Fourier-type integral transforms in modeling of transversal oscillation

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Abstract. The model of transversal oscillation for an elastic piecewise-homogeneous rod is constructed. In order to find a solution of this model a Fourier-type integral transforms method for the fourth-order differential equations is developed. The decomposition theorem is proved by Cauchy contour integration method. The conditions of existence for fundamental solutions of the initial - boundary value problem are established and explicit expressions of these fundamental solutions are found.

Keywords: eigenfunction, fourth-order differential equation, fundamental solution, Green's function.

1. Introduction

An eigenfunction of a linear operator A, defined on some function space, is any non-zero function f in that space that returns from the operator exactly as it is, except for a multiplicative scaling factor. Eigenfunctions play an important role in many branches of physics. Using eigenfunctions the laws of mathematical physics can be described, for example, the law of the transversal oscillations of semi-limited piecewise-homogeneous rod.

Laws of the transverse oscillations of rod are found theoretically by Euler. These laws on the experience are checked by Chladni, Lissajous, Mercadier etc. Transversal oscillations modeling of semi-limited piecewise-homogeneous rod

\[ I^*_a = \{x: x \in U_{i+1}^{i} (l_i, l_j), l_0 > 0, l_{n+1} = \infty \}, \]

leads to solve the system of fourth-order differential equations

\[ \left( \frac{\partial^2}{\partial t^2} + A^1 \frac{\partial^4}{\partial t^4} \right) y_j(x,t) = 0, \quad 0 < t < l_j, \quad j = 1, \ldots, n+1. \] (1)

Here function \( y_j(x,t) \) is the ordinate of the deformed axis of \( j \)-layer at the point \( x \) in time \( t \):

\[ y_j(x,t), \quad x \in (l_{j-1}, l_j), \quad j = 1, \ldots, n+1, \quad t > 0 \]

\( E_j \) - an elastic modulus (or Young's modulus) of \( j \)-layer rod;
\( I_j \) - the moment of inertia of \( j \)-layer rod, a line perpendicular to the plane of the rod and passing through the gravity center of areas \( S_j \);
\( \rho_j \) - the density of \( j \)-layer rod;

\[ A^1_j = \frac{E_j I_j}{\rho_j S_j}. \]

The boundary conditions have the form:

\[ y = f(t), \quad \frac{\partial y}{\partial t} = 0 \]

(2)

when

\[ x = 0, t > 0. \]

Further formulate the conditions at the points of coupling intervals.

The conditions for continuity of the ordinates and for the tangent of the bending moment and clipping efforts must be true:

\[ y_j(l_j, t) = y_{j+1}(l_j, t), \quad y'_j(l_j, t) = y'_{j+1}(l_j, t), \]

\[ E_j I_j y''_{j+1}(l_j, t) = E_{j+1} I_{j+1} y''_{j+1}(l_j, t). \] (3)

Let's assume, the rod was in an equilibrium state before the hanging point was set into motion. The initial conditions have the form

\[ y = 0, \quad \frac{\partial y}{\partial t} = 0 \]

when

\[ t = 0, x \geq 0. \]

2. Problem Statement

We construct the integral transform by a method of delta-shaped sequences. Integrated transform is generated on the set \( I_n^* \) by the fourth - order differential operator \( \hat{B} \):

\[ B = \sum_{i=1}^{n+1} A_i \delta(x-l_i) \theta(x-l_i) \frac{d^4}{dx^4} + A^* \delta(x-l_{n+1}) \frac{d^4}{dx^4}, \]

here \( A_k \) - matrix of size \( \sigma \times \sigma \), all eigenvalues are the real positive number, \( \theta(x) \) - the Heaviside step function.

Let's consider solving problem of separate matrix system for \((n+1)\) iterated parabolic equations

\[ \left( A^*_m + \frac{d^4}{dx^4} \right) y_m(x, t) = 0, \quad t > 0, \quad x \in I_m, \quad m = 1, n+1, \]

limited on the set \( D \).

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on the initial conditions \[ v_m(x, t) = g_m(x), x \in I_m^*, \]
\[ \frac{\partial v_m(x, t)}{\partial t} = 0, x \in I_m^*, m = 1, n. \] (5)

on the boundary conditions
\[ \sum_{i=0}^{3} a_{m,i} \frac{\partial^i}{\partial x^i} v_i |_{x=\xi_i} = 0, \quad m = 1, n \] (6)

and the coupling conditions
\[ \sum_{i=0}^{3} a_{m,i} \frac{\partial^i}{\partial x^i} v_i = \sum_{i=0}^{3} a_{n,i} \frac{\partial^i}{\partial x^i} v_i, \quad x = \xi_j, \quad k = 1, n; m = 1, 4. \] (7)

Here \( v_m = v_m(x, t) \) is the unknown vector-valued function of size \( \sigma \times 1 \), \( g_m(x) = \{ \text{set of vector-valued functions of size } \sigma \times 1 \}, \)
\( a_{m,i}^{(j)} ; j = 1, 4, \quad e = 1, 2, \quad i = 0, 3 \) - matrices of size \( \sigma \times \sigma \).

The special solution \( H_{j, s} (t, x, \xi) \), meeting conditions
\[ H_{j, s}(t, x, \xi) |_{t=\delta} = \delta(x - \xi), \quad x, \xi \in I_s^*, \quad j = 1, n + 1 \]
\[ \sum_{i=0}^{3} a_{m,i} \frac{\partial^i}{\partial x^i} H_{j,s} |_{x=\xi} = 0, \quad i = 1, 2 \]
\[ \frac{\partial^i}{\partial x^i} H_{j,s} |_{x=\xi} = 0, \quad i = 0, 3 \]
\[ \left[ a_{m,s} \frac{d}{dx} + \beta_{m,s} \right] I_{j,s} = \left[ a_{n,s} \frac{d}{dx} + \beta_{n,s} \right] I_{j,s}, \quad x = \xi, \quad k = 1, n, \quad m = 1, 2. \]

is called as the matrix fundamental solution or the Green's function.

We can write the solution of the General boundary value problem \((4)-(7)\), if we know influence function \( H_{j, s} = H_{j, s}(t, x, \xi) \). Our next goal is to clarify the conditions of existence of the influence functions \( H_{j, s} \) and finding explicit expressions for these functions.

We introduce notations:
\[ M_{m,s} = \begin{pmatrix} a_{m,0} & \cdots & a_{m,3} \\ \vdots & \ddots & \vdots \\ a_{m,0} & \cdots & a_{m,3} \end{pmatrix}, \quad k = 1, n; m = 1, 2. \]
and demand fulfillment of the following condition:
\[ \det M_{m,s} = C_{m,s} = 0, k = 1, n; m = 1, 2. \] (8)

Let's assume that the desired vector-valued function \( U_j(t, x) \) is Laplace's original on \( t \). In the images of the Laplace we get a problem about a construction of limited on the set \( I_{m,s} \) solutions of the separate matrix system of ordinary differential equations
\[ \left( \frac{d^i}{dx^i} - q_i \right) U_j(t, x) = p g_j(x), \quad j = 1, n + 1 \]
\[ q_i = -A_i p^2, \quad g_i(x) = A_i^2 f_i(x) \]

on the boundary conditions
\[ \sum_{i=0}^{3} a_{m,i} \frac{d^i}{dx^i} v_i |_{x=\xi} = 0, \quad e = 1, 2 \]

and the contact conditions in the points of joint
\[ \sum_{i=0}^{3} a_{m,i} \frac{d^i}{dx^i} v_i = \sum_{i=0}^{3} a_{n,i} \frac{d^i}{dx^i} v_i, \quad x = \xi, \quad k = 1, n. \]

We define the images \( H_{j, s}^* \) of matrix fundamental solution as solutions of the following boundary value problem for the separate matrix system of ordinary differential equations \((9)\):
\[ \left( \frac{d^i}{dx^i} - q_i \right) H_{j, s}^*(t, x, \xi) = A_j^s \delta(x - \xi), \quad j = 1, n + 1, \]
\[ q_i = -A_i^s p^2, \quad \xi \in I_s. \]

For images of matrix influence functions \( H_{j, s}^* \) formulas are fair:
if \( k < s \)
\[ H_{j,s}^* = \begin{pmatrix} \theta_i(x) & \theta_i(x) & \theta_i(x) & \theta_i(x) \end{pmatrix} A_j^s (\xi) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]
\[ k = 1, n + 1, j = 1, n + 1. \]

if \( k > s \)
\[ H_{j,s}^* = \begin{pmatrix} \theta_i(x) & \theta_i(x) & \theta_i(x) & \theta_i(x) \end{pmatrix} A_j^s (\xi) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]
\[ k = 1, n + 1, j = 1, n + 1. \]

Matrix functions are present in the expressions for images of matrix influence functions \( H_{j,s}^* \)
\[ \varphi(x, \lambda) = \sum_{i=0}^{3} \theta_i(x - \xi_i) \varphi_i(x, \lambda) + \theta_i(x - \xi_i) \varphi_{j, i}(x, \lambda). \]
\[ \psi(x, \lambda) = \sum_{i=0}^{3} \theta_i(x - \xi_i) \psi_i(x, \lambda) + \theta_i(x - \xi_i) \psi_{j, i}(x, \lambda) \]

For \( \varphi_{j, i} = (\varphi_{j, i}^{(0)}, \varphi_{j, i}^{(1)}); \psi_{j, i} = (\psi_{j, i}^{(0)}, \psi_{j, i}^{(1)}) \) we have values
where the j radical branch is designated by a symbol $\sqrt[j]{\cdot}$. Other pairs of functions $\phi_{\omega}, \psi_{\omega}$ uniquely defined by coupling conditions:

$$\sum_{\omega} \frac{d}{dt} (\phi_{\omega}, v_{\omega}) - \sum_{\omega} \frac{d}{dt} (\phi_{\omega}, \psi_{\omega}) = 0, \quad s = 1 - \frac{1}{n}, \quad \omega = 1, \ldots, n - 1$$

Furthermore, $\phi_{1}, \psi_{1}$ and $\Omega_k$ are defined by relations:

$$\Omega_k = \begin{pmatrix}
\phi_{1, 1} & \phi_{1, 2} & \cdots & \phi_{1, n} \\
\phi_{2, 1} & \phi_{2, 2} & \cdots & \phi_{2, n} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{n, 1} & \phi_{n, 2} & \cdots & \phi_{n, n}
\end{pmatrix}$$

In the future, the condition will be considered fulfilled:

$$\det \left( \begin{array}{cc}
E_{\sigma_2} & 0_{\sigma_2} \\
0_{\sigma_2} & 0_{\sigma_2}
\end{array} \right) \neq 0, \quad \sigma_2 \geq 2 \sigma$$

(12)

Proof. Verification of the first statement of the lemma is trivial. It is enough to establish that $\det(\Omega_1) \neq 0$. The last requirement immediately follows from the identity:

$$\begin{pmatrix}
\phi_{1, 1} & \phi_{1, 2} & \cdots & \phi_{1, n} \\
\phi_{2, 1} & \phi_{2, 2} & \cdots & \phi_{2, n} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{n, 1} & \phi_{n, 2} & \cdots & \phi_{n, n}
\end{pmatrix} = \begin{pmatrix}
E_{\sigma_2} & 0_{\sigma_2} \\
0_{\sigma_2} & 0_{\sigma_2}
\end{pmatrix} \Omega_k (\xi)$$

Lemma 2. If the condition (12) is satisfied, then conditions of unlimited resolvability of the problem (9)-(11) are satisfied for $p = \sigma + i \tau \in \mathbb{R} + \mathbb{C}$, $\Re \geq 0$, where

- $\sigma_0$ - abscess of convergence of Laplace integral, and
- $\mathbb{Jnp} \in \mathbb{C}$ - nondegenerate:

$$\det(\Omega(\xi)) \neq 0$$

(13)

Lemma 3. If conditions of unlimited resolvability of the problem are satisfied, then limited on the set $I_n^j$ the solution of separate system (9) has the form:

$$v_j = \sum_{\omega} \mathcal{Q}_{\omega} \phi_{\omega} \psi_{\omega} + \sum_{\omega} \mathcal{Q}_{\omega} \phi_{\omega} \psi_{\omega}$$

(14)

Proof. On the basis of the Lemma 2, the right part of the formula (14) has the meaning. Therefore direct Validate of each conditions (9)-(11) is possible.

Let's return to originals in the formula (14). We apply the inverse Laplace transform formula, we have

$$v_j (t, \xi) = \frac{1}{2 \pi i} \int_{\mathbb{C}} \mathcal{L}^{-1} (\phi (s, \xi)) \mathcal{L}^{-1} (\psi (s, \xi)) ds$$

(15)

We consider that the function

$$\phi (s, \xi) \Omega_k (\sigma, \xi) \begin{pmatrix} 0 \\ \xi \end{pmatrix}$$

is analytic in the half-plane $\Re (p) \geq 0$ according to condition (12). We make the change $p = i \lambda$ in the first integral, and in the second integral $p = -i \lambda$ we will present the problem solution (4)-(7) in the form:

$$v_j (t, \xi) = -\frac{1}{2 \pi i} \int_{0}^{\infty} \mathcal{Q} \left( \xi \right)$$

(16)

In the found representation for the mixed boundary problem (4)-(7) we will pass to the limit at $t \to 0$. We obtain the integral representation for initial conditions (5):

$$u_j (x, \xi) = -\frac{1}{2 \pi i} \int_{0}^{\infty} \mathcal{Q} \left( \xi \right)$$

(17)

If to put

$$u_j (x, \xi) = -\frac{1}{2 \pi i} \int_{0}^{\infty} \mathcal{Q} \left( \xi \right)$$

(18)

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\[ g(x) = \sum_{i=1}^{n} \theta(x-l_{i-1}) \theta(l_{i}-x) g_{i}(x) + \theta(x-l_{n}) g_{n}(x). \]

Following the decomposition theorem can be proved by the Cauchy's method of contour integration [14].

**Theorem 1.** If the function \( g(x) \) is defined, piecewise continuous, absolutely summable, and has a limited variation on \( I_{n}^{+} \), then for \( x \in I_{n}^{+} \) the integral representation is true:

\[
\int_{0}^{1} [g(x+\alpha) - g(x-\alpha)] - \int_{0}^{1} [u(x,\lambda) \int_{0}^{1} u(x,\lambda) g(x) d\lambda] \, d\alpha.
\]

(19)

We will receive the main integrated transformation identity of the differential operator for the application of the integral transformations for the problems solution of mathematical physics

\[ B = \sum_{i=1}^{n} A_{n}^{i} \theta(x-l_{i-1}) \theta(l_{i}-x) \frac{d^{n}}{dx^{n}} + A_{n}^{i} \theta(x-l_{n}) \frac{d^{n}}{dx^{n}}. \]

**Theorem 2.** If the function \( g(x) \) is defined, four times differentiable on the set \( I_{n}^{+} \), \( g(x) \), then the main identity for the function satisfying to the coupling conditions (7) and vanishing at infinity together with the derivatives to the third order holds:

\[
F_{n}^{+}[u(x)](\lambda) = \lambda^{2} F_{n}^{+}[u(x)](\lambda) - \sum_{i=1}^{n} A_{n}^{i} \theta(x-l_{i-1}) \theta(l_{i}-x) \frac{d^{n}}{dx^{n}} + A_{n}^{i} \theta(x-l_{n}) \frac{d^{n}}{dx^{n}}.
\]

(20)

Applies the formula of integration by parts in the form

\[
\int_{0}^{1} u(x,v)^{(n)}(x) \, dx = [u(x,v)^{(n)}(x)]_{0}^{1} + \int_{0}^{1} \frac{d^{n}}{dx^{n}} u(x,v) \, dx
\]

(1)

+ \frac{d^{n}}{dx^{n}} v(x) u^{(n)}(x) \, dx.

Let's prove that all members outside the integral except the first member in the right part of the written-out formula will disappear. Let's use coupling conditions

\[ M_{1} \Omega_{n} = M_{1} \Omega_{n+1}, x = l_{i} \]

and its consequence

\[ \Omega_{n}^{+} M_{1}^{+} \Omega_{n} = \Omega_{n+1}^{+} M_{1}^{+} \Omega_{n+1}, x = l_{i}. \]

Not the zero member in the formula of integration by parts in the form

\[
- \phi u_{x} (\lambda), \psi (\lambda) \) \Omega_{n}^{+} (\lambda) = \begin{pmatrix} g_{i} (l_{i}) \\ g'_{i} (l_{i}) \\ g''_{i} (l_{i}) \\ g^{(n)}_{i} (l_{i}) \end{pmatrix}
\]

When it is considered that

\[
\phi (\lambda), \psi (\lambda) = \begin{pmatrix} a_{n}^{0} a_{n}^{1} a_{n}^{2} a_{n}^{3} \\ a_{n}^{0} a_{n}^{1} a_{n}^{2} a_{n}^{3} \\ a_{n}^{0} a_{n}^{1} a_{n}^{2} a_{n}^{3} \\ a_{n}^{0} a_{n}^{1} a_{n}^{2} a_{n}^{3} \end{pmatrix} \Omega_{n} (\lambda, x).
\]

we obtain the formula for member outside the integral in the form

\[
\begin{pmatrix} a_{n+1}^{0} a_{n+1}^{1} a_{n+1}^{2} a_{n+1}^{3} \\ a_{n+1}^{0} a_{n+1}^{1} a_{n+1}^{2} a_{n+1}^{3} \\ a_{n+1}^{0} a_{n+1}^{1} a_{n+1}^{2} a_{n+1}^{3} \\ a_{n+1}^{0} a_{n+1}^{1} a_{n+1}^{2} a_{n+1}^{3} \end{pmatrix} \Omega (l_{i}, x) \int_{0}^{1} \left( \begin{pmatrix} g_{i} (l_{i}) \\ g'_{i} (l_{i}) \\ g''_{i} (l_{i}) \\ g^{(n)}_{i} (l_{i}) \end{pmatrix} \right) \Omega (l_{i}, \lambda) \right).
\]

The theorem is proved.

3. Modeling of transversal oscillation for an elastic piecewise-homogeneous rod

Let's apply the method of integral Fourier transforms from item 2 to the problem solving (1)-(3). Let's choose the parameters in conditions (9),(10),(11):

\[ A_{n}^{i} = E_{n} I_{n} \rho_{n} \]

\[
\begin{pmatrix} a_{n+1}^{0} a_{n+1}^{1} a_{n+1}^{2} a_{n+1}^{3} \\ a_{n+1}^{0} a_{n+1}^{1} a_{n+1}^{2} a_{n+1}^{3} \\ a_{n+1}^{0} a_{n+1}^{1} a_{n+1}^{2} a_{n+1}^{3} \\ a_{n+1}^{0} a_{n+1}^{1} a_{n+1}^{2} a_{n+1}^{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & E_{n} I_{n} \\ 0 & 0 & 0 & E_{n} I_{n} \end{pmatrix}
\]

Theorem 2. If the function \( g(x) \) is defined, four times differentiable on the set \( I_{n}^{+} \), then the main identity for the function satisfying to the coupling conditions (7) and vanishing at infinity together with the derivatives to the third order holds:

\[ F_{n}^{+}[u(x)](\lambda) = \lambda^{2} F_{n}^{+}[u(x)](\lambda) - \sum_{i=1}^{n} A_{n}^{i} \theta(x-l_{i-1}) \theta(l_{i}-x) \frac{d^{n}}{dx^{n}} + A_{n}^{i} \theta(x-l_{n}) \frac{d^{n}}{dx^{n}}. \]

(20)

In Fourier’s images considering integral identity (20) problem (1)-(3) takes the form:

\[
\frac{d^{2}}{dt^{2}} + \lambda^{2} \tilde{y}(t, \lambda) = \begin{pmatrix} f(t) \\ 0 \end{pmatrix}, \quad t > 0,
\]

\[ \tilde{y}(0) = \frac{\partial \tilde{y}}{\partial t}(0) = 0 \]

Its solution in Fourier’s images has the form

\[ \tilde{y}(t, \lambda) = \int_{0}^{1} \left( \begin{pmatrix} f(t) \\ 0 \end{pmatrix} \right) \tilde{d} \tau. \]

According to the theorem 1 we will find a formula for displacement \( y_{n}(x, t) \)

\[ y_{n}(x, t) = \frac{1}{2\pi i} \int_{0}^{1} u_{n}(x, \lambda) \int_{0}^{1} \left( \begin{pmatrix} f(t) \\ 0 \end{pmatrix} \right) \tilde{d} \tau. \]

After change of the order of integration the formula takes the form:

\[ y_{n}(x, t) = \int_{0}^{1} \left( \begin{pmatrix} f(t) \\ 0 \end{pmatrix} \right) \tilde{d} \tau. \]
where

\[
W_n(x,t-r) = -\frac{1}{2\pi i} \int_{\gamma} \frac{u_n(x,\lambda)}{\lambda} \sin(t-r)\lambda d\lambda.
\]

References


