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EM-HERMITE RINGS

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ABSTRACT. A ring R is called EM-Hermite if for each $a, b \in R$, there exist $a_1, b_1, d \in R$ such that $a = a_1 d, b = b_1 d$ and the ideal (a_1, b_1) is regular. We give several characterizations of EM-Hermite rings analogue to those for K-Hermite rings, for example, R is an EM-Hermite ring if and only if any matrix in $M_{n,m}(R)$ can be written as a product of a lower triangular matrix and a regular $m \times m$ matrix. We relate EM-Hermite rings to Armendariz rings, rings with a.c. condition, rings with property A, EM-rings, generalized morphic rings, and PP-rings. We show that for an EM-Hermite ring, the polynomial ring and localizations are also EM-Hermite rings, and show that any regular row can be extended to regular matrix. We relate EM-Hermite rings to weakly semi-Steinitz rings, and characterize the case at which every finitely generated R-module with finite free resolution of length 1 is free.

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1. Introduction

All rings are assumed to be commutative with unity 1. For any ring R, let $Z(R)$ be the set of all zero-divisors, and $reg(R) = R \setminus Z(R)$ be the set of all regular elements, and let $U(R)$ be the set of all units in R. Recall that if R is a commutative ring with unity, then the total quotient ring of R is the localization $T(R) = (reg(R))^{-1}R$. Let $M_{n,m}(R)$ be the ring of all $n \times m$ matrices defined on R. It is well known that $A \in U(M_{n,n}(R))$ if and only if $\det(A) \in U(R)$, $A \in reg(M_{n,n}(R))$ if and only if $\det(A) \in reg(R)$, and A is left zero-divisor if and only if it is right zero-divisor, see [3]. The row $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ is called unimodular if the ideal $(a_1, a_2, \cdots, a_n) = R$, and it is called regular if the ideal $(a_1, a_2, \cdots, a_n) \nsubseteq Z(R)$, in this case the ideal $(a_1, a_2, \cdots, a_n)$ is called a regular ideal. Similar definitions are for columns.

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A ring R is called a K-Hermite ring if for each $a, b \in R$, there exist $a_1, b_1, d \in R$ such that $a = a_1 d, b = b_1 d$ and the ideal $(a_1, b_1) = R$, see [6] and [8]. It is clear that if R is a K-Hermite ring, then it is a Bézout ring (every finitely generated ideal is principal). A ring R is called Hermite if any unimodular row over R can be completed to an invertible matrix by adding a suitable number of new rows. Any K-Hermite is Hermite, but the converse is not true, see [10].

We generalize the concept of K-Hermite rings in the following sense: we call a ring R EM-Hermite, if for each $a, b \in R$, there exist $a_1, b_1, d \in R$ such that $a = a_1 d, b = b_1 d$ and the ideal (a_1, b_1) is regular. We find that this ring has some nice properties; it is preserved by the direct products and localizations, and unlike the case of K-Hermite rings, if R is EM-Hermite, then so is $R[x]$. We give several characterizations of EM-Hermite rings analogue to those for K-Hermite rings, for example, R is an EM-Hermite ring if and only if any matrix in $M_{n,m}(R)$ can be written as a product of a lower triangular matrix and a regular $m \times m$ matrix. We also show that any regular row can be extended to a regular matrix by adding a suitable number of rows. We prove that EM-Hermite rings are non-comparable with Bézout rings, nor Hermite rings, but R is K-Hermite if and only if it is Bézout EM-Hermite. We also relate EM-Hermite rings to Armendariz rings, rings with a.c. condition, rings with property A, PP-rings, weakly semi-Steinitz rings, EM-rings, and generalized morphic rings. Finally, we characterize when an R-module with finite free resolution of length 1 is free.

2. EM-Hermite rings

In this section, we define EM-Hermite rings, and give several characterizations for it, and study some cases at which an EM-Hermite ring is K-Hermite.

Definition 2.1. A ring R is called EM-Hermite if for each $a, b \in R$, there exist $a_1, b_1, d \in R$ such that $a = a_1 d, b = b_1 d$ and the ideal (a_1, b_1) is regular.

We now give some examples of EM-Hermite rings.

Example 2.2. (1) Since any principal ideal ring is K-Hermite, see [10], it is also EM-Hermite.

(2) It is clear that any integral domain is an EM-Hermite ring, and so, $\mathbb{Z}[x]$ is an EM-Hermite ring that is not K-Hermite, being non-Bézout.

(3) Consider the idealization $\mathbb{Z}_4(+)\mathbb{Z}_4$, and consider the two elements $(2, 0)$ and $(0, 1)$. Assume $(2, 0) = (a, b)(c, d)$ and $(0, 1) = (a, b)(x, y)$.

If $x \neq 0$, then we must have $a = 2 = x$, and so we have $1 = 2y + 2b$, and hence $2 = 0$, a contradiction.

So, we must have $x = 0$, and hence, $1 = ay$, i.e. a is a unit in $\mathbb{Z}_4$. Thus we have $c = 2$. Now,

$$(2, d)(0, 2) = (0, 0),$$
$$(0, y)(0, 2) = (0, 0).$$

Hence $Ann((2, d), (0, y)) \neq \{(0, 0)\}$, and $\mathbb{Z}_4(+)\mathbb{Z}_4$ is not an EM-Hermite ring. Since any finite ring is Hermite, then $\mathbb{Z}_4(+)\mathbb{Z}_4$ is Hermite that is not EM-Hermite.

(4) Let $R = \mathbb{Z}[x_1, x_2, x_3, y_1, y_2, y_3] / (x_1y_1 + x_2y_2 + x_3y_3 - 1)$. Then R is an integral domain, and hence EM-Hermite that is not a Hermite ring, see [12].

We now give equivalent characterizations of EM-Hermite rings, parallel to those for K-Hermite, see [10].

Theorem 2.3. *The following statements are equivalent for a ring R.*

(1) *R is an EM-Hermite ring.*
(2) *For any finite set $\{a_1, a_2, \ldots, a_n\} \subseteq R$, there exists $\{b_1, b_2, \ldots, b_n, d\} \subseteq R$ such that $a_i = b_i d$, for each i, and the ideal $(b_1, b_2, \ldots, b_n)$ is regular.*
(3) *For any finite set $\{a_1, a_2, \ldots, a_n\} \subseteq R$, there exist $d \in R$ and a regular matrix $Q \in M_{n,n}(R)$ such that $[a_1\ a_2\ \ldots\ a_n] = [d\ 0\ 0\ \ldots\ 0]Q$.*
(4) *For any matrix $B \in M_{m,n}(R)$, there exists a regular matrix $Q \in M_{n,n}(R)$ such that $B = LQ$, with L a lower triangular matrix.*

Proof. (1) $\Rightarrow$ (2) Assume R is an EM-Hermite ring, and let $a, b, c \in R$. Then there exist $a_1, b_1, d \in R$ such that $a = a_1 d, b = b_1 d$ and $r_1 = \alpha_1 a_1 + \beta_1 b_1 \in (a_1, b_1) \cap reg(R)$. Also there exist $a_2, b_2, k \in R$ such that $d = a_2 k, c = b_2 k$ and $r_2 = \alpha_2 a_2 + \beta_2 b_2 \in (a_2, b_2) \cap reg(R)$.

But $a = a_1 d = a_1 a_2 k$ and $b = b_1 d = b_1 a_2 k$. Also we have $r_1 r_2 = (\alpha_1\alpha_2)(a_1 a_2) + (\alpha_2\beta_1)(a_2 b_1) + (\alpha_1\beta_2 a_1 + \beta_1\beta_2 b_1)(b_2) \in (b_2, a_1 a_2, a_2 b_1) \cap reg(R)$. So, the condition can be applied to any finite subset of R.

(2) $\Rightarrow$ (3) Let $\{a_1, a_2, \ldots, a_n\} \subset R$. Then there exits $\{b_{n-1}, b_n, d_1\} \subseteq R$ such that $a_i = b_i d_1$, for $i \in \{n, n-1\}$, and $r_1 = \alpha_{n-1} b_{n-1} + \alpha_n b_n \in (b_{n-1}, b_n) \cap reg(R)$. So we have

$$[a_1\ a_2\ \ldots\ a_n] = [a_1\ a_2\ \ldots\ a_{n-2}\ d_1\ 0]Q_1,$$

where $Q_1 = \begin{bmatrix} I_{n-2} & 0 \\ 0 & \begin{bmatrix} b_{n-1} & b_n \\ -\alpha_n & \alpha_{n-1} \end{bmatrix} \end{bmatrix}$,

and note that $\det(Q_1) = r_1 \in reg(R)$.

There exists $\{b_{n-3}, b_{n-2}, d_2\} \subset R$ such that $a_{n-2} = b_{n-2}d_2, d_1 = b_{n-3}d_2$ and $r_2 = \alpha_{n-2}b_{n-2} + \alpha_{n-3}b_{n-3} \in (b_{n-2}, b_{n-3})R \cap reg(R)$. So we have

$[a_1\ a_2\ \dots\ a_{n-2}\ d_1\ 0] = [a_1\ a_2\ \dots\ a_{n-3}\ d_2\ 0\ 0]Q_2$,

where $Q_2 = \begin{bmatrix} I_{n-3} & 0 \\ 0 & \begin{bmatrix} b_{n-2} & b_{n-3} & 0 \\ -\alpha_{n-3} & \alpha_{n-2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{bmatrix}$,

and note that $\det(Q_2) = r_2 \in reg(R)$.

In this case we have $[a_1\ a_2\ \dots\ a_n] = [a_1\ a_2\ \dots\ a_{n-3}\ d_2\ 0\ 0]Q_2Q_1$, and $\det(Q_2Q_1) = r_2r_1 \in reg(R)$.

Continue to get $[a_1\ a_2\ \dots\ a_n] = [d\ 0\ 0\ \dots\ 0]Q$, and $\det(Q) = r \in reg(R)$.

(3) $\Rightarrow$ (4) Let $B \in M_{m,n}(R)$. We will proceed by induction on m. By (3) the result is true when $m = 1$. So assume it is true for all $k < m$, and let $B = [b_{ij}]_{m\times n}$. It follows by (3) that $[b_{11}\ b_{12}\ \dots\ b_{1n}] = [d\ 0\ 0\ \dots\ 0]Q_1$, where Q_1 is a regular matrix. So, $[b_{11}\ b_{12}\ \dots\ b_{1n}]adj(Q_1) = \det(Q_1)[d\ 0\ 0\ \dots\ 0]$. Thus, $B\ adj(Q_1) = \det(Q_1)\begin{bmatrix} d & 0 \\ C & D \end{bmatrix}$. By induction hypothesis we have $D = L_1Q_2$, where L_1 is a lower triangular matrix and Q_2 is regular matrix in $M_{(n-1),(n-1)}(R)$. Substituting we get

$$B\ adj(Q_1) = \det(Q_1)\begin{bmatrix} d & 0 \\ C & L_1Q_2 \end{bmatrix} = \det(Q_1)\begin{bmatrix} d & 0 \\ C & L_1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & Q_2 \end{bmatrix},$$

and so,

$$B = \begin{bmatrix} d & 0 \\ C & L_1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & Q_2 \end{bmatrix} Q_1.$$

Now, let $L = \begin{bmatrix} d & 0 \\ C & L_1 \end{bmatrix}$, and $Q = \begin{bmatrix} 1 & 0 \\ 0 & Q_2 \end{bmatrix} Q_1$. Then L is lower triangular, $\det(Q) = \det(Q_2)\det(Q_1) \in reg(R)$, and $B = LQ$.

(4) $\Rightarrow$ (1) Let $a, b \in R$, Then there exist $d \in R$, and a regular matrix $Q \in M_{2,2}(R)$ such that $[a\ b] = [d\ 0\ ]Q$.

So, $a = dq_{11}, b = dq_{12}$, and $\det(Q) = q_{11}q_{22} - q_{12}q_{21} \in (q_{11}, q_{12}) \cap reg(R)$. Thus, R is an EM-Hermite ring. □

If we extend our work to non-commutative rings, we will have:

Corollary 2.4. *If R is an EM-Hermite ring, then $M_{n,n}(R)$ is also EM-Hermite.*

Proof. Assume R is an EM-Hermite ring, and let $A, B \in M_{n,n}(R)$. Then there exist lower triangular matrix $L \in M_{n,2n}(R)$ and a regular matrix $Q \in M_{2n,2n}(R)$ such that

$$\begin{bmatrix} A & B \end{bmatrix} = LQ = \begin{bmatrix} L_1 & 0 \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}.$$

So it follows by (3) in Theorem 2.3 that $M_{n,n}(R)$ is EM-Hermite. □

We can follow the proof of [10] to show that the following statements are equivalent.

Proposition 2.5. *The following statements are equivalent for a ring R.*

(1) *For any matrix $B \in M_{m,n}(R)$, there exists a regular matrix $Q \in M_{n,n}(R)$ such that $BQ = L$ a lower triangular matrix.*
(2) *For any vector $[a_1\ a_2\ \ldots\ a_n] \in M_{1,n}(R)$, there exists a regular matrix $Q \in M_{n,n}(R)$ and $d \in R$ such that $[a_1\ a_2\ \ldots\ a_n]Q = [d\ 0\ 0\ \ldots\ 0]$.*
(3) *For any $a, b \in R$, there exists a regular matrix $Q \in M_{2,2}(R)$ and $d \in R$ such that $[a_1\ a_2]Q = [d\ 0\ ]$.*
(4) *For any $a, b \in R$, there exist $x, y \in R$ such that $ax + by = 0$ and (x, y) is a regular ideal in R.*

Assume that R is an EM-Hermite ring, and let $a, b, d, -x, y \in R$ such that $a = dy, b = d(-x)$ and $\beta(-x) + \alpha y = r \in reg(R)$. Then $ax + by = 0$. So, R satisfies condition (4) in Proposition 2.5, and hence it satisfies all the conditions. Moreover we have:

$$\begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} d & 0 \end{bmatrix} \begin{bmatrix} y & -x \\ -\beta & \alpha \end{bmatrix},$$

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} \alpha & x \\ \beta & y \end{bmatrix} = \begin{bmatrix} dr & 0 \end{bmatrix},$$

with $\det \begin{bmatrix} y & -x \\ -\beta & \alpha \end{bmatrix} = \det \begin{bmatrix} \alpha & x \\ \beta & y \end{bmatrix} = r \in reg(R)$.

To give a more general result, let $B \in M_{m,n}(R)$. There exists a regular matrix $Q \in M_{n,n}(R)$ such that $B = LQ$ with L a lower triangular matrix. Then $B\ adj(Q) = \det(Q)L$. Moreover, $\det(Q)L$ is a lower triangular matrix and $\det(adj(Q)) = (\det(Q))^{n-1} \in reg(R)$.

Although EM-Hermite rings are in general not K-Hermite, the following Theorem shows that for some rings they are equivalent.

Theorem 2.6. *If every regular element in R is a unit, then R is a K-Hermite ring if and only if it is an EM-Hermite ring.*

The condition in the above Theorem is not necessary, since $\mathbb{Z}$ has regular elements that are not units, but it is K-Hermite.

Corollary 2.7. *If R is a finite ring, then R is K-Hermite ring if and only if it is an EM-Hermite ring.*

Corollary 2.8. *For any ring R, $T(R)$ is K-Hermite ring if and only if it is an EM-Hermite ring.*

We now continue the investigation started in [7], [8] and [11] for the cases at which a Bézout ring is K-Hermite.

Theorem 2.9. *A ring R is K-Hermite if and only if it is a Bézout EM-Hermite ring.*

Proof. If R is K-Hermite, then clearly it is a Bézout EM-Hermite ring. So assume that R is a Bézout EM-Hermite ring, and let $a, b \in R$. Then there exist $a_1, b_1, d \in R$ such that $a = a_1 d, b = b_1 d$ and $(d_1) = (a_1, b_1)$ is a regular ideal in R, and so $d_1 \in reg(R)$. Thus we have:

$$d_1 = a_1 x + b_1 y,$$

$$a_1 = \alpha d_1,$$

$$b_1 = \beta d_1.$$

Hence we get

$$d_1 = d_1(\alpha x + \beta y),$$

and since $d_1 \in reg(R)$, we would have

$$1 = \alpha x + \beta y.$$

Therefore, $a = \alpha(d_1 d), b = \beta(d_1 d\ )$ and $(\alpha, \beta) = R$, i.e. R is K-Hermite. □

3. Relations with other rings

In this section, we relate EM-Hermite rings to Armendariz rings, rings with a.c. condition, rings with property A, EM-rings, generalized morphic rings, and PP-rings.

A ring R is said to be Armendariz if the product of two polynomials in $R[x]$ is zero if and only if the product of their coefficients is zero.

Theorem 3.1. *If R is an EM-Hermite ring, then it is Armendariz.*

Proof. Let $f(x) = \sum_{i=0}^{n} f_i x^i$. Then it follows by Theorem 2.3 that $f_i = k_i h$ for each i and $Ann(k_0, \dots, k_n) = \{0\}$. So it follows by McCoy's Theorem that $\sum_{i=0}^{n} k_i x^i$ is not a zero-divisor in $R[x]$, and $f(x) = h \sum_{i=0}^{n} k_i x^i$. If $g(x) = \sum_{i=0}^{m} g_i x^i = k \sum_{i=0}^{m} l_i x^i$ with $\sum_{i=0}^{m} l_i x^i$ is not a zero-divisor in $R[x]$. Then $f(x)g(x) = 0$ if and only if $hk = 0$. Thus we have $f_i g_j = (hk)(k_i l_j) = 0$ for each i and j. Hence R is Armendariz. □

A ring R is said to have a.c. condition, if for any $a, b \in R$ there exists $c \in R$ such that $Ann(a, b) = Ann(c)$.

Theorem 3.2. *If R is an EM-Hermite ring, then it has a.c. condition.*

Proof. Let $a, b \in R$. Then there exist d, x, y such that $a = dx, b = dy$ and the ideal (x, y) is regular. Thus we have $Ann(x, y) = \{0\}$ and so, $Ann(a, b) = Ann(d)$. □

A ring R is said to have property A, if any finitely generated ideal contained in $Z(R)$ has nonzero annihilator. It was shown in [9] that any Noetherian ring has property A, see Theorem 82.

Theorem 3.3. *If R is an EM-Hermite ring, then it has property A.*

Proof. Let $a, b \in R$ such that $Ann(a, b) = \{0\}$. Then there exist d, x, y such that $a = dx, b = dy$ and the ideal (x, y) is regular. Let $r = \alpha x + \beta y \in reg(R)$. But $d \in reg(R)$ since $Ann(d) = Ann(a, b) = \{0\}$. Thus we have

$$a\alpha + b\beta = dx\alpha + dy\beta = dr \in (a, b) \cap reg(R).$$

Therefore, $(a, b) \nsubseteq Z(R)$. □

Let R be a ring, and let $f(x) \in Z(R[x])$ such that $f(x) = c_f f_1(x)$, where $c_f \in R$ and $f_1(x) \in reg(R[x])$. Then c_f is called an annihilating content for $f(x)$. It is clear that $\deg(f) \leq \deg(f_1)$. If every zero-divisor polynomial in $R[x]$ has an annihilating content, R is called an EM-ring. A ring R is called generalized morphic ring if $Ann(a)$ is a principal ideal for each $a \in R$, see [1]. Using Theorem 2.3, one can see easily that any EM-Hermite ring is an EM-ring. But the following Theorem shows that the two properties are equivalent if the ring was Noetherian. But first we need the following important lemma.

Lemma 3.4 ([1, Lemma 3.25])**.** *Assume that R is a Noetherian ring, and bR is a prime principal ideal with $b \in Z(R)$. If $a \in bR \setminus \{0\}$, then $a = b^n s$ for some $n \in \mathbb{N}$ and $s \in R \setminus bR$.*

Theorem 3.5. *Assume that R is a Noetherian ring. Then the following are equivalent:*

(1) *R is an EM-ring.*
(2) *R is a generalized morphic ring.*
(3) *R is an EM-Hermite ring.*

Proof. For the equivalence of (1) and (2), see [1].

(2) $\Rightarrow$ (3) Recall first that since R is a Noetherian ring, then $Ann(a_1, a_2) \neq \{0\}$ if and only if the ideal $(a_1, a_2) \subseteq Z(R)$.

Let $a_1, a_2 \in R$. If $Ann(a_1, a_2) = \{0\}$, then $a_1 = a_1.1, a_2 = a_2.1$, and $Ann(a_1, a_2) = \{0\}$. If $0 \neq m \in Ann(a_1, a_2)$, then $(a_1, a_2) \subseteq Ann(m) \subseteq M_1 = c_1R \subseteq Z(R)$, where M_1 is a maximal ideal in $Z(R)$, and so it is prime, see [9, Theorem 6]. Hence, using Lemma 3.4, $a_i = \alpha_i c_1^{k_i}$ with $\alpha_i \notin c_1R$, and $k_i \geq 1$ for each $i = 1, 2$. Let $k_{11} = Min\{k_i\}$, $b_i = \alpha_i c_1^{k_i - k_{11}}$. Then $a_i = c_1^{k_{11}} b_i$ and $(a_1, a_2) \subset (b_1, b_2)$. Then repeat the work to write $b_i = c_2^{k_{22}} d_i$ and $(a_1, a_2) \subset (b_1, b_2) \subset (d_1, d_2)$. Continue to get an ascending chain in the Noetherian ring R, and thus it must terminate. Hence there exits $f_i \in R$ and $a_i = c_1^{k_{11}} c_2^{k_{22}} c_3^{k_{33}} \ldots {}_n^{k_{nn}} f_i = cf_i$ with $Ann(f_1, f_2) = \{0\}$.

(3) $\Rightarrow$ (1) Clear. $\square$

It was shown in [5] that if $X = \beta\mathbb{R}^+ - \mathbb{R}^+$, then $C(X)$ is a K-Hermite, and hence EM-Hermite ring, and since X is connected, $C(X)$ is not generalized morphic ring. Also it was shown in [5] that if $X = [-1, 1] \times [0, \infty)$, then $C(\beta X - X)$ is a Bézout ring that is not K-Hermite, then it follows by Theorem 2.9 that $C(\beta X - X)$ is not an EM-Hermite ring. Also it follows by [1] that $C(\beta X - X)$ is an EM-ring.

We note that the Bézout property and the EM-Hermite property are non-comparable, but adding them together would give the K-Hermite property, unlike the case of Hermite property and the EM-Hermite property, they are non-comparable, and adding them together need not be K-Hermite as in the case of $\mathbb{Z}[x]$.

Recall that a ring R is called a PP-ring if every principal ideal in R is a projective R-module. While any von Neumann regular ring is K-Hermite, $\mathbb{Z}[x]$ is a PP-ring that is not K-Hermite.

Theorem 3.6. *If R is a PP-ring, then it is an EM-Hermite ring.*

Proof. Let $a_1, a_2 \in R$. Then $a_i = u_i e_i$, where $u_i \in reg(R)$ and e_i is an idempotent for each i, see [4, Lemma 2]. Let $e = e_1 + e_2 - e_1e_2$. Then e is also an idempotent and $e_ie = e_i$ for $i = 1, 2$. Thus $a_i = eu_i(e_i + 1 - e)$, and since $1 = (e_1 + 1 - e) + (e_2 + 1 - e) - (e_1 + 1 - e)(e_2 + 1 - e)$, we have $u_1u_2 = (u_1(e_1 + 1 - e))\, u_2 + (u_2(e_2 + 1 - e))\, u_1 - u_1(e_1 + 1 - e)u_2(e_2 + 1 - e) \in (u_1(e_1 + 1 - e), u_2(e_2 + 1 - e)) \cap reg(R)$. $\square$

The converse of this theorem needs not be true, since $\mathbb{Z}_8$ is an EM-Hermite ring which is not a PP-ring, being non-reduced.

4. Some properties of EM-Hermite rings

In this section, we study some properties of EM-Hermite rings, such as polynomial rings and localizations of EM-Hermite rings, and extending regular rows to regular matrices.

The ring $\mathbb{Z}$ is K-Hermite, but $\mathbb{Z}[x]$ is not, and it is conjectured that if R is Hermite, then $R[x]$ is Hermite. We now show that if R is an EM-Hermite ring, then $R[x]$ is EM-Hermite.

Theorem 4.1. *If R is an EM-Hermite ring, then $R[x]$ is an EM-Hermite ring.*

Proof. Let $f(x) = \sum_{i=0}^{n} f_i x^i, g(x) = \sum_{i=0}^{m} g_i x^i \in R[x]$. Then it follows by Theorem 2.3 that $f_i = k_i h, g_i = l_i h$, for each i and the ideal $(k_0, \ldots, k_n, l_0, \ldots, l_m) \nsubseteq Z(R)$. Thus, $f(x) = h \sum_{i=0}^{n} k_i x^i, g(x) = h \sum_{i=0}^{m} l_i x^i$. If $\sum_{i=0}^{l} h_i x^i \in Ann(\sum_{i=0}^{n} k_i x, \sum_{i=0}^{m} l_i x^i)$, then since R is Armendariz, $h_i \in Ann(k_0, \ldots, k_n, l_0, \ldots, l_m) = \{0\}$ for each i, and so, $Ann(\sum_{i=0}^{n} k_i x, \sum_{i=0}^{m} l_i x^i) = \{0\}$, and since $R[x]$ has property A for any ring R, see [9], we have $(\sum_{i=0}^{n} k_i x, \sum_{i=0}^{m} l_i x^i) R[x] \nsubseteq Z(R[x])$. □

Corollary 4.2. *Let R be an EM-Hermite ring. Then $R[x_1, x_2, \ldots, x_n]$ is an EM-Hermite ring.*

Theorem 4.3. *Let R be an EM-Hermite ring, and let S be a multiplicatively closed subset of R. Then $S^{-1}R$ is an EM-Hermite ring.*

Proof. Let $a, b \in S^{-1}R$. Then there exist $t, s \in S$ such that $ta, sb \in R$. Since R is an EM-Hermite ring, there exist $d, a_1, b_1 \in R$ such that $ta = da_1$ and $sb = db_1$ and (a_1, b_1) is a regular ideal in R. There exist $x, y \in R$ such that $r = xa_1 + yb_1 \in reg(R)$. Thus we have $a = d(\frac{a_1}{t})$ and $b = d(\frac{b_1}{s})$.

Now, $\frac{x}{s} \frac{a_1}{t} + \frac{y}{t} \frac{b_1}{s} = \frac{r}{st} \in (\frac{a_1}{t}, \frac{b_1}{s}) \cap reg(S^{-1}R)$. □

Corollary 4.4. *Let R be an EM-Hermite ring. Then $T(R)$ is K-Hermite.*

The converse of this Corollary is not in general true as illustrated in the following example.

Example 4.5. It was shown in [1] that if $R = \mathbb{Z}_6[x, y]/(xy)$, then $T(R)$ is a von Neumann regular ring, and hence it is K-Hermite. But R is not an EM-Hermite ring, since $x, 3 \in R$, and if $x = ah, 3 = bh$ with $Ann(a, b) = \{0\}$, then $0 = a(2yh) =$

$b(2yh)$, which implies that $0 = 2yh$, and so, $(h) \subseteq Ann(2y) = (3, x) \subseteq (h)$, and so, $(h) = (3, x)$, a contradiction.

Theorem 4.6. *If R is an EM-Hermite ring, then any regular row can be completed to a regular square matrix by adding a suitable number of rows.*

Proof. We will proceed by induction on n, and make some modifications on the proof of [10, page 28].

If $n = 2$, and $\begin{bmatrix} a_1 & a_2 \end{bmatrix}$ is regular, then $a_1 t + a_2 s = r \in reg(R)$, and $\det \begin{bmatrix} a_1 & a_2 \\ -s & t \end{bmatrix} = r \in reg(R)$. So, assume that the result is true for all $m < n$, and consider the regular row $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$. Since R is an EM-Hermite ring, $a_i = dc_i$, $1 \leq i < n$, and $(c_1, c_2, \cdots, c_{n-1}) \nsubseteq Z(R)$, and so, the regular row $\begin{bmatrix} c_1 & c_2 & \cdots & c_{n-1} \end{bmatrix}$ can be extended to an $(n-1) \times (n-1)$ regular matrix C. Again, since R is an EM-Hermite ring, $a_n = k\alpha$, $d = k\beta$, with $\alpha t + \beta s = r \in reg(R)$. Note that if $wk = 0$, then $w \in Ann(a_1, a_2, \cdots, a_n) = \{0\}$, and hence we have $k \in reg(R)$. Thus $a_n t + ds = k\alpha t + k\beta s = kr \in reg(R)$. Now consider the matrix,

$$B = \begin{bmatrix} d & 0 & a_n \\ 0 & I_{n-2} & 0 \\ -t & 0 & s \end{bmatrix}.$$

Then $\det(B) = kr \in reg(R)$, and the $n \times n$ matrix

$$A = B \begin{bmatrix} C & 0 \\ 0 & 1 \end{bmatrix}$$

is regular and has first row $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$. □

Corollary 4.7. *If R is an EM-Hermite ring, then any regular column can be completed to a regular square matrix by adding a suitable number of columns.*

Proof. Just take transpose, and the result follows immediately by the previous Theorem. □

Corollary 4.8. *If R is an EM-Hermite ring, then any unimodular row can be completed to a regular square matrix by adding a suitable number of rows.*

Note that in the ring $\mathbb{Z}_4(+)\mathbb{Z}_4$ any regular row is extendable to a regular matrix, being a finite Hermite ring, although it is not an EM-Hermite ring.

5. Applications to finitely presented modules

In this section, we relate EM-Hermite rings to weakly semi-Steinitz rings, and characterize the case at which every finitely generated R-module with finite free resolution of length 1 is free.

An R-module M satisfies property P if any two maximal independent subsets of M have the same cardinality. It was shown in [2] that every free R-module satisfies property P if and only if whenever $a_1, \dots, a_n \in R$ such that $Ann_R(a_1, \dots, a_n) = \{0\}$, then the row $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ can be completed to a square regular matrix.

A ring R is called a weakly semi-Steinitz ring if every finite independent subset of a finitely generated free R-module can be extended to a basis. The following two propositions characterize weakly semi-Steinitz rings, see [2] and [12].

Proposition 5.1. *The following statements are equivalent:*

(1) *R is a weakly semi-Steinitz ring.*
(2) *R is Hermite and every finitely generated proper ideal of R has non-zero annihilator.*
(3) *Every finitely generated proper ideal of R has non-zero annihilator and any finitely generated stably free R-module is a direct sum of cyclic modules.*
(4) *For each $n \geq 1$, every linearly independent element of R^n can be extended to a basis of R^n.*
(5) *$reg(R) = U(R)$ and every free R-module satisfies property P.*

Proposition 5.2. *Let R be a Noetherian ring. Then R is a weakly semi-Steinitz ring if and only if $reg(R) = U(R)$. If in addition, R is reduced, then R is a weakly semi-Steinitz ring if and only if R is a finite direct product of fields.*

We now give extra two characterizations of weakly semi-Steinitz rings.

Theorem 5.3. *R is a weakly semi-Steinitz ring if and only if whenever $a_1, \dots, a_n \in R$ such that $Ann_R(a_1, \dots, a_n) = \{0\}$, then the row $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ can be completed to a square invertible matrix.*

Proof. Assume R is a weakly semi-Steinitz ring and assume that $a_1, \dots, a_n \in R$ such that $Ann_R(a_1, \dots, a_n) = \{0\}$. Then $\overline{x}_1 = (a_1, \dots, a_n) \in R^n$ is linearly independent, and so R^n has a basis $\{\overline{x}_1, \dots, \overline{x}_n\}$. Let $A = \begin{bmatrix} \overline{x}_1 \\ \vdots \\ \overline{x}_n \end{bmatrix}$. There exist $c_{ij} \in R$ such that $\sum_{j=1}^{n} c_{ij}\overline{x}_j = \overline{e}_i$ for $i = 1, 2, \dots, n$, where $\{\overline{e}_1, \dots, \overline{e}_n\}$ is the standard basis for R^n. Let $C = [c_{ij}]$. Then $CA = I_n$. Thus, A is a regular matrix, with the ideal $(\det(A))$ is non-proper. Thus A is invertible.

Conversely, it is clear that R is Hermite. Assume that $a_1, \ldots, a_n \in R$ such that $Ann_R(a_1, \ldots, a_n) = \{0\}$. Let $\overline{x} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$. Then there exists an invertible $n \times n$ matrix $A = \begin{bmatrix} \overline{x}_1 \\ \vdots \\ \overline{x}_n \end{bmatrix}$. But $\det(A) \in \sum_{i=1}^{n} a_i R \cap U(R)$. Thus, the ideal $(a_1, \ldots, a_n)$ is non-proper, and R is a weakly semi-Steinitz ring. □

Recall that a finitely generated R-module P is said to have finite free resolution of length 1 if we have the short exact sequence

$$0 \longrightarrow R^m \stackrel{\alpha}{\longrightarrow} R^n \longrightarrow P \longrightarrow 0.$$

If the sequence splits, then P is a finitely generated stably free module.

Theorem 5.4. *R is a weakly semi-Steinitz ring if and only if every finitely generated R-module with finite free resolution of length 1 is free.*

Proof. Assume that R is a weakly semi-Steinitz ring, and consider the short exact sequence

$$0 \longrightarrow R^m \stackrel{\alpha}{\longrightarrow} R^n \longrightarrow P \longrightarrow 0.$$

If $\{\overline{a}_i\}_{i=1}^m$ is a basis for R^m, then $\{\alpha(\overline{a}_i)\}_{i=1}^m$ is a linear independent subset of the weakly semi-Steinitz ring R^n and so it can be extended to a basis $\{\alpha(\overline{a}_i)\}_{i=1}^m \cup \{\overline{b}_i\}_{i=1}^{n-m}$. Now, define the R-module homomorphism $T : R^n \longrightarrow R^m$ such that $T(\alpha(\overline{a}_i)) = \overline{a}_i$, and $T(\overline{b}_i) = 0$. Then $T \circ \alpha = Id_{R^m}$, and so, the exact sequence splits. Thus P is a finitely generated stably free R-module, and hence it is free, since R is Hermite.

Conversely, it is clear that R is Hermite. Assume $a_1, \ldots, a_n \in R$ such that $Ann_R(a_1, \ldots, a_n) = \{0\}$. Then $\overline{x} = (a_1, \ldots, a_n) \in R^n$ is linearly independent, and so $\alpha : R \longrightarrow R^n$ defined by $\alpha(r) = r\overline{x}$ is an injective R-homomorphism. Thus the sequence

$$0 \longrightarrow R \stackrel{\alpha}{\longrightarrow} R^n \longrightarrow R^n / \operatorname{Im} \alpha \longrightarrow 0$$

is short exact, and so, $R^n / \operatorname{Im}\ \alpha$ is a free R-module. Thus there exists an R-homomorphism $\beta : R^n \longrightarrow R$ such that $\beta \circ \alpha = Id_R$, and so, $\beta \circ \alpha(1) = 1$, and hence

$$1 = M(\beta)M(\alpha)(1) = \begin{bmatrix} \beta_1 & \cdots & \beta_n \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} (1),$$

where $M(\beta)$and $M(\alpha)$ are the corresponding matrices for β and α respectively. Therefore, the ideal $(a_1, \ldots, a_n)$ is non-proper.

Thus R is a weakly semi-Steinitz ring. □

It follows by Theorem 4.6 that if R is an EM-Hermite ring, then every free R-module satisfies property P. Thus we have the following result:

Theorem 5.5. *If R is an EM-Hermite ring, then $T(R)$ is a weakly semi-Steinitz ring.*

It is clear that $\mathbb{Z}_4(+)\mathbb{Z}_4$ is a weakly semi-Steinitz ring that is not EM-Hermite, while $\mathbb{Z}$ is a K-Hermite ring that has $\mathbb{Z}$-modules of finite resolution that are not free.

References

[1] E. Abuosba and M. Ghanem, *Annihilating content in polynomial and power series rings*, J. Korean Math. Soc., 56(5) (2019), 1403-1418.

[2] A. Bouanane and F. Kourki, *On weakly semi-Steinitz rings*, Commutative Ring Theory, Lecture Notes in Pure and Appl. Math., Dekker, New York, 185 (1997), 131-139.

[3] W. C. Brown, Matrices over Commutative Rings, Monographs and Textbooks in Pure and Applied Mathematics, 169, Marcel Dekker, Inc., New York, 1993.

[4] S. Endo, *Note on p.p. rings*, A supplement to Hattori's paper, Nagoya Math. J., 17 (1960), 167-170.

[5] L. Gillman and M. Henriksen, *Rings of continuous functions in which every finitely generated ideal is principal*, Trans. Amer. Math. Soc., 82 (1956), 366-391.

[6] L. Gillman and M. Henriksen, *Some remarks about elementary divisor rings*, Trans. Amer. Math. Soc., 82 (1956), 362-365.

[7] M. Henriksen, *Some remarks on elementary divisor rings II*, Michigan Math. J., 3(2) (1955), 159-163.

[8] I. Kaplansky, *Elementary divisors and modules*, Trans. Amer. Math. Soc., 66 (1949), 464-491.

[9] I. Kaplansky, Commutative Rings, Revised Edition, The University of Chicago Press, Chicago, 1974.

[10] T. Y. Lam, Serre's Problem on Projective Modules, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2006.

[11] M. D. Larsen, W. J. Lewis and T. S. Shores, *Elementary divisor rings and finitely presented modules*, Trans. Amer. Math. Soc., 187 (1974), 231-248.

[12] B. Nashier and W. Nichols, *On Steinitz properties*, Arch. Math. (Basel), 57(3) (1991), 247-253.

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