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The Exact Solutions of Conformable Fractional Partial Differential Equations Using New Sub Equation Method

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Abstract

In this article, authors employed the new sub equation method to attain new traveling wave solutions of conformable time fractional partial differential equations. Conformable fractional derivative is a well behaved, applicable and understandable definition of arbitrary order derivation. Also this derivative obeys the basic properties that Newtonian concept satisfies. In this study authors obtained the exact solution for KDV equation where the fractional derivative is in conformable sense. New solutions are obtained in terms of the generalized version of the trigonometric functions.

1. Introduction

Fractional differential equations (FDEs) are generalized form of the integer order differential equations. In the last decades, researchers have worked hard for obtaining analytic solutions of nonlinear FDEs. Nonlinear FDEs are often used to describe many problems arising in many fields such as physics, chemistry, engineering, heat transfer, applied mathematics, control theory et all. [1]-[4]. So, many authors presented very strong methods to solve FDEs. For instance Kurt et. al. [5] studied the solutions of time fractional Whitham-Broer-Kaup Equation by using homotopy analysis method where the fractional terms are described in Caputo sense. Tasbozan et. al. [6] employed the finite element method for attaining the approximate solutions of diffusion equation where the derivatives are in Riemann-Liouville sense. Celik et. al. [7] utilized Crank-Nicolson scheme to get the the numerical solutions of fractional diffusion equation. As it is seen from the given references, all the obtained results are numerical solutions for the considered nonlinear equations. Because, the analytical methods can not be applied to the nonlinear equations which involves Caputo, Riemann-Liouville and Riesz fractional derivative definitions. On the contrary, conformable fractional detivative definition gives us chance to get the exact solutions of nonlinear FDEs by using new wave transformation [8] and the chain rule [9]. For example Eslami and Rezazadeh [10] used the first integral method to obtain analytic solutions of time fractional Wu-Zhang system. Aminikhah et. al. [11] obtained analytic solutions of fractional regularized long-wave equations using sub-equation method. Osman et al. [12] employed the unified method to get the analytic solutions of conformable time fractional Schrödinger equation with perturbation terms. For further details please see the references [13]-[34]. In this paper, we handle the Korteweg-de Vries equation with a source that provides a sixth order differential equation.

$$D_x^6 u + 20 D_x u D_x^4 u + 40 D_x^2 u D_x^3 u + 120 D_x u^2 D_x^2 u + D_x^3 D_t^{\mu} u + 8 D_x u D_x D_t^{\mu} u + 4 D_t^{\mu} u D_x^2 u = 0. \tag{1.1}$$

2. Conformable fractional calculus

R. Khalil et. al. [32] presented the definition of conformable fractional derivative as follows.

Definition 2.1. *μ^{th} order "conformable fractional derivative" of function g which is defined as $g : [0,\infty) \to \mathbb{R}$ can be dedicated as*

$$T_{\mu}(g)(t) = \lim_{\varepsilon \to 0} \frac{g(t+\varepsilon t^{1-\mu}) - g(t)}{\varepsilon}$$

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for all $t>0,\ \alpha\in(0,1)$. Assuming thatg is μ- differentiable over some $(0,a)$ where $a>0$ and $\lim_{t\to 0^+} g^{(\mu)}(t)$ exists, then $g^{(\mu)}(0)=\lim_{t\to 0^+} g^{(\mu)}(t)$.

The other fractional derivative definitions such as Caputo, Riemann-Liouville, Grünwald-Letnikov and etc. do not satisfy basic principles which are provided by Newtonian type derivative. For instance

1. Assume that λ is a constant and $\alpha\in R$. Then $D_a^{\mu}(\lambda)\neq 0$ for Riemann-Liouville derivative.
2. The Riemann-Liouville and Caputo derivatives do not provide the derivative of the product of two functions.
3. $D_a^{\mu}(fg)\neq fD_a^{\mu}(g)+gD_a^{\mu}(f)$.
4. The Riemann-Liouville and Caputo derivatives do not do not provide the derivative of the quotient of two functions
5. $D_a^{\mu}\left(\frac{f}{g}\right)\neq \frac{gD_a^{\mu}(f)-fD_a^{\mu}(g)}{g^2}$.

This new definition satisfies the properties which are given in the following theorem.

Theorem 2.2. *Let $\mu\in(0,1)$ and f,g be $\mu-$ differentiable at point $t>0$. Then*

1. $T_{\mu}(af+bg)=aT_{\mu}(f)+bT_{\mu}(g)$, *for all* $a,b\in\mathbb{R}$
2. $T_{\mu}(t^p)=pt^{p-\mu}$ *for all* $p\in\mathbb{R}$.
3. $T_{\mu}(\lambda)=0$ *for all constant function* $f(t)=\lambda$.
4. $T_{\mu}(fg)=fT_{\mu}(g)+gT_{\mu}(f)$.
5. $T_{\mu}\left(\frac{f}{g}\right)=\frac{gT_{\mu}(g)-fT_{\mu}(f).}{g^2}$.
6. *If f is differentiable, then* $T_{\mu}(f)(t)=t^{1-\mu}\frac{df}{dt}$.

3. The new sub-equation method

Consider that the general form of nonlinear fractional partial differential equation can be expressed as

$$H\left(u,\frac{\partial^{\mu}u}{\partial t^{\mu}},\frac{\partial u}{\partial x},u\frac{\partial u}{\partial x},u^2\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\dots\right)=0. \tag{3.1}$$

Using the wave transform $\xi=kx+w\frac{t^{\mu}}{\mu}$ where k and w are constants and chain rule [9] in Eq. (3.1), the independent variables and can be changed into single variable. So Eq. (3.1) can be rewritten as

$$P(u,u'(\xi),u''(\xi),...). \tag{3.2}$$

Consider that $u(\xi)$ can be written as a polynomial in $Q(\xi)$

$$u(\xi)=\sum_{j=0}^{n}a_jQ^j(\xi), \tag{3.3}$$

where $a_j\,(0\le j\le n)$ are constant coefficients to be determined after and $Q(\xi)$ provides first order linear ODE of the form

$$Q'(\xi)=Ln(A)\left(\alpha+\beta Q(\xi)+\sigma Q^2(\xi)\right),\quad A\neq 0,1, \tag{3.4}$$

where α,β,σ are constants. Moreover , Eq. has the following traveling wave solutions.
Family 1. If $\beta^2-4\alpha\sigma<0$ and $\sigma\neq 0$, then we have

$$\begin{aligned}
Q_1(\xi) &= -\frac{\beta}{2\sigma}+\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2\sigma}\tan_A\left(\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2}\xi\right),\\
Q_2(\xi) &= -\frac{\beta}{2\sigma}-\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2\sigma}\cot_A\left(\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2}\xi\right),\\
Q_3(\xi) &= -\frac{\beta}{2\sigma}+\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2\sigma}\left(\tan_A\left(\sqrt{-(\beta^2-4\alpha\sigma)}\xi\right)\pm\sqrt{pq}\sec_A\left(\sqrt{-(\beta^2-4\alpha\sigma)}\xi\right)\right),\\
Q_4(\xi) &= -\frac{\beta}{2\sigma}+\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2\sigma}\left(-\cot_A\left(\sqrt{-(\beta^2-4\alpha\sigma)}\xi\right)\pm\sqrt{pq}\csc_A\left(\sqrt{-(\beta^2-4\alpha\sigma)}\xi\right)\right),\\
Q_5(\xi) &= -\frac{\beta}{2\sigma}+\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{4\sigma}\left(\tan_A\left(\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{4}\xi\right)-\cot_A\left(\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{4}\xi\right)\right).
\end{aligned}$$

Family 2. Suppose that $\beta^2-4\alpha\sigma>0$ and $\sigma\neq 0$,

$$\begin{aligned}
Q_6(\xi) &= -\frac{\beta}{2\sigma}-\frac{\sqrt{\beta^2-4\alpha\sigma}}{2\sigma}\tanh_A\left(\frac{\sqrt{\beta^2-4\alpha\sigma}}{2}\xi\right),\\
Q_7(\xi) &= -\frac{\beta}{2\sigma}-\frac{\sqrt{\beta^2-4\alpha\sigma}}{2\sigma}\coth_A\left(\frac{\sqrt{\beta^2-4\alpha\sigma}}{2}\xi\right),\\
Q_8(\xi) &= -\frac{\beta}{2\sigma}+\frac{\sqrt{\beta^2-4\alpha\sigma}}{2\sigma}\left(-\tanh_A\left(\sqrt{\beta^2-4\alpha\sigma\xi}\right)\pm i\sqrt{pq}\operatorname{sech}_A\left(\sqrt{\beta^2-4\alpha\sigma\xi}\right)\right),\\
Q_9(\xi) &= -\frac{\beta}{2\sigma}+\frac{\sqrt{\beta^2-4\alpha\sigma}}{2\sigma}\left(-\coth_A\left(\sqrt{\beta^2-4\alpha\sigma\xi}\right)\pm \sqrt{pq}\operatorname{csch}_A\left(\sqrt{\beta^2-4\alpha\sigma\xi}\right)\right),\\
Q_{10}(\xi) &= -\frac{\beta}{2\sigma}-\frac{\sqrt{\beta^2-4\alpha\sigma}}{4\sigma}\left(\tanh_A\left(\frac{\sqrt{\beta^2-4\alpha\sigma}}{4}\xi\right)+\coth_A\left(\frac{\sqrt{\beta^2-4\alpha\sigma}}{4}\xi\right)\right).
\end{aligned}$$

Family 3. Consider that $\alpha\sigma>0$ and $\beta=0$,

$$\begin{aligned}
Q_{11}(\xi) &= \sqrt{\frac{\alpha}{\sigma}}\tan_A\left(\sqrt{\alpha\sigma}\xi\right),\\
Q_{12}(\xi) &= -\sqrt{\frac{\alpha}{\sigma}}\cot_A\left(\sqrt{\alpha\sigma}\xi\right),\\
Q_{13}(\xi) &= \sqrt{\frac{\alpha}{\sigma}}\left(\tan_A\left(2\sqrt{\alpha\sigma}\xi\right)\pm\sqrt{pq}\sec_A\left(2\sqrt{\alpha\sigma}\xi\right)\right),\\
Q_{14}(\xi) &= \sqrt{\frac{\alpha}{\sigma}}\left(-\cot_A\left(2\sqrt{\alpha\sigma}\xi\right)\pm\sqrt{pq}\csc_A\left(2\sqrt{\alpha\sigma}\xi\right)\right),\\
Q_{15}(\xi) &= \frac{1}{2}\sqrt{\frac{\alpha}{\sigma}}\left(\tan_A\left(\frac{\sqrt{\alpha\sigma}}{2}\xi\right)-\cot_A\left(\frac{\sqrt{\alpha\sigma}}{2}\xi\right)\right).
\end{aligned}$$

Family 4. Regard that $\alpha\sigma<0$ and $\beta=0$,

$$\begin{aligned}
Q_{16}(\xi) &= -\sqrt{-\frac{\alpha}{\sigma}}\tanh_A\left(\sqrt{-\alpha\sigma}\xi\right),\\
Q_{17}(\xi) &= -\sqrt{-\frac{\alpha}{\sigma}}\coth_A\left(\sqrt{-\alpha\sigma}\xi\right),\\
Q_{18}(\xi) &= \sqrt{-\frac{\alpha}{\sigma}}\left(-\tanh_A\left(2\sqrt{-\alpha\sigma}\xi\right)\pm i\sqrt{pq}\operatorname{sech}_A\left(2\sqrt{-\alpha\sigma}\xi\right)\right),\\
Q_{19}(\xi) &= \sqrt{-\frac{\alpha}{\sigma}}\left(-\coth_A\left(2\sqrt{-\alpha\sigma}\xi\right)\pm \sqrt{pq}\operatorname{csch}_A\left(2\sqrt{-\alpha\sigma}\xi\right)\right),\\
Q_{20}(\xi) &= -\frac{1}{2}\sqrt{-\frac{\alpha}{\sigma}}\left(\tanh_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\xi\right)+\coth_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\xi\right)\right).
\end{aligned}$$

Family 5. When $\beta=0$ and $\sigma=\alpha$,

$$\begin{aligned}
Q_{21}(\xi) &= \tan_A(\alpha\xi),\\
Q_{22}(\xi) &= -\cot_A(\alpha\xi),\\
Q_{23}(\xi) &= \tan_A(2\alpha\xi)\pm\sqrt{pq}\sec_A(2\alpha\xi),\\
Q_{24}(\xi) &= -\cot_A(2\alpha\xi)\pm\sqrt{pq}\csc_A(2\alpha\xi),\\
Q_{25}(\xi) &= \frac{1}{2}\left(\tan_A\left(\frac{\alpha}{2}\xi\right)-\cot_A\left(\frac{\alpha}{2}\xi\right)\right).
\end{aligned}$$

Family 6. If $\beta=0$ and $\sigma=-\alpha$, chosen

$$\begin{aligned}
Q_{26}(\xi) &= -\tanh_A(\alpha\xi),\\
Q_{27}(\xi) &= -\coth_A(\alpha\xi),\\
Q_{28}(\xi) &= -\tanh_A(2\alpha\xi)\pm i\sqrt{pq}\operatorname{sech}_A(2\alpha\xi),\\
Q_{29}(\xi) &= -\coth_A(2\alpha\xi)\pm \sqrt{pq}\operatorname{csch}_A(2\alpha\xi),\\
Q_{30}(\xi) &= -\frac{1}{2}\left(\tanh_A\left(\frac{\alpha}{2}\xi\right)+\coth_A\left(\frac{\alpha}{2}\xi\right)\right).
\end{aligned}$$

Family 7. While $\beta^2=4\alpha\sigma$,

$$Q_{31}(\xi) = \frac{-2\alpha(\beta\xi Ln(A)+2)}{\beta^2\xi Ln(A)}.$$

Family 8. When $beta = k$, $\alpha = mk(m \neq 0)$ and $\sigma = 0$,

$$Q_{32}(\xi) = A^{k\xi} - m.$$

Family 9. When $\beta = \sigma = 0$,

$$Q_{33}(\xi) = \alpha\xi Ln(A).$$

Family 10. When $\beta = \alpha = 0$,

$$Q_{34}(\xi) = \frac{-1}{\sigma\xi Ln(A)}.$$

Family 11. When $\alpha = 0$ and $\beta \neq 0$,

$$\begin{aligned}
Q_{35}(\xi) &= -\frac{p\beta}{\sigma(\cosh_A(\beta\xi) - \sinh_A(\beta\xi) + p)}, \\
Q_{36}(\xi) &= -\frac{q\beta}{\sigma(\cosh_A(\beta\xi) - \sinh_A(\beta\xi) + q)}, \\
Q_{37}(\xi) &= -\frac{\beta(\sinh_A(\beta\xi) + \cosh_A(\beta\xi))}{\sigma(\sinh_A(\beta\xi) + \cosh_A(\beta\xi) + q)},
\end{aligned}$$

Family 12. When $\beta = k$, $\sigma = mk(m \neq 0), p = q$ and $\alpha = 0$,

$$Q_{38}(\xi) = \frac{pA^{k\xi}}{p - mqA^{k\xi}}.$$

Remark 3.1. *The generalized version of the trigonometric functions and the generalized types of the hypergeometric functions are declared as [33]*

$$\begin{aligned}
\sinh_A(\xi) &= \frac{pA^{\xi} - qA^{-\xi}}{2}, & \cosh_A(\xi) &= \frac{pA^{\xi} + qA^{-\xi}}{2}, \\
\tanh_A(\xi) &= \frac{pA^{\xi} - qA^{-\xi}}{pA^{\xi} + qA^{-\xi}}, & \coth_A(\xi) &= \frac{pA^{\xi} + qA^{-\xi}}{pA^{\xi} - qA^{-\xi}}, \\
\text{sech}_A(\xi) &= \frac{2}{pA^{\xi} + qA^{-\xi}}, & \text{csch}_A(\xi) &= \frac{2}{pA^{\xi} - qA^{-\xi}}, \\
\sin_A(\xi) &= \frac{pA^{i\xi} - qA^{-i\xi}}{2i}, & \cos_A(\xi) &= \frac{pA^{i\xi} + qA^{-i\xi}}{2}, \\
\tan_A(\xi) &= -i\frac{pA^{i\xi} - qA^{-i\xi}}{pA^{i\xi} + qA^{-i\xi}}, & \cot_A(\xi) &= i\frac{pA^{i\xi} + qA^{-i\xi}}{pA^{i\xi} - qA^{-i\xi}}, \\
\sec_A(\xi) &= \frac{2}{pA^{i\xi} + qA^{-i\xi}}, & \csc_A(\xi) &= \frac{2i}{pA^{i\xi} - qA^{-i\xi}},
\end{aligned}$$

where $p,q > 0$ are constants and ξ is an independent variable. In addition, by considering the balance between the highest order derivative linear term and nonlinear terms appearing in ODE (3.2), the positive integer n can be defined. Replacing Eq. (3.3) into ODE (3.2), using Eq. (3.4), and equalizing the coefficients of all the powers of $Q(\xi)$ to zero, we will obtain an equation system in terms of k,w and a_j $(0 \leq j \leq n)$. From this obtained system the values for k,w and a_j can be found with the aid of a computer software. Replacing the obtained values of k,w and a_j into Eq.(3.3), we may acquire all possible solutions of Eq. (3.1).

4. Analytic results for time fractional KdV6 equation with conformable derivative

Using the wave transformation and applying chain rule [9]

$$u(x,t) = u(\xi), \qquad \xi = kx + w\frac{t^{\mu}}{\mu}. \tag{4.1}$$

Eq. (1.1) is transferred to

$$k^6u^{(vi)}(\xi) + k^3wu^{iv}(\xi) + 6k^2w(u'(\xi))^2 + 20k^5u^{iv}(\xi)u'(\xi) + 40k^5u''(\xi)u'''(\xi) + 12k^2wu'(\xi)u''(\xi) = 0$$

where the prime symbolizes the known derivative of function $u(\xi)$ with respect to ξ. Integrating the above equation once and making some algebraic calculations led to

$$k^6u^{(v)}(\xi)+k^3wu'''(\xi)+3k^2w(u')^2+5k^5u'''u+20k^5(u'')^2+12k^2wuu'=0. \tag{4.2}$$

Considering the homogeneous balance between u^2u' and $u^{(5)}$ in Eq. (4.2) we obtain $n+5=3(n+1)$; then $n=1$;so we can write Eq. (3.3) as

$$u(\xi)=a_0+a_1Q(\xi). \tag{4.3}$$

Subrogating Eq. (4.3) with (3.4) into Eq. (4.2) and gathering all the same power of $Q(\xi)$ together, the left hand side of Eq. (4.2) turns into a polynomial of $Q(\xi)$. Equalizing the each coefficient of the same power of $Q(\xi)$ to zero led to an equation system. Solving the obtained system due to unknowns variables $a_0,\ a_1$ and w, the solutions can be concluded as

$$w=\frac{{a_1}^3(\beta^2-4\alpha\sigma)}{\sigma^3Ln(A)},\qquad k=-\frac{a_1}{\sigma Ln(A)}. \tag{4.4}$$

Putting the solution set (4.4) with (4.1) into (4.3) and solutions of Eq. (1.1), can be expressed as

Case 1. If $\beta^2-4\alpha\sigma<0$ and$\sigma\neq0$, then we have

$$\begin{aligned}
u_1(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}+\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2\sigma}\tan_A\left(\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2}\xi\right)\right),\\
u_2(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}-\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2\sigma}\cot_A\left(\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2}\xi\right)\right),\\
u_3(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}+\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2\sigma}\left(\tan_A\left(\sqrt{-(\beta^2-4\alpha\sigma)}\xi\right)\pm\sqrt{pq}\sec_A\left(\sqrt{-(\beta^2-4\alpha\sigma)}\xi\right)\right)\right),\\
u_4(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}+\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2\sigma}\left(-\cot_A\left(\sqrt{-(\beta^2-4\alpha\sigma)}\xi\right)\pm\sqrt{pq}\csc_A\left(\sqrt{-(\beta^2-4\alpha\sigma)}\xi\right)\right)\right),\\
u_5(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}+\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{4\sigma}\left(\tan_A\left(\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{4}\xi\right)-\cot_A\left(\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{4}\xi\right)\right)\right)
\end{aligned}$$

where $\xi=-\frac{a_1}{\sigma Ln(A)}x+\frac{{a_1}^3(\beta^2-4\alpha\sigma)^2}{\sigma^3\mu Ln(A)}t^{\mu}$.

Case 2. Suppose that $\beta^2-4\alpha\sigma>0$ and $\sigma\neq0$,

$$\begin{aligned}
u_6(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}-\frac{\sqrt{\beta^2-4\alpha\sigma}}{2\sigma}\tanh_A\left(\frac{\sqrt{\beta^2-4\alpha\sigma}}{2}\xi\right)\right),\\
u_7(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}-\frac{\sqrt{\beta^2-4\alpha\sigma}}{2\sigma}\coth_A\left(\frac{\sqrt{\beta^2-4\alpha\sigma}}{2}\xi\right)\right),\\
u_8(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}+\frac{\sqrt{\beta^2-4\alpha\sigma}}{2\sigma}\left(-\tanh_A\left(\sqrt{\beta^2-4\alpha\sigma\xi}\right)\pm i\sqrt{pq}\operatorname{sech}_A\left(\sqrt{\beta^2-4\alpha\sigma\xi}\right)\right)\right),\\
u_9(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}+\frac{\sqrt{\beta^2-4\alpha\sigma}}{2\sigma}\left(-\coth_A\left(\sqrt{\beta^2-4\alpha\sigma\xi}\right)\pm\sqrt{pq}\operatorname{csch}_A\left(\sqrt{\beta^2-4\alpha\sigma\xi}\right)\right)\right),\\
u_{10}(\xi) &= a_0+a_1\left(-\frac{\beta}{2\sigma}-\frac{\sqrt{\beta^2-4\alpha\sigma}}{4\sigma}\left(\tanh_A\left(\frac{\sqrt{\beta^2-4\alpha\sigma}}{4}\xi\right)+\coth_A\left(\frac{\sqrt{\beta^2-4\alpha\sigma}}{4}\xi\right)\right)\right)
\end{aligned}$$

where $\xi=-\frac{a_1}{\sigma Ln(A)}x+\frac{{a_1}^3(\beta^2-4\alpha\sigma)^2}{\sigma^3\mu Ln(A)}t^{\mu}$.

Case 3. Consider that $\alpha\sigma>0$ and $\beta=0$,

$$\begin{aligned}
u_{11}(\xi) &= a_0+a_1\left(\sqrt{\frac{\alpha}{\sigma}}\tan_A\left(\sqrt{\alpha\sigma}\xi\right)\right),\\
u_{12}(\xi) &= a_0-a_1\left(\sqrt{\frac{\alpha}{\sigma}}\cot_A\left(\sqrt{\alpha\sigma}\xi\right)\right),\\
u_{13}(\xi) &= a_0+a_1\left(\sqrt{\frac{\alpha}{\sigma}}\left(\tan_A\left(2\sqrt{\alpha\sigma}\xi\right)\pm\sqrt{pq}\sec_A\left(2\sqrt{\alpha\sigma}\xi\right)\right)\right),\\
u_{14}(\xi) &= a_0+a_1\left(\sqrt{\frac{\alpha}{\sigma}}\left(-\cot_A\left(2\sqrt{\alpha\sigma}\xi\right)\pm\sqrt{pq}\csc_A\left(2\sqrt{\alpha\sigma}\xi\right)\right)\right),\\
u_{15}(\xi) &= a_0+a_1\left(\frac{1}{2}\sqrt{\frac{\alpha}{\sigma}}\left(\tan_A\left(\frac{\sqrt{\alpha\sigma}}{2}\xi\right)-\cot_A\left(\frac{\sqrt{\alpha\sigma}}{2}\xi\right)\right)\right)
\end{aligned}$$

where $\xi=-\frac{a_1}{\sigma Ln(A)}x+\frac{a_1{}^3(\beta^2-4\alpha\sigma)^2}{\sigma^3\mu Ln(A)}t^{\mu}$.
Case 4. Regard that $\alpha\sigma<0$ and $\beta=0$,

$$\begin{aligned}
u_{16}(\xi) &= a_0-a_1\left(\sqrt{-\frac{\alpha}{\sigma}}\tanh_A\left(\sqrt{-\alpha\sigma}\xi\right)\right),\\
u_{17}(\xi) &= a_0-a_1\left(\sqrt{-\frac{\alpha}{\sigma}}\coth_A\left(\sqrt{-\alpha\sigma}\xi\right)\right),\\
u_{18}(\xi) &= a_0+a_1\left(\sqrt{-\frac{\alpha}{\sigma}}\left(-\tanh_A\left(2\sqrt{-\alpha\sigma}\xi\right)\pm i\sqrt{pq}\operatorname{sech}_A\left(2\sqrt{-\alpha\sigma}\xi\right)\right)\right),\\
u_{19}(\xi) &= a_0+a_1\left(\sqrt{-\frac{\alpha}{\sigma}}\left(-\coth_A\left(2\sqrt{-\alpha\sigma}\xi\right)\pm\sqrt{pq}\operatorname{csch}_A\left(2\sqrt{-\alpha\sigma}\xi\right)\right)\right),\\
u_{20}(\xi) &= a_0-a_1\left(\frac{1}{2}\sqrt{-\frac{\alpha}{\sigma}}\left(\tanh_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\xi\right)+\coth_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\xi\right)\right)\right)
\end{aligned}$$

where $\xi=-\frac{a_1}{\sigma Ln(A)}x+\frac{a_1{}^3(\beta^2-4\alpha\sigma)^2}{\sigma^3\mu Ln(A)}t^{\mu}$.
Case 5. When $\beta=0$ and $\sigma=\alpha$,

$$u_{21}(\xi)=a_0+a_1\tan_A(\alpha\xi),$$

$$\begin{aligned}
u_{22}(\xi) &= a_0-a_1\cot_A(\alpha\xi),\\
u_{23}(\xi) &= a_0+a_1\left(\tan_A(2\alpha\xi)\pm\sqrt{pq}\sec_A(2\alpha\xi)\right),\\
u_{24}(\xi) &= a_0+a_1\left(-\cot_A(2\alpha\xi)\pm\sqrt{pq}\csc_A(2\alpha\xi)\right),\\
u_{25}(\xi) &= a_0+a_1\left(\frac{1}{2}\left(\tan_A\left(\frac{\alpha}{2}\xi\right)-\cot_A\left(\frac{\alpha}{2}\xi\right)\right)\right)
\end{aligned}$$

where $\xi=-\frac{a_1}{\sigma Ln(A)}x+\frac{a_1{}^3(\beta^2-4\alpha\sigma)^2}{\sigma^3\mu Ln(A)}t^{\mu}$.
Case 6. If $\beta=0$ and $\sigma=-\alpha$, chosen

$$\begin{aligned}
u_{26}(\xi) &= a_0-a_1\tanh_A(\alpha\xi),\\
u_{27}(\xi) &= a_0-a_1\coth_A(\alpha\xi),\\
u_{28}(\xi) &= a_0+a_1\left(-\tanh_A(2\alpha\xi)\pm i\sqrt{pq}\operatorname{sech}_A(2\alpha\xi)\right),\\
u_{29}(\xi) &= a_0+a_1\left(-\coth_A(2\alpha\xi)\pm\sqrt{pq}\operatorname{csch}_A(2\alpha\xi)\right),\\
u_{30}(\xi) &= a_0-\frac{a_1}{2}\left(\tanh_A\left(\frac{\alpha}{2}\xi\right)+\coth_A\left(\frac{\alpha}{2}\xi\right)\right)
\end{aligned}$$

where $\xi=-\frac{a_1}{\sigma Ln(A)}x+\frac{a_1{}^3(\beta^2-4\alpha\sigma)^2}{\sigma^3\mu Ln(A)}t^{\mu}$.
Case 11. When $\alpha=0$ and $\beta\neq 0$,

$$\begin{aligned}
u_{31}(\xi) &= a_0-\frac{pa_1\beta}{\sigma\left(\cosh_A(\beta\xi)-\sinh_A(\beta\xi)+p\right)},\\
u_{32}(\xi) &= a_0-\frac{qa_1\beta}{\sigma\left(\cosh_A(\beta\xi)-\sinh_A(\beta\xi)+q\right)},\\
u_{33}(\xi) &= a_0-\frac{a_1\beta\left(\sinh_A(\beta\xi)+\cosh_A(\beta\xi)\right)}{\sigma\left(\sinh_A(\beta\xi)+\cosh_A(\beta\xi)+q\right)},
\end{aligned}$$

where $\xi=-\frac{a_1}{\sigma Ln(A)}x+\frac{a_1{}^3(\beta^2-4\alpha\sigma)^2}{\sigma^3\mu Ln(A)}t^{\mu}$.

Case 12. When $\beta=k$, $\sigma=mk(m\neq 0), p=q$ and$\alpha=0$,

$$u_{34}(\xi)=a_0+\frac{pa_1A^{k\xi}}{p-mqA^{k\xi}}.$$

5. Conclusion

In this manuscript the new sub-equation method successfully applied to time fractional KdV6 equation. Analytic solutions of the nonlinear KdV6 equation are successfully obtained. Also wave transform and chain rule are used, so the nonlinear conformable FDE changes into differential equation with integer order derivative. As it can be from the obtained results new sub-equation method is a reliable, efficient and applicable tool for obtaining the exact solutions of fractional partial differential equations in conformable sense.

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