \frac{1+q(x, fx)q(y, fy)}{1+q(x, y)} \right\},$$

for all $x, y$ in $X$. Then $f$ has a unique fixed point in $X$.

References


