International Electronic Journal of Algebra
Volume 27 (2020) 77-87
DOI: 10.24330/ieja.662963

ψ-SECONDARY SUBMODULES OF A MODULE

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Received: 11 January 2019; Revised: 25 September 2019; Accepted: 23 October 2019
Communicated by Roger A. Wiegand

Abstract. Let R be a commutative ring with identity and M be an R-module. Let $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function, where $S(M)$ denote the set of all submodules of M. The main purpose of this paper is to introduce and investigate the notion of ψ-secondary submodules of an R-module M as a generalization of secondary submodules of M.

Mathematics Subject Classification (2010): 13C05
Keywords: Secondary submodule, ϕ-prime ideal, weak secondary submodule, ψ-secondary submodule

1. Introduction

Throughout this paper, R will denote a commutative ring with identity, $\mathbb{Z}$ and $\mathbb{N}$ will denote the ring of integers and the set of positive integers, respectively. We will denote the set of ideals of R by $S(R)$ and the set of all submodules of M by $S(M)$, where M is an R-module.

Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [5]. A non-zero R-module M is said to be *secondary* if for each $a \in R$ the endomorphism of M given by multiplication by a is either surjective or nilpotent [8]. A non-zero submodule N of M is said to be *second* if for each $a \in R$, the endomorphism of N given by multiplication by a is either surjective or zero [9].

Anderson and Bataineh in [1] defined the notation of ϕ-prime ideals as follows: let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. Then, a proper ideal P of R is *ϕ-prime* if for r, $s \in R$, $rs \in P \setminus \phi(P)$ implies that $r \in P$ or $s \in P$ [1]. A proper ideal I of R is said to be *ϕ-primary* if for $a, b \in R$ with $ab \in I \setminus \phi(I)$, then either $a \in I$ or $b \in \sqrt{I}$ [1].

Zamani in [10] extended this concept to prime submodule. For a function $\phi : S(M) \to S(M) \cup \{\emptyset\}$, a proper submodule N of M is called *ϕ-prime* if whenever $r \in R$ and $x \in M$ with $rx \in N \setminus \phi(N)$, then $r \in (N :_R M)$ or $x \in N$. Bataineh and Kuhail in [4] generalized the concept of ϕ-prime submodules to ϕ-primary submodules. For a function $\phi : S(M) \to S(M) \cup \{\emptyset\}$, a proper submodule N of M is called *ϕ-primary* if whenever $r \in R$ and $x \in M$ with $rx \in N \setminus \phi(N)$, then $x \in N$ or $r^n \in (N :_R M)$ for some $n \in \mathbb{N}$.

Let $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Farshadifar and Ansari-Toroghy in [6], defined the notation of ψ-second submodules of M as a dual notion of ϕ-prime submodules of M. A non-zero submodule N of M is said to be a *ψ-second submodule of* M if $r \in R$, K a submodule of M, $rN \subseteq K$, and $r\psi(N) \not\subseteq K$, then $N \subseteq K$ or $rN = 0$.

The main purpose of this paper is to introduce and study the concept of ψ-secondary submodules of M as a generalization of the notion of secondary submodules of M. Also, the notion of ψ-secondary submodules of M can be regarded as a generalization of the notion of ψ-second submodules of M. We say that a non-zero submodule N of M is a *ψ-secondary submodule of* M if $r \in R$, K a submodule of M, $rN \subseteq K$, and $r\psi(N) \not\subseteq K$, then $N \subseteq K$ or $r^nN = 0$ for some $n \in \mathbb{N}$. In fact the notion of ψ-secondary submodules is a dual notion of ϕ-primary submodules. There are some works about ϕ-primary submodules. It is natural to ask the following question: To what extent does the dual of these results hold for ψ-secondary submodules of an R-module? The aim of this paper is to provide some information in this case. Among the other results, we have shown that if N is a ψ-secondary submodule of M such that $Ann_R(N)\psi(N) \not\subseteq N$, then N is a secondary submodule of M (see Theorem 2.5). Also, we have proved that if H is a submodule of M such that far all ideals I and J of R, $(H :_M I) \subseteq (H :_M J)$ implies that $J \subseteq I$, then H is a secondary submodule of M if and only if H is a ψ_1-secondary submodule of M (see Corollary 2.9). In Theorem 2.10, it is shown that for a submodule S of M, we have

(a) If S is a ψ-secondary submodule of M such that $Ann_R(\psi(S)) \subseteq \phi(Ann_R(S))$, then $Ann_R(S)$ is a ϕ-primary ideal of R.

(b) If $\psi(S) = (0 :_M \phi(Ann_R(S))$, M is a comultiplication R-module and $Ann_R(S)$ is a ϕ-primary ideal of R, then S is a ψ-secondary submodule of M.

The Example 2.11 shows that the condition "M is a comultiplication R-module" in Theorem 2.10 (b) can not be omitted. Also, it is shown that if a is an element of R such that $(0 :_M a) \subseteq a(0 :_M aAnn_R((0 :_M a)))$ and $(0 :_M a)$ is a ψ_1-secondary submodule of M, then $(0 :_M a)$ is a secondary submodule of M (see Theorem 2.17). Finally, in Theorem 2.18, we characterize ψ-secondary submodules of M.

2. Main results

Definition 2.1. Let M be an R-module. We say that a non-zero submodule N of M is a *weak secondary submodule of* M if $r \in R$, K a submodule of M, $rN \subseteq K$, and $rM \not\subseteq K$, then $N \subseteq K$ or $r^n N = 0$ for some $n \in \mathbb{N}$.

Clearly, every secondary submodule of an R-module M is a weak secondary submodule of M. But the converse is not true in general, as we see in the following example.

Example 2.2. Due to the fact that in logic if P is false, then $P \Rightarrow Q$ is true, every R-module is a weak secondary submodule of itself but not every R-module is a secondary R-module. For example, the $\mathbb{Z}$-module $\mathbb{Z}$ is weak secondary which is not secondary.

Definition 2.3. Let M be an R-module, $S(M)$ be the set of all submodules of M, and let $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. We say that a non-zero submodule N of M is a *ψ-secondary submodule of* M if $r \in R$, K a submodule of M, $rN \subseteq K$, and $r\psi(N) \not\subseteq K$, then $N \subseteq K$ or $r^n N = 0$ for some $n \in \mathbb{N}$.

In Definition 2.3, since $r\psi(N) \not\subseteq K$ implies that $r(\psi(N) + N) \not\subseteq K$, there is no loss of generality in assuming that $N \subseteq \psi(N)$ in the rest of this paper. Let M be an R-module. We use the following functions $\psi : S(M) \to S(M) \cup \{\emptyset\}$.

$$\psi_i(N) = (N :_M Ann^i_R(N)), \ \forall N \in S(M), \ \forall i \in \mathbb{N},$$

$$\psi_\sigma(N) = \sum_{i=1}^{\infty} \psi_i(N), \ \forall N \in S(M).$$

$$\psi_M(N) = M, \ \forall N \in S(M).$$

Then it is clear that the set of all ψ_M-secondary submodules is exactly the set of all weakly secondary submodules. Clearly, for any submodule and every positive integer n, we have the following implications:

$$secondary \Rightarrow \psi_{n-1} - secondary \Rightarrow \psi_n - secondary \Rightarrow \psi_\sigma - secondary.$$

For functions $\psi, \theta : S(M) \to S(M) \cup \{\emptyset\}$, we write $\psi \leq \theta$ if $\psi(N) \subseteq \theta(N)$ for each $N \in S(M)$. So whenever $\psi \leq \theta$, any ψ-secondary submodule is θ-secondary.

Theorem 2.4. [3, 2.8]. *For a submodule S of an R-module M the following statements are equivalent.*

(a) *S is a secondary submodule of M.*

(b) *$S \neq 0$ and $rS \subseteq K$, where $r \in R$ and K is a submodule of M, implies either $r^n S = 0$ for some $n \in \mathbb{N}$ or $S \subseteq K$.*

Theorem 2.5. *Let M be an R-module and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Let N be a ψ-secondary submodule of M such that $Ann_R(N)\psi(N) \not\subseteq N$. Then N is a secondary submodule of M.*

Proof. Let $a \in R$ and K be a submodule of M such that $aN \subseteq K$. If $a\psi(N) \not\subseteq K$, then we are done because N is a ψ-secondary submodule of M. Thus suppose that $a\psi(N) \subseteq K$. If $a\psi(N) \not\subseteq N$, then $a\psi(N) \not\subseteq N \cap K$. Hence $aN \subseteq N \cap K$ implies that $N \subseteq N \cap K \subseteq K$ or $a^n N = 0$ for some $n \in \mathbb{N}$, as required. So let $a\psi(N) \subseteq N$. If $Ann_R(N)\psi(N) \not\subseteq K$, then $(a + Ann_R(N))\psi(N) \not\subseteq K$. Hence, there exists $x \in Ann_R(N)$ such that $(a + x)\psi(N) \not\subseteq K$. Thus $(a + x)N \subseteq K$ implies that $N \subseteq K$ or $a^n N = (a^n + x^n)N \subseteq (a + x)^n N = 0$ for some $n \in \mathbb{N}$, since N is a ψ-secondary submodule of M. So suppose that $Ann_R(N)\psi(N) \subseteq K$. Since by assumption, $Ann_R(N)\psi(N) \not\subseteq N$, there exists $b \in Ann_R(N)$ such that $b\psi(N) \not\subseteq N$. Hence $b\psi(N) \not\subseteq N \cap K$. This in turn implies that $(a + b)\psi(N) \not\subseteq N \cap K$. Thus $(a+b)N \subseteq N \cap K$ implies that $N \subseteq N \cap K \subseteq K$ or $a^n N = (a^n + b^n)N \subseteq (a+b)^n N = 0$ for some $n \in \mathbb{N}$, as desired. □

Corollary 2.6. *Let N be a weak secondary submodule of an R-module M such that $Ann_R(N)M \not\subseteq N$. Then N is a secondary submodule of M.*

Proof. In Theorem 2.5 set $\psi = \psi_M$. □

Corollary 2.7. *Let M be an R-module and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. If N is a ψ-secondary submodule of M such that $(N :_M Ann^2_R(N)) \subseteq \psi(N)$, then N is a ψ_σ-secondary submodule of M.*

Proof. If N is a secondary submodule of M, then the result is clear. So suppose that N is not a secondary submodule of M. Then by Theorem 2.5, we have $Ann_R(N)\psi(N) \subseteq N$. Therefore, by assumption,

$$(N :_M Ann^2_R(N)) \subseteq \psi(N) \subseteq (N :_M Ann_R(N)).$$

This implies that $\psi(N) = (N :_M Ann^2_R(N)) = (N :_M Ann_R(N))$ because always $(N :_M Ann_R(N)) \subseteq (N :_M Ann^2_R(N))$. Now

$$(N :_M Ann^3_R(N)) = ((N :_M Ann^2_R(N)) :_M Ann_R(N)) =$$

$$((N :_M Ann_R(N)) :_M Ann_R(N)) = (N :_M Ann^2_R(N)) = \psi(N).$$

By continuing, we get that $\psi(N) = (N :_M Ann^i_R(N))$ for all $i \geq 1$. Therefore, $\psi(N) = \psi_\sigma(N)$ as needed. □

Theorem 2.8. *Let M be an R-module and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Let H be a submodule of M such that far all ideals I and J of R, $(H :_M I) \subseteq (H :_M J)$ implies that $J \subseteq I$. If H is not a secondary submodule of M, then H is not a ψ_1-secondary submodule of M.*

Proof. As H is not a secondary submodule of M, there exists $r \in R$ and a submodule K of M such that $r^n H \neq 0$ for each $n \in \mathbb{N}$ and $H \not\subseteq K$, but $rH \subseteq K$ by Theorem 2.4. We have $H \not\subseteq K \cap H$ and $rH \subseteq K \cap H$. If $r(H :_M Ann_R(H)) \not\subseteq K \cap H$, then by our definition H is not a ψ_1-secondary submodule of M. So let $r(H :_M Ann_R(H)) \subseteq K \cap H$. Then $r(H :_M Ann_R(H)) \subseteq K \cap H \subseteq H$. Thus $(H :_M Ann_R(H)) \subseteq (H :_M r)$ and so by assumption, $r \in Ann_R(H)$. This is a contradiction. □

Corollary 2.9. *Let M be an R-module and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Let H be a submodule of M such that far all ideals I and J of R, $(H :_M I) \subseteq (H :_M J)$ implies that $J \subseteq I$. Then H is a secondary submodule of M if and only if H is a ψ_1-secondary submodule of M.*

An R-module M is said to be a *comultiplication module* if for every submodule N of M, there exists an ideal I of R such that $N = (0 :_M I)$ [2]. It is easy to see that M is a comultiplication module if and only if $N = (0 :_M Ann_R(N))$ for each submodule N of M.

Theorem 2.10. *Let M be an R-module, $\phi : S(R) \to S(R) \cup \{\emptyset\}$, and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be functions.*

(a) *If S is a ψ-secondary submodule of M such that $Ann_R(\psi(S)) \subseteq \phi(Ann_R(S))$, then $Ann_R(S)$ is a ϕ-primary ideal of R.*

(b) *If $\psi(S) = (0 :_M \phi(Ann_R(S))$, M is a comultiplication R-module and $Ann_R(S)$ is a ϕ-primary ideal of R, then S is a ψ-secondary submodule of M.*

Proof. (a) Let $ab \in Ann_R(S) \setminus \phi(Ann_R(S))$ for some $a, b \in R$. Then $ab\psi(S) \neq 0$ by assumption. If $a\psi(S) \subseteq (0 :_M b)$, then $ab\psi(S) = 0$, a contradiction. Thus $a\psi(S) \not\subseteq (0 :_M b)$. Therefore, $S \subseteq (0 :_M b)$ or $a^n S = 0$ for some $n \in \mathbb{N}$ because S is a ψ-secondary submodule of M.

(b) Let $a \in R$ and K be a submodule of M such that $aS \subseteq K$ and $a\psi(S) \not\subseteq K$. As $aS \subseteq K$, we have $S \subseteq (K :_M a)$. It follows that

$$S \subseteq ((0 :_M Ann_R(K)) :_M a) = (0 :_M aAnn_R(K)).$$

This implies that $aAnn_R(K) \subseteq Ann_R((0 :_M aAnn_R(K))) \subseteq Ann_R(S)$. Hence, $aAnn_R(K) \subseteq Ann_R(S)$. If $aAnn_R(K) \subseteq \phi(Ann_R(S))$, then $\psi(S) = (0 :_M \phi(Ann_R(S)) \subseteq ((0 :_M Ann_R(K) :_M a)$. As M is a comultiplication R-module, we have $a\psi(S) \subseteq K$, a contradiction. Thus $aAnn_R(K) \not\subseteq \phi(Ann_R(S))$ and so as $Ann_R(S)$ is a ϕ-primary ideal of R, we conclude that $a^nS = 0$ for some $n \in \mathbb{N}$ or

$$S = (0 :_M Ann_R(S)) \subseteq (0 :_M Ann_R(K)) = K,$$

as needed. □

The following example shows that the condition "M is a comultiplication R-module" in Theorem 2.10 (b) can not be omitted.

Example 2.11. Let $R = \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z}$, and $S = 2\mathbb{Z} \oplus 2\mathbb{Z}$. Clearly, M is not a comultiplication R-module. Suppose that $\phi : S(R) \to S(R) \cup \{\emptyset\}$ and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be functions such that $\phi(I) = I$ for each ideal I of R and $\psi(S) = M$. Then clearly, $Ann_R(S) = 0$ is a ϕ-primary ideal of R and $\psi(S) = M = (0 :_M \phi(Ann_R(S))$. But as $3S \subseteq 6\mathbb{Z} \oplus 6\mathbb{Z}$, $S \not\subseteq 6\mathbb{Z} \oplus 6\mathbb{Z}$, and $3^nS \neq 0$ for each $n \in \mathbb{N}$, we have that S is not a ψ-secondary submodule of M.

The following lemma is known, but we write it here for the sake of reference.

Lemma 2.12. *Let M be an R-module, S a multiplicatively closed subset of R, and N be a finitely generated submodule of M. If $S^{-1}N \subseteq S^{-1}K$ for a submodule K of M, then there exists an $s \in S$ such that $sN \subseteq K$.*

Proof. This is straightforward. □

Proposition 2.13. *Let M be an R-module, $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function, and N be a ψ-secondary submodule of M. Then we have the following statements.*

(a) *If K is a submodule of M with $K \subset N$ and $\psi_K : S(M/K) \to S(M/K) \cup \{\emptyset\}$ is a function such that $\psi_K(N/K) = \psi(N)/K$, then N/K is a ψ_K-secondary submodule of M/K.*

(b) *If N is a finitely generated submodule of M, S is a multiplicatively closed subset of R with $Ann_R(N) \cap S = \emptyset$, and $S^{-1}\psi : S(S^{-1}M) \to S(S^{-1}M) \cup \{\emptyset\}$ is a function such that $(S^{-1}\psi)(S^{-1}N) = S^{-1}\psi(N)$, then $S^{-1}N$ is a $S^{-1}\psi$-secondary submodule of $S^{-1}M$.*

Proof. (a) This is straightforward.

(b) As N is a ψ-secondary submodule of M, we have $N \neq 0$. This implies that $S^{-1}N \neq 0$ since N is finitely generated and $Ann_R(N) \cap S = \emptyset$ by using Lemma 2.12.

Let $a/s \in S^{-1}R$ and $S^{-1}K$ be a submodule of $S^{-1}M$ such that $(a/s)S^{-1}N \subseteq S^{-1}K$ and $(a/s)S^{-1}(\psi(S^{-1}N)) \not\subseteq S^{-1}K$. It follows that $(a/s)S^{-1}(\psi(N)) \not\subseteq S^{-1}K$. Now the result follows from the fact that N is a ψ-secondary submodule of M and Lemma 2.12. □

Proposition 2.14. *Let M and $\acute{M}$ be R-modules and $f : M \to \acute{M}$ be an R-monomorphism. Let $\psi : S(M) \to S(M) \cup \{\emptyset\}$ and $\acute{\psi} : S(\acute{M}) \to S(\acute{M}) \cup \{\emptyset\}$ be functions such that $\psi(f^{-1}(\acute{N})) = f^{-1}(\acute{\psi}(\acute{N}))$, for each submodule $\acute{N}$ of $\acute{M}$. If $\acute{N}$ is a $\acute{\psi}$-secondary submodule of $\acute{M}$ such that $\acute{N} \subseteq Im(f)$, then $f^{-1}(\acute{N})$ is a ψ-secondary submodule of M.*

Proof. As $\acute{N} \neq 0$ and $\acute{N} \subseteq Im(f)$, we have $f^{-1}(\acute{N}) \neq 0$. Let $a \in R$ and K be a submodule of M such that $af^{-1}(\acute{N}) \subseteq K$ and $a\psi(f^{-1}(\acute{N})) \not\subseteq K$. Then by using assumptions, $a\acute{N} \subseteq f(K)$ and $a\acute{\psi}(\acute{N}) \not\subseteq f(K)$. Thus $a^n\acute{N} = 0$ for some $n \in \mathbb{N}$ or $\acute{N} \subseteq f(K)$ since $\acute{N}$ is a $\acute{\psi}$-secondary submodule of $\acute{M}$. This implies that $a^n f^{-1}(\acute{N}) = 0$ or $f^{-1}(\acute{N}) \subseteq K$, as needed. □

A proper submodule N of an R-module M is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [7].

Remark 2.15. Let N and K be two submodules of an R-module M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$.

Proposition 2.16. *Let M be an R-module, $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function, and let N be a ψ_1-secondary submodule of M. Then we have the following statements.*

(a) *If for $a \in R$, $aN \neq N$, then $(N :_M \sqrt{Ann_R(N)}) \subseteq (N :_M a)$.*
(b) *If J is an ideal of R such that $\sqrt{Ann_R(N)} \subseteq J$ and $JN \neq N$, then $(N :_M \sqrt{Ann_R(N)}) = (N :_M J)$.*

Proof. (a) Let $a \in R$ such that $aN \neq N$. If $a^nN = 0$ for some $n \in \mathbb{N}$, then clearly $(N :_M \sqrt{Ann_R(N)}) \subseteq (N :_M a)$. So let $a^nN \neq 0$ for each $n \in \mathbb{N}$. Now let $\acute{L}$ be a completely irreducible submodule of M such that $N \subseteq \acute{L}$. Then $N \not\subseteq \acute{L} \cap aN$ and $aN \subseteq \acute{L} \cap aN$. Hence as N is a ψ_1-secondary submodule of M, we have $a(N :_M Ann_R(N)) \subseteq \acute{L} \cap aN \subseteq \acute{L}$. Therefore, $a(N :_M Ann_R(N)) \subseteq N$ by Remark 2.15. Hence, $a(N :_M \sqrt{Ann_R(N)}) \subseteq a(N :_M Ann_R(N))$ implies that $a(N :_M \sqrt{Ann_R(N)}) \subseteq N$. Thus $(N :_M \sqrt{Ann_R(N)}) \subseteq (N :_M a)$.

(b) As $JN \neq N$, we have $aN \neq N$ for each $a \in J$. Thus by part (a), for each $a \in J$, $(N :_M \sqrt{Ann_R(N)}) \subseteq (N :_M a)$. This implies that

$$(N :_M J) = \cap_{a \in J}(N :_M a) \supseteq (N :_M \sqrt{Ann_R(N)}).$$

The inverse inclusion follows from the fact that $\sqrt{Ann_R(N)} \subseteq J$. □

Theorem 2.17. *Let M be an R-module, $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function, and let a be an element of R such that $(0 :_M a) \subseteq a(0 :_M aAnn_R((0 :_M a)))$. If $(0 :_M a)$ is a ψ_1-secondary submodule of M, then $(0 :_M a)$ is a secondary submodule of M.*

Proof. Let $N := (0 :_M a)$ be a ψ_1-secondary submodule of M. Then $(0 :_M a) \neq 0$. Now let $t \in R$ and K be a submodule of M such that $t(0 :_M a) \subseteq K$. If $t(N :_M Ann_R(N)) \not\subseteq K$, then $t^n(0 :_M a) = 0$ for some $n \in \mathbb{N}$ or $(0 :_M a) \subseteq K$ since $(0 :_M a)$ is a ψ_1-secondary submodule of M. So suppose that $t(N :_M Ann_R(N)) \subseteq K$. Now we have $(t + a)(0 :_M a) \subseteq K$. If $(t + a)(N :_M Ann_R(N)) \not\subseteq K$, then as $(0 :_M a)$ is a ψ_1-secondary submodule of M,

$$t^n(0 :_M a) = (t^n + a^n)(0 :_M a) \subseteq (t + a)^n(0 :_M a) = 0$$

for some $n \in \mathbb{N}$ or $(0 :_M a) \subseteq K$, and we are done. So assume that $(t + a)(N :_M Ann_R(N)) \subseteq K$. Then $t(N :_M Ann_R(N)) \subseteq K$ gives that $a(N :_M Ann_R(N)) \subseteq K$. Hence by assumption, $(0 :_M a) \subseteq K$ and the result follows from Theorem 2.4. □

Theorem 2.18. *Let N be a non-zero submodule of an R-module M and let $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Then the following are equivalent:*

(a) *N is a ψ-secondary submodule of M;*

(b) *for a submodule K of M with $N \not\subseteq K$, we have*

$$\sqrt{(K :_R N)} = \sqrt{Ann_R(N)} \cup \sqrt{(K :_R \psi(N))};$$

(c) *for a submodule K of M with $N \not\subseteq K$, we have $\sqrt{(K :_R N)} = \sqrt{Ann_R(N)}$ or $\sqrt{(K :_R N)} = \sqrt{(K :_R \psi(N))}$;*

(d) *for any ideal I of R and any submodule K of M, if $IN \subseteq K$ and $I \not\subseteq \sqrt{(K :_R \psi(N))}$, then $IN = 0$ or $N \subseteq K$;*

(e) *for each $a \in R$ with $a\psi(N) \not\subseteq aN$, we have $aN = N$ or $a^nN = 0$ for some $n \in \mathbb{N}$.*

Proof. (a) $\Rightarrow$ (b) Let for a submodule K of M with $N \not\subseteq K$, we have $a \in \sqrt{(K :_R N)} \setminus \sqrt{(K :_R \psi(N))}$. Then $a^nN \subseteq K$ for some $n \in \mathbb{N}$ and $a^n\psi(N) \not\subseteq K$. Since N is a ψ-secondary submodule of M, we have $a \in \sqrt{Ann_R(N)}$. As we may assume that $N \subseteq \psi(N)$, the other inclusion always holds.

(b) $\Rightarrow$ (c) This follows from the fact that if a subgroup is a union of two subgroups, it is equal to one of them.

(c) $\Rightarrow$ (d) Let I be an ideal of R and K be a submodule of M such that $IN \subseteq K$ and $I \not\subseteq \sqrt{(K :_R \psi(N))}$. Suppose $I \not\subseteq \sqrt{Ann_R(N)}$ and $N \not\subseteq K$. We show that $I \subseteq \sqrt{(K :_R \psi(N))}$. Let $a \in I$ and first let $a \notin \sqrt{Ann_R(N)}$. Then, since $aN \subseteq K$, we have $\sqrt{(K :_R N)} \neq \sqrt{Ann_R(N)}$. Hence by our assumption $\sqrt{(K :_R N)} = \sqrt{(K :_R \psi(N))}$. So $a \in \sqrt{(K :_R \psi(N)}$. Now assume that $a \in I \cap \sqrt{Ann_R(N)}$. Let $u \in I \setminus \sqrt{Ann_R(N)}$. Then $a + u \in I \setminus \sqrt{Ann_R(N)}$. So by the first case, we have $u \in \sqrt{(K :_R \psi(N)}$ and $u + a \in \sqrt{(K :_R \psi(N)}$. This gives that $a \in \sqrt{(K :_R \psi(N)}$. Thus in any case $a \in \sqrt{(K :_R \psi(N)}$. Therefore, $I \subseteq \sqrt{(K :_R \psi(N)}$, as desired.

(d) $\Rightarrow$ (a) This is clear.

(a) $\Rightarrow$ (e) Let $a \in R$ such that $a\psi(N) \not\subseteq aN$. Then $aN \subseteq aN$ implies that $N \subseteq aN$ or $a^n N = 0$ for some $n \in \mathbb{N}$ by part (a). Thus $N = aN$ or $a^n N = 0$ for some $n \in \mathbb{N}$, as requested.

(e) $\Rightarrow$ (a) Let $a \in R$ and K be a submodule of M such that $aN \subseteq K$ and $a\psi(N) \not\subseteq K$. If $a\psi(N) \subseteq aN$, then $aN \subseteq K$ implies that $a\psi(N) \subseteq K$, a contradiction. Thus by part (e), $aN = N$ or $a^n N = 0$ for some $n \in \mathbb{N}$. Therefore, $N \subseteq K$ or $a^n N = 0$ for some $n \in \mathbb{N}$, as needed. □

Example 2.19. Let N be a non-zero submodule of an R-module M and let $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. If $\psi(N) = N$, then N is a ψ-secondary submodule of M by Theorem 2.18 $(e) \Rightarrow (a)$.

Let M be an R-module and let $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. The following example shows that if N_1 and N_2 are two ψ-secondary submodules of M, then $N_1 + N_2$ and $N_1 \cap N_2$ are not ψ-secondary submodules of M in general.

Example 2.20. (a) Let p, q be two prime numbers, $N =< 1/p + \mathbb{Z} >$, and $K =< 1/q + \mathbb{Z} >$. Then clearly, $N \oplus 0$ and $0 \oplus K$ are weak secondary submodules of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty}$ but as $p(N + K) \subseteq K$, $p(\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty}) \not\subseteq K$, $N + K \not\subseteq K$, and $p^n(N + K) \neq 0$ for each $n \in \mathbb{N}$ we have that $N + K$ is not a weak secondary submodule of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty}$.

(b) Clearly, the submodules $\bar{2}\mathbb{Z}_6$ and $\bar{3}\mathbb{Z}_6$ are ψ-secondary submodules of $\mathbb{Z}_6$, where $\psi : S(\mathbb{Z}_6) \to S(\mathbb{Z}_6) \cup \{\emptyset\}$ is a function. But $\bar{2}\mathbb{Z}_6 \cap \bar{3}\mathbb{Z}_6 = 0$ is not a ψ-secondary submodule of $\mathbb{Z}_6$.

Proposition 2.21. *Let M be an R-module and let N_1 and N_2 be weak secondary submodules of M such that $N_1 \cap N_2 \neq 0$ and $r(N_1 \cap N_2) = rN_1 \cap rN_2$ for each $r \in R$, then $N_1 \cap N_2$ is a weak secondary submodule of M.*

Proof. Let $a \in R$ with $aM \not\subseteq a(N_1 \cap N_2)$. If $aM \subseteq aN_1$ and $aM \subseteq aN_2$, then $aM \subseteq a(N_1 \cap N_2)$, a contradiction. If $aM \not\subseteq aN_1$ and $aM \not\subseteq aN_2$, then by Theorem 2.18 $(a) \Rightarrow (e)$, $aN_1 = N_1$ or $a^n N_1 = 0$ for some $n \in \mathbb{N}$ and $aN_2 = N_2$ or $a^m N_2 = 0$ for some $m \in \mathbb{N}$. If $a^m N_2 = 0$ or $a^n N_1 = 0$, then $a^t(N_1 \cap N_2) = 0$ for some $t \in \mathbb{N}$ and we are done. So suppose that $aN_1 = N_1$ and $aN_2 = N_2$. Then $a(N_1 \cap N_2) = N_1 \cap N_2$. Finally if $aM \not\subseteq aN_1$, $aM \subseteq aN_2$, and $aN_1 = N_1$, then $aN_1 \subseteq aM \subseteq aN_2$. Hence, $N_1 \cap N_2 \subseteq N_1 = aN_1 = aN_1 \cap aN_2 = a(N_1 \cap N_2)$. It follows that $a(N_1 \cap N_2) = N_1 \cap N_2$, as needed. □

Let R_1 and R_2 be two commutative rings with identity. Let M_1 and M_2 be R_1 and R_2-module, respectively and put $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module and each submodule of M is of the form $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2. Suppose that $\psi^i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function for $i = 1, 2$. One can see that the $R = R_1 \times R_2$-module $S_1 \times 0$ and $0 \times S_2$, where S_1 is a secondary submodule of M_1 and S_2 is a secondary submodule of M_2, are secondary submodules of M. The following example, shows that this is not true for correspondence $\psi^1 \times \psi^2$-secondary submodules in general.

Example 2.22. Let $R_1 = R_2 = M_1 = M_2 = S_1 = \mathbb{Z}_6$. Then clearly, S_1 is a weak secondary submodule of M_1. However, $(\bar{2}, \bar{1})(\mathbb{Z}_6 \times 0) \subseteq \bar{2}\mathbb{Z}_6 \times \bar{3}\mathbb{Z}_6$ and $(\bar{2}, \bar{1})(\mathbb{Z}_6 \times \mathbb{Z}_6) \not\subseteq \bar{2}\mathbb{Z}_6 \times \bar{3}\mathbb{Z}_6$. But $(\bar{2}, \bar{1})^n(\mathbb{Z}_6 \times 0) = \bar{2}\mathbb{Z}_6 \times 0 \neq 0 \times 0$ for each $n \in \mathbb{N}$, and $\mathbb{Z}_6 \times 0 \not\subseteq \bar{2}\mathbb{Z}_6 \times \bar{3}\mathbb{Z}_6$. Therefore, $S_1 \times 0$ is not a weak secondary submodule of $M_1 \times M_2$.

Theorem 2.23. *Let $R = R_1 \times R_2$ be a ring and $M = M_1 \times M_2$ be an R-module, where M_1 is an R_1-module and M_2 is an R_2-module. Suppose that $\psi^i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function for $i = 1, 2$. Then $S_1 \times 0$ is a $\psi^1 \times \psi^2$-secondary submodule of M, where S_1 is a ψ^1-secondary submodule of M_1 and $\psi^2(0) = 0$.*

Proof. Let $(r_1, r_2) \in R$ and $K_1 \times K_2$ be a submodule of M such that $(r_1, r_2)(S_1 \times 0) \subseteq K_1 \times K_2$ and

$$(r_1, r_2)((\psi^1 \times \psi^2)(S_1 \times 0)) = r_1\psi^1(S_1) \times r_2\psi^2(0) = r_1\psi^1(S_1) \times 0 \not\subseteq K_1 \times K_2.$$

Then $r_1 S_1 \subseteq K_1$ and $r_1\psi^1(S_1) \not\subseteq K_1$. Hence, $(r_1)^n S_1 = 0$ for some $n \in \mathbb{N}$ or $S_1 \subseteq K_1$ since S_1 is a ψ^1-secondary submodule of M_1. Therefore, $(r_1, r_2)^n(S_1 \times 0) = 0 \times 0$ or $S_1 \times 0 \subseteq K_1 \times K_2$, as requested. □

Acknowledgement. The authors would like to thank the referee for careful reading our manuscript and his/her valuable comments which improved this work.

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