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Some properties of Sadik transform and its applications of fractional-order dynamical systems in control theory

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Abstract

In this paper, we study some new properties of Sadik transform such as integration, time delay, initial value theorem, and final value theorem. Moreover, we prove the theorem of Sadik transform for Caputo fractional derivative and we also establish sufficient conditions for the existence of the Sadik transform of Caputo fractional derivatives. At the end, the fractional-order dynamical systems in control theory as application of this transform is discussed, in addition, some numerical examples to justify our results.

Keywords: A fractional differential equation Caputo fractional derivative Integral transforms Sadik transform .
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1. Introduction

The integral transformation method is excessively used to solve different kinds of differential equations in a simple way. As integral transforms converts differential equations into algebraic equation algebraic equations are more simple than differential equations. In the literature, there are several integral transforms

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and all of them are acceptable to resolve numerous types differential equations. Recently some new integral transforms were introduced, see [3, 7, 8, 18]. Recently Shaikh [14, 15, 16, 17], proposed a new integral transform that known as Sadik transform. This transform is an unification of some famous transforms such as Laplace transform, Sumudu transform, Kamal transform, Tarig and Laplace- Carson transform and Elzaki transform. For example, Shaikh in [15], presented some properties of this transform, like the existing theorem of Sadik transform and duality theorem. Further, the author proved that above mentioned transforms are particular cases of Sadik transform. Shaikh in [16], proved properties of Sadik transform for derivative of functions, shifting theorem for Sadik transform. Also, the author obtained transfer function of dynamical system in control theory using Sadik transform. Moreover, he solved some applications in control theory by Sadik transform. At the outsight, integral transform method is useful and effective tool for solving fractional differential equations. But it is also true that all types of fractional differential equations are not solvable by integral transform technique see [4, 5, 13] and the references therein. Abhale and Pawar in [10], proved some fundamental properties of Sadik transform and used these properties of Sadik transform to solve first order and second order ordinary differential equations.

On the other hand, the fractional calculus is a generalization of classical differentiation and integration into non-integer order. Some fundamental definitions of fractional derivatives were given by Riemann-Liouville, Hadamard, Caputo, Hilfer, Liouville-Caputo, Grünwald-Letnikov, Riesz, Coimbra and Weyl. The fractional derivatives describe the property of memory and heredity of many materials. see [6, 9].

Fractional ordinary and partial differential equations, as a generalization of classical integer order differential equations. Fractional differential equations have enabled the investigation of multiple phenomena such as diffusion processes, electrodynamics, fluid flow, elasticity and it increasingly used to model problems in biology, viscoelasticity, fluid mechanics, physics, engineering, and others applications [6, 9, 11, 12].

In this paper, we introduce Sadik transform of fractional order (Caputo derivative operator) and some new properties of this transform such as time delay, initial and final value theorems are proved. Further, we established a sufficient condition for the existence of Sadik transform of Caputo fractional derivatives and have solved Caputo fractional differential equations. Finally, the time-domain analysis of dynamical systems involving fractional-order is presented to solving systems of control theory.

2. preliminaries:

In this section, we recall some notions, definitions and lemmas that used through this paper. Let $[a,b]\subset\mathbb{R}^+$ and $\mathcal{C}[a,b]$ be the space of all continuous functions $\varphi:[a,b]\longrightarrow\mathbb{R}$ with the norm $\|\varphi\|_\infty=\max\{|\varphi(t)|: t\in[a,b]\}$ for any $\varphi\in\mathcal{C}[a,b]$. Denote $L^1[a,b]$ the labesgue integrable functions with the norm $\|\varphi\|_{L^1}=\int_a^b|\varphi(t)|\,dt<\infty$.

Definition 2.1. *[6] Let φ be a locally integrable function on $[0,+\infty)$. The Riemann-Liouville fractional integral of order $\gamma>0$ of function φ is given by*

$$I_{0^+}^{\gamma}\varphi(t)=\begin{cases}\frac{1}{\Gamma(\gamma)}\int_0^t(t-\tau)^{\gamma-1}\varphi(\tau)d\tau, & \gamma>0,\\ \varphi(t), & \gamma=0.\end{cases}$$

where $\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt$ is a Gamma function of Euler, $z\in\mathbb{C}$.

Definition 2.2. *[6] Let $n-1<\gamma<n\in\mathbb{N}$, and $\varphi(t)$ has absolutely continuous derivatives up to order $(n-1)$. Then the left sided Riemann-Liouville fractional derivative of order γ of φ is defined by*

$$\begin{aligned}D_{0^+}^{\gamma}\varphi(t) &= \left(\frac{d}{dt}\right)^n I_{0^+}^{n-\gamma}\varphi(t)\\ &= \left(\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\gamma)}\int_0^t(t-\tau)^{n-\gamma-1}\varphi(\tau)d\tau,\end{aligned}$$

where $n=[\gamma]+1$ and $[\gamma]$ denotes the integer part of the real number γ.

Definition 2.3. *[6] The left sided Caputo derivative of fractional order γ $(n-1<\gamma<n\in\mathbb{N})$ is given by*

$$^{c}D_{0^+}^{\gamma}\varphi(t)=I_{0^+}^{n-\gamma}\frac{d^n}{dt^n}\varphi(t)=\frac{1}{\Gamma(n-\gamma)}\int_0^t(t-\tau)^{n-\gamma-1}\varphi^{(n)}(\tau)d\tau,$$

where the function $\varphi(t)$ has absolutely continuous derivatives up to order $(n-1)$. In particular, if $0<\gamma<1$, we have

$$^{c}D_{0^+}^{\gamma}\varphi(t)=I_{0^+}^{1-\gamma}\frac{d}{dt}\varphi(t)=\frac{1}{\Gamma(1-\gamma)}\int_0^t(t-\tau)^{-\gamma}\varphi'(\tau)d\tau.$$

Definition 2.4. *[15] (Sadik transform) Assume that φ is piecewise continuous on the interval $[0,A]$ for any $A>0$ and satisfies $|\varphi(t)|\leq Ke^{at}$ when $t\geq M$, for any real constant a, and some positive constants K and M. Then the Sadik transform of $\varphi(t)$ is defined by*

$$\Phi(v,\alpha,\beta)=\mathcal{S}[\varphi(t)]=\frac{1}{v^{\beta}}\int_0^{\infty}e^{-tv^{\alpha}}\varphi(t)dt,$$

where v is complex variable, α is any non zero real number, and β is any real number.

Definition 2.5. *[12] (Mittag-Leffer function) Let $p,q\in\mathbb{C}$, $Re(p)>0$, $Re(q)>0$. Then the Mittag-Leffer function of one variable is given by*

$$E_p(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(pk+1)}.$$

The Mittag-leffer function of two variables is given by

$$E_{p,q}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(pk+q)}.$$

Properties: [15] Let $\Phi(v,\alpha,\beta)$ be a Sadik transform of $\varphi(t)$, i.e. $\mathcal{S}[\varphi(t)]=\Phi(v,\alpha,\beta)$. Then

1. If $\varphi(t)=1$, then $\mathcal{S}[1]=\frac{1}{v^{\alpha+\beta}}$.
2. If $\varphi(t)=t^n$, then $\mathcal{S}[t^n]=\frac{n!}{v^{n\alpha+(\alpha+\beta)}}$.
3. If $\varphi(t)=e^{at}$, then $\mathcal{S}[e^{at}]=\frac{v^{-\beta}}{v^{\alpha}-a}$.
4. If $\varphi(t)=\sin(at)$, then $\mathcal{S}[\sin(at)]=\frac{av^{-\beta}}{v^{2\alpha}+a^2}$.

Lemma 2.6. *[15] Let φ_1 and φ_2 two functions belong to $L^1[a,b]$ the usuall convolution product is given by*

$$(\varphi_1*\varphi_2)(t)=\int_{-\infty}^{\infty}\varphi_1(\tau)\varphi_2(t-\tau)d\tau,\ t>0.$$

Lemma 2.7. *[15] Let $\Phi_1(v,\alpha,\beta)$ and $\Phi_2(v,\alpha,\beta)$ are Sadik Transforms of $\varphi_1(t)$ and $\varphi_2(t)$ respectively, and $(\varphi_1*\varphi_2)(t)$ is a convolution of $\varphi_1(t)$ and $\varphi_2(t)$. Then, Sadik transform of $(\varphi_1*\varphi_2)(t)$ is*

$$\mathcal{S}[(\varphi_1*\varphi_2)(t)]=v^{\beta}\Phi_1(v,\alpha,\beta).\Phi_2(v,\alpha,\beta),$$

where $*$ denotes convolution.

Lemma 2.8. *[10] Let $\mathcal{S}[\varphi(t)]=\Phi(v,\alpha,\beta)$. Then*

$$\mathcal{S}[t^n\varphi(t)]=(-1)^n\left(\frac{1}{\alpha v^{\alpha-1}}\frac{d}{dv}+\frac{\beta}{\alpha v^{\alpha}}\right)^n\Phi(v,\alpha,\beta).$$

3. Main results

In this section, we prove a new some properties of Sadik transform that is, the Mittag-Liffler function, the integration, the time delay, the initial and final value theorems. Moreover, we demonstrate the Sadik transform of Caputo fractional differential equations, and the existence theorem of Sadik transform.

Theorem 3.1. *Let $\Phi(v,\alpha,\beta)$ is a Sadik transform of $\varphi(t)$ and $\varphi(t)$, $\varphi'(t)$, $\varphi''(t)$,....,$\varphi^{(n-1)}(t)$ are continuous on $[0,\infty)$. Then*

$$\mathcal{S}[\varphi^{(n)}(t)] = v^{n\alpha}\Phi(v,\alpha,\beta) - \sum_{k=0}^{n-1} v^{k\alpha-\beta}\varphi^{(n-1-k)}(0).$$

Proof. For the first order derivative of $\varphi(t)$, we starting with the definition of Sadik transform,

$$\mathcal{S}[\varphi'(t)] = v^{-\beta}\int_0^\infty v^{-tv^\alpha}\varphi'(t)dt.$$

By using integration of parts, we obtain,

$$= v^{-\beta}\left[v^{-tv^\alpha}\varphi(t)\Big|_0^\infty - \int_0^\infty v^{-tv^\alpha}(-v^\alpha)\varphi(t)dt\right],$$

Assuming $Re(v^\alpha) > 0$, we get

$$\begin{aligned}\mathcal{S}[\varphi'(t)] &= -v^{-\beta}\varphi(0) + v^\alpha\frac{1}{v^\beta}\int_0^\infty v^{-tv^\alpha}\varphi(t)dt\\ &= v^\alpha\Phi(v,\alpha,\beta) - v^{-\beta}\varphi(0). \qquad (1)\end{aligned}$$

By using a minner way of second order derivative of $\varphi(t)$ we get

$$\begin{aligned}\mathcal{S}[\varphi''(t)] &= v^{-\beta}\int_0^\infty v^{-tv^\alpha}\varphi''(t)dt\\ &= v^{-\beta}\left[v^{-tv^\alpha}\varphi'(t)\Big|_0^\infty - \int_0^\infty v^{-tv^\alpha}(-v^\alpha)\varphi'(t)dt\right]\\ &= -v^{-\beta}\varphi'(0) + v^\alpha v^{-\beta}\int_0^\infty v^{-tv^\alpha}\varphi'(t)dt\\ &= -v^{-\beta}\varphi'(0) + v^\alpha\mathcal{S}[\varphi'(x)],\end{aligned}$$

from Eq.(1), we get

$$\begin{aligned}\mathcal{S}[\varphi''(t)] &= -v^{-\beta}\varphi'(0) + v^\alpha\left[v^\alpha U(v,\alpha,\beta) - v^{-\beta}\varphi(0)\right]\\ &= v^{2\alpha}U(v,\alpha,\beta) - v^{\alpha-\beta}\varphi(0) - v^{-\beta}\varphi'(0). \qquad (2)\end{aligned}$$

Similarly, for third order partial derivative of $\varphi(t)$ and using Eq.(2), we get

$$\begin{aligned}\mathcal{S}[\varphi'''(t)] &= v^{3\alpha}U(v,\alpha,\beta) - v^{2\alpha-\beta}\varphi(0) - v^{\alpha-\beta}\varphi'(0) - v^{-\beta}\varphi''(0)\\ &= v^{n\alpha}U(v,\alpha,\beta) - \sum_{k=0}^{n-1} v^{k\alpha-\beta}\varphi^{(n-k-1)}(0),\ n=3, k=0,1,2.\end{aligned}$$

Continually in the general, we get

$$\mathcal{S}[\varphi^{(n)}(t)] = v^{n\alpha}U(v,\alpha,\beta) - \sum_{k=0}^{n-1} v^{k\alpha-\beta}\varphi^{(n-k-1)}(0).$$

□

Lemma 3.2. *Let* $\varphi(t) = t^{pm+q-1}E_{p,q}^{(m)}(\pm at^p)$*. Then the Sadik Transform of* $\varphi(t)$ *is given by*

$$\frac{1}{v^\beta}\int_0^\infty e^{-v^\alpha t}t^{pm+q-1}E_{p,q}^{(m)}(\pm at^p)dt = \frac{m!v^{\alpha p-(\alpha q+\beta)}}{(v^{\alpha p}\mp a)^{m+1}},$$

where $\alpha,\beta \in \mathbb{C}, \mathcal{R}e(p) > 0,\ \mathcal{R}e(q) > 0,\ \mathcal{R}e(v) > |a|^{\frac{1}{\mathcal{R}e(\alpha p)}}$ *and* $E_{p,q}^{(m)}(t) = \frac{d^m}{dt^m}E_{p,q}(t)$.

Proof. In view of definitions of Mittag-leffler function and Sadik transform with the help of classical calculus, we have

$$\begin{aligned}
&\frac{1}{v^\beta}\int_0^\infty e^{-v^\alpha t}t^{pm+q-1}E_{p,q}^{(m)}(\pm at^p)dt\\
=\ &\frac{1}{v^\beta}\int_0^\infty e^{-v^\alpha t}t^{pm+q-1}\frac{d^m}{dt^m}\sum_{k=0}^\infty\frac{(\pm at^p)^k}{\Gamma(pk+q)}dt\\
=\ &\frac{1}{v^\beta}\int_0^\infty e^{-v^\alpha t}t^{pm+q-1}\sum_{k=0}^\infty\frac{(k+m)!(\pm a)^k t^{pk}}{k!\Gamma(pk+pm+q)}dt\\
=\ &\sum_{k=0}^\infty\frac{(k+m)!(\pm a)^k}{k!\Gamma(p\,(k+m)+q)}\frac{1}{v^\beta}\int_0^\infty e^{-v^\alpha t}t^{p(m+k)+q-1}dt\\
=\ &\frac{v^{-\alpha(pm+q)}}{v^\beta}\sum_{k=0}^\infty\frac{(k+m)!}{k!}\left(\frac{\pm a}{v^{\alpha p}}\right)^k\\
=\ &\frac{v^{-\alpha(pm+q)}}{v^\beta}\sum_{k=0}^\infty(k+m).......(k+1)\left(\frac{\pm a}{v^{\alpha p}}\right)^k.
\end{aligned}$$

Now let $k = k - m$, it follows that

$$\begin{aligned}
&\frac{1}{v^\beta}\int_0^\infty e^{-v^\alpha t}t^{pm+q-1}E_{p,q}^{(m)}(\pm at^p)dt\\
=\ &\frac{v^{-\alpha(pm+q)}}{v^\beta}\sum_{k=m}^\infty(k)(k-1)...(k-m-1)\left(\frac{\pm a}{v^{\alpha p}}\right)^k\\
=\ &v^{-\alpha pm-\alpha q-\beta}\frac{d^m}{da^m}\sum_{k=m}^\infty\left(\frac{\pm a}{v^{\alpha p}}\right)^k\\
=\ &v^{-\alpha pm-\alpha q-\beta}\frac{d^m}{da^m}\sum_{k=m}^\infty\left(\frac{1}{1\mp\frac{a}{v^{\alpha p}}}\right)\\
=\ &v^{-\alpha pm-\alpha q-\beta}\frac{m!}{\left(1\mp\frac{a}{v^{\alpha p}}\right)^{m+1}}\\
=\ &\frac{m!v^{\alpha p-(\alpha q+\beta)}}{(v^{\alpha p}\mp a)^{m+1}}.
\end{aligned}\tag{3}$$

□

Lemma 3.3. *(Integration) Let* $\mathcal{S}[\varphi(t)] = \Phi(v,\alpha,\beta)$ *is a Sadik transform of* $\varphi(t)$*. Then Sadik transform of integration of* $\varphi(t)$ *is*

$$\mathcal{S}\left[\int_0^t\varphi(\tau)d\tau\right] = \frac{1}{v^\alpha}\Phi(v,\alpha,\beta).$$

Proof. According to the definition of Sadik transform, we have

$$\mathcal{S}\left[\int_0^t\varphi(\tau)d\tau\right] = \frac{1}{v^\beta}\int_0^\infty e^{-tv^\alpha}\left[\int_0^t\varphi(\tau)d\tau\right]dt.$$

Then by the integration of parts, we get

$$\begin{aligned}
\mathcal{S}\left[\int_0^t \varphi(\tau)d\tau\right] &= \frac{1}{v^\beta}\left[\int_0^t \varphi(\tau)d\tau \frac{e^{-tv^\alpha}}{v^\alpha}\Big|_0^\infty - \int_0^\infty \frac{e^{-tv^\alpha}}{-v^\alpha}\right]\varphi(t)dt \\
&= \frac{1}{v^\alpha}\left[\frac{1}{v^\beta}\int_0^\infty e^{-tv^\alpha}\varphi(t)dt\right] \\
&= \frac{1}{v^\alpha}\Phi(v,\alpha,\beta).
\end{aligned}$$

□

Theorem 3.4. *Let $n-1<\gamma<n$, $(n=[\gamma]+1)$ and $\varphi(t), \varphi'(t),\varphi''(t), \ldots.., \varphi^{(n-1)}(t)$ are continuous on $[0,\infty)$ and of exponential order, while ${}^cD_{0+}^\gamma\varphi(t)$ is piecewise continuous on $[0,\infty)$. Then Sadik transform of Caputo fractional derivative of order γ of function φ is given by*

$$\mathcal{S}[{}^cD_{0+}^\gamma\varphi(t)] = v^{\gamma\alpha}\Phi(v,\alpha,\beta) - \sum_{k=0}^{n-1} v^{(\gamma-n+k)\alpha-\beta}\varphi^{(n-1-k)}(0^+).$$

Proof. In light of Definitions 2.3, 2.4, then using Theorem 3.1, and property 2, we find that

$$\begin{aligned}
\mathcal{S}[{}^cD_{0+}^\gamma\varphi(t)] &= \frac{1}{v^\beta}\int_0^\infty e^{-tv^\alpha}\left[{}^cD_{0+}^\gamma\varphi(t)\right]dt \\
&= \frac{1}{v^\beta}\int_0^\infty e^{-tv^\alpha}\left[\frac{1}{\Gamma(n-\gamma)}\int_0^t (t-\tau)^{n-\gamma-1}\varphi^{(n)}(\tau)d\tau\right]dt \\
&= \frac{1}{\Gamma(n-\gamma)}\frac{1}{v^\beta}\int_0^\infty\int_\tau^\infty e^{-tv^\alpha}(t-\tau)^{n-\gamma-1}\varphi^{(n)}(\tau)dtd\tau \\
&= \frac{1}{\Gamma(n-\gamma)}\frac{1}{v^\beta}\int_0^\infty \varphi^{(n)}(\tau)\int_0^\infty e^{-v^\alpha(u+\tau)}u^{n-\gamma-1}dud\tau \\
&= \frac{1}{\Gamma(n-\gamma)}\frac{1}{v^\beta}\int_0^\infty e^{-\tau v^\alpha}\varphi^{(n)}(\tau)d\tau\int_0^\infty e^{-uv^\alpha}u^{n-\gamma-1}du \\
&= \frac{1}{\Gamma(n-\gamma)}\int_0^\infty e^{-\tau v^\alpha}\varphi^{(n)}(\tau)d\tau\frac{1}{v^\beta}\int_0^\infty e^{-uv^\alpha}u^{n-\gamma-1}du \\
&= \frac{v^\beta}{\Gamma(n-\gamma)}\mathcal{S}\left[\varphi^{(n)}(t)\right]\mathcal{S}\left[t^{n-\gamma-1}\right] \\
&= \frac{v^\beta}{\Gamma(n-\gamma)}\mathcal{S}\left[\varphi^{(n)}(t)\right]\frac{\Gamma(n-\gamma)}{v^{(n-\gamma-1)\alpha+(\alpha+\beta)}} \\
&= \frac{v^\beta}{v^{(n-\gamma)\alpha+\beta}}\left[v^{n\alpha}\Phi(v,\alpha,\beta) - \sum_{k=0}^{n-1} v^{k\alpha-\beta}\varphi^{(n-1-k)}(0)\right] \\
&= v^{\alpha\gamma}\Phi(v,\alpha,\beta) - \sum_{k=0}^{n-1} v^{(\gamma+k-n)\alpha-\beta}\varphi^{(n-1-k)}(0).
\end{aligned}$$

□

Theorem 3.5. *Assume that a linear Caputo fractional differential equation*

$$ {}^cD_{0+}^\gamma u(t) = \varphi(t), \quad 0<\gamma<1, \tag{4}$$

with intial condition

$$u(0) = u_0, \tag{5}$$

has a unique continuous solution

$$u(t) = u_0 + \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} \varphi(\tau) d\tau, \tag{6}$$

if $\varphi(t)$ is continuous on $[0, \infty)$ and exponentially bounded, then $u(t)$ and ${}^cD^{\gamma}_{0+}u(t)$ are both exponentially bounded, thus their Sadik transform exists.

Proof. Since $\varphi(t)$ is exponentially bounded, there exist two positive constants M, σ and enough large T such that $\|\varphi(t)\| \leq Me^{\sigma t}$ for all $t \geq T$. It is easy to see that Eq.(4) is equivalent to the Volterra integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} \varphi(\tau) d\tau, \;\; 0 \leq t < \infty. \tag{7}$$

For $t \geq T$, Eq.(7) can be rewritten as

$$u(t) = u_0 + \frac{1}{\Gamma(\gamma)} \int_0^T (t-\tau)^{\gamma-1} \varphi(\tau) d\tau + \frac{1}{\Gamma(\gamma)} \int_T^t (t-\tau)^{\gamma-1} \varphi(\tau) d\tau.$$

In view of assumptions, $u(t)$ is unique continuous solution on $[0, \infty)$, with $u(0) = u_0$, then $\varphi(t)$ is bounded on $[0, T]$, i.e. there exists a constant $k > 0$ such that $\|\varphi(t)\| \leq k$. Now, we have

$$\|u(t)\| \leq \|u_0\| + \frac{k}{\Gamma(\gamma)} \int_0^T (t-\tau)^{\gamma-1} d\tau + \frac{1}{\Gamma(\gamma)} \int_T^t (t-\tau)^{\gamma-1} \|\varphi(\tau)\| d\tau.$$

Multiply the last inequality by $e^{-\sigma t}$ then from fact that $e^{-\sigma t} \leq e^{-\sigma T}$, $e^{-\sigma t} \leq e^{-\sigma \tau}$, and $\|\varphi(t)\| \leq Me^{\sigma t}$ $(t \geq T)$, we obtain

$$\begin{aligned}
\|u(t)\| e^{-\sigma t} &\leq \|u_0\| e^{-\sigma t} + \frac{ke^{-\sigma t}}{\Gamma(\gamma)} \int_0^T (t-\tau)^{\gamma-1} d\tau + \frac{e^{-\sigma t}}{\Gamma(\gamma)} \int_T^t (t-\tau)^{\gamma-1} \|\varphi(\tau)\| d\tau \\
&\leq \|u_0\| e^{-\sigma T} + \frac{ke^{-\sigma T}}{\Gamma(\gamma+1)} [(t)^{\gamma} - (t-T)^{\gamma}] + \frac{M}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} e^{\sigma(\tau - t)} d\tau \\
&\leq \|u_0\| e^{-\sigma T} + \frac{ke^{-\sigma T}}{\Gamma(\gamma+1)} T^{\gamma} + \frac{M}{\Gamma(\gamma)} \int_0^t s^{\gamma-1} e^{-\sigma s} ds \\
&\leq \|u_0\| e^{-\sigma T} + \frac{ke^{-\sigma T}}{\Gamma(\gamma+1)} T^{\gamma} + \frac{M}{\Gamma(\gamma)} \int_0^{\infty} s^{\gamma-1} e^{-\sigma s} ds \\
&\leq \|u_0\| e^{-\sigma T} + \frac{ke^{-\sigma T}}{\Gamma(\gamma+1)} T^{\gamma} + \frac{M}{\sigma^{\gamma}}.
\end{aligned}$$

Denote

$$A = \|u_0\| e^{-\sigma T} + \frac{ke^{-\sigma T}}{\Gamma(\gamma+1)} T^{\gamma} + \frac{M}{\sigma^{\gamma}},$$

we get

$$\|u(t)\| \leq Ae^{\sigma t}, \;\; t \geq T.$$

□

From Eq.(4) and hypothesis of φ, we conclude that

$$\left\| {}^cD^{\gamma}_{0+}u(t) \right\| = \|\varphi(t)\| \leq Me^{\sigma t} \;\, t \geq T.$$

Applying Sadik transform on both sides of Eq.(4) and using Theorem 3.4, we have

$$v^{\alpha\gamma} U(v, \alpha, \beta) - v^{(\gamma-1)\alpha - \beta} u(0) = \Phi(v, \alpha, \beta).$$

Since $u(0)=u_0$, it follows

$$U(v,\alpha,\beta)=u_0\frac{1}{v^{\alpha+\beta}}+\frac{\Phi(v,\alpha,\beta)}{v^{\alpha\gamma}}.$$

Take the inverse of Sadik transform to both sides of the above equation, and using property 1, and Lemma 2.7, we get

$$\begin{aligned} u(t) &= u_0\mathcal{S}^{-1}\left[\frac{1}{v^{\alpha+\beta}}\right]+\mathcal{S}^{-1}\left[\frac{1}{v^{\alpha\gamma}}\Phi(v,\alpha,\beta)\right] \\ &= u_0+\mathcal{S}^{-1}\left[v^{\beta}\frac{1}{v^{\alpha\gamma+\beta}}\Phi(v,\alpha,\beta)\right] \\ &= u_0+\mathcal{S}^{-1}\left[v^{\beta}\frac{1}{v^{(\gamma-1)\alpha+\alpha+\beta}}\Phi(v,\alpha,\beta)\right] \\ &= u_0+(\varphi_1*\varphi)(t). \end{aligned} \tag{8}$$

Put $\Phi_1(v,\alpha,\beta):=\frac{1}{v^{(\gamma-1)\alpha+\alpha+\beta}}$, such that $\mathcal{S}^{-1}\left[\Phi_1(v,\alpha,\beta)\right]=\varphi_1(t)$ and $\mathcal{S}^{-1}[\Phi(v,\alpha,\beta)]=\varphi(t)$. Applying the inverse Sadik transform of $\Phi_1(v,\alpha,\beta)$, with using property 2, we find that

$$\mathcal{S}^{-1}\left[\Phi_1(v,\alpha,\beta)\right]=\mathcal{S}^{-1}\left[\frac{1}{v^{(\gamma-1)\alpha+\alpha+\beta}}\right]=\frac{t^{\gamma-1}}{\Gamma(\gamma)}=\varphi_1(t).$$

Therefore Eq.(8) becomes as follows

$$\begin{aligned} u(t) &= u_0+(\varphi_1*\varphi)(t) \\ &= u_0+\frac{1}{\Gamma(\gamma)}\int_0^t (t-\tau)^{\gamma-1}\varphi(\tau)d\tau. \end{aligned}$$

Theorem 3.6. *(Time Delay) Let $\Phi(v,\alpha,\beta)=\mathcal{S}[\varphi(t)]$. Then Sadik transform of time delay is given by*

$$\mathcal{S}\left[\varphi(t-a)\cdot\eta(t-a)\right]=e^{-av^{\alpha}}\Phi(v,\alpha,\beta),$$

where

$$\eta(t-a)=\begin{cases} 0, & t<a \\ 1, & t\geq a \end{cases}. \tag{9}$$

Proof. We prove by going back to the original definition of the Sadik transform

$$\mathcal{S}\left[\varphi(t-a)\cdot\eta(t-a)\right]=\frac{1}{v^{\beta}}\int_0^{\infty}e^{-tv^{\alpha}}\left[\varphi(t-a)\cdot\eta(t-a)\right]dt,$$

Changing the lower limit of the integral from 0 to a and drop the step function gives

$$\begin{aligned} \mathcal{S}\left[\varphi(t-a)\cdot\eta(t-a)\right] &= \frac{1}{v^{\beta}}\int_0^{a}e^{-tv^{\alpha}}\left[\varphi(t-a)\cdot\eta(t-a)\right]dt \\ &\quad +\frac{1}{v^{\beta}}\int_a^{\infty}e^{-tv^{\alpha}}\left[\varphi(t-a)\cdot\eta(t-a)\right]dt \\ &= \frac{1}{v^{\beta}}\int_a^{\infty}e^{-tv^{\alpha}}\varphi(t-a)dt. \end{aligned}$$

By change of variable $u=t-a$, it follows

$$\begin{aligned} \mathcal{S}\left[\varphi(t-a)\cdot\eta(t-a)\right] &= \frac{1}{v^{\beta}}\int_0^{\infty}e^{-(u+a)v^{\alpha}}\varphi(u)du \\ &= e^{-av^{\alpha}}\Phi(v,\alpha,\beta). \end{aligned}$$

□

Example 3.7. *The Sadik transform of $\eta(t-a)$ is $\frac{e^{-av^{\alpha}}}{v^{\alpha+\beta}}$. Indeed, from the definition of Sadik transform and the relation Eq.(9), we have*

$$\mathcal{S}[\eta(t-a)] = \frac{1}{v^{\beta}}\int_{0}^{a} e^{-tv^{\alpha}}\eta(t-a)dt + \frac{1}{v^{\beta}}\int_{a}^{\infty} e^{-tv^{\alpha}}\eta(t-a)dt = \frac{e^{-av^{\alpha}}}{v^{\alpha+\beta}}.$$

Setting $\varphi(t-a) = 1$. Then

$$\mathcal{S}\left[\varphi(t-a)\cdot\eta(t-a)\right] = \mathcal{S}\left[\eta(t-a)\right] = \frac{e^{-av^{\alpha}}}{v^{\alpha+\beta}} = e^{-av^{\alpha}}\Phi(v,\alpha,\beta).$$

This satisfies Theorem 3.6.

Example 3.8. *Let $\varphi(t-a) = t-a$. Then, from Lemma 2.8, we have*

$$\mathcal{S}[t\varphi(t)] = -\left(\frac{1}{\alpha v^{\alpha-1}}\frac{d}{dv} + \frac{\beta}{\alpha v^{\alpha}}\right)\Phi(v,\alpha,\beta).$$

Hence, with using Example 3.7, we get

$$\begin{aligned}
&\mathcal{S}\left[\varphi(t-a).\eta(t-a)\right] \\
= & -\left(\frac{1}{\alpha v^{\alpha-1}}\frac{d}{dv} + \frac{\beta}{\alpha v^{\alpha}}\right)\frac{e^{-av^{\alpha}}}{v^{\alpha+\beta}} - a\frac{e^{-av^{\alpha}}}{v^{\alpha+\beta}} \\
= & -\frac{1}{\alpha v^{\alpha-1}}\left[\frac{(-a\alpha v^{\alpha-1} - (\alpha+\beta)\,v^{-1})e^{-av^{\alpha}}}{v^{\alpha+\beta}}\right] + \frac{\beta}{\alpha}\frac{e^{-av^{\alpha}}}{v^{2\alpha+\beta}} - a\frac{e^{-av^{\alpha}}}{v^{\alpha+\beta}} \\
= & \frac{1}{v^{2\alpha+\beta}}e^{-av^{\alpha}} \\
= & e^{-av^{\alpha}}\Phi(v,\alpha,\beta).
\end{aligned}$$

Theorem 3.9. *(Initial Value Theorem) Let $\Phi(v,\alpha,\beta) = \mathcal{S}[\varphi(t)]$. Then the sadik transform of initial value given by*

$$\lim_{v^{\alpha}\to\infty}\left[v^{\alpha}\Phi(v,\alpha,\beta)\right] = v^{-\beta}\varphi(0^{+}).$$

Proof. We first start with the derivative rule:

$$\mathcal{S}\left[\frac{d\varphi(t)}{dt}\right] = v^{\alpha}\Phi(v,\alpha,\beta) - v^{-\beta}\varphi(0^{-}).$$

From the definition of Sadik transform, with splitting the integral into two parts:

$$\begin{aligned}
v^{\alpha}\Phi(v,\alpha,\beta) - v^{-\beta}\varphi(0^{-}) &= \frac{1}{v^{\beta}}\int_{0^{-}}^{\infty}\frac{d\varphi(t)}{dt}e^{-tv^{\alpha}}dt = \frac{1}{v^{\beta}}\int_{0^{-}}^{\infty}\varphi'(t)e^{-tv^{\alpha}}dt \\
&= \frac{1}{v^{\beta}}\int_{0^{-}}^{0^{+}}\varphi'(t)e^{-tv^{\alpha}}dt + \frac{1}{v^{\beta}}\int_{0^{+}}^{\infty}\varphi'(t)e^{-tv^{\alpha}}dt.
\end{aligned}$$

Take the limit as $v^{\alpha}\to\infty$,

$$\begin{aligned}
&\lim_{v^{\alpha}\to\infty}\left[v^{\alpha}\Phi(v,\alpha,\beta) - v^{-\beta}\varphi(0^{-})\right] \\
= &\lim_{v^{\alpha}\to\infty}\left[\frac{1}{v^{\beta}}\int_{0^{-}}^{0^{+}}\varphi'(t)e^{-tv^{\alpha}}dt + \frac{1}{v^{\beta}}\int_{0^{+}}^{\infty}\varphi'(t)e^{-tv^{\alpha}}dt\right].
\end{aligned}$$

Several facilitations are as follows:
In the expression

$$\lim_{v^{\alpha}\to\infty}\left[v^{\alpha}\Phi(v,\alpha,\beta) - v^{-\beta}\varphi(0^{-})\right],$$

we can take the second term out of the limit, since it doesn't depend on v^α.

In the expression

$$\lim_{v^\alpha\to\infty}\left[\frac{1}{v^\beta}\int_{0^-}^{0^+}\varphi'(t)e^{-tv^\alpha}dt+\frac{1}{v^\beta}\int_{0^+}^{\infty}\varphi'(t)e^{-tv^\alpha}dt\right],$$

we can take the first term out of the limit for the same reason, and when $v^\alpha\to\infty$ the exponential term in the second term goes to zero. Hence

$$\begin{aligned}\left(\lim_{v^\alpha\to\infty}[v^\alpha\Phi(v,\alpha,\beta)]\right)-v^{-\beta}\varphi(0^-) &= \frac{1}{v^\beta}\int_{0^-}^{0^+}\varphi'(t)dt+\frac{1}{v^\beta}\int_{0^+}^{\infty}\varphi'(t)(0)dt\\ &= \frac{1}{v^\beta}\int_{0^-}^{0^+}\varphi'(t)dt\\ &= v^{-\beta}\varphi(0^+)-v^{-\beta}\varphi(0^-).\end{aligned}$$

This gives

$$\lim_{v^\alpha\to\infty}[v^\alpha\Phi(v,\alpha,\beta)]=v^{-\beta}\varphi(0^+).$$

□

Remark 3.10. *Theorem 3.9 is true only if $\varphi(t)$ is a strictly proper fraction in which the numerator order is lower than the denominator order.*

Theorem 3.11. *(Final Value Theorem) Let $\Phi(v,\alpha,\beta)=\mathcal{S}[\varphi(t)]$. Then the sadik transform of final value given by*

$$\lim_{v^\alpha\to 0}[v^\alpha\Phi(v,\alpha,\beta)]=\lim_{t\to\infty}\left[v^{-\beta}\varphi(t)\right].$$

Proof. We start as we did for the initial value theorem, with the Sadik transform of the derivative

$$\mathcal{S}\left[\frac{d\varphi(t)}{dt}\right]=\frac{1}{v^\beta}\int_{0^-}^{\infty}\frac{d\varphi(t)}{dt}e^{-tv^\alpha}dt=v^\alpha\Phi(v,\alpha,\beta)-v^{-\beta}\varphi(0^-).$$

Take the limit on both sides as $v^\alpha\to 0$, we have

$$\lim_{v^\alpha\to 0}\left[\frac{1}{v^\beta}\int_{0^-}^{\infty}\frac{d\varphi(t)}{dt}e^{-tv^\alpha}dt\right]=\lim_{v^\alpha\to 0}\left[v^\alpha\Phi(v,\alpha,\beta)-v^{-\beta}\varphi(0^-)\right].$$

As $v^\alpha\to 0$, e^{-tv^α}vanishes from the integral. Also, the term $v^{-\beta}\varphi(0^-)$ in the right side we can take it out of the limit since it independent of v^α. Hence, by the theory of fundamental calculus, we have

$$\begin{aligned}\lim_{v^\alpha\to 0}\left(\frac{1}{v^\beta}\varphi(\infty)-\frac{1}{v^\beta}\varphi(0^-)\right) &= \lim_{v^\alpha\to 0}\left[\frac{1}{v^\beta}\int_{0^-}^{\infty}\frac{d\varphi(t)}{dt}dt\right]\\ &= \left(\lim_{v^\alpha\to 0}[v^\alpha\Phi(v,\alpha,\beta)]\right)-v^{-\beta}\varphi(0^-).\end{aligned}$$

Since the term on the left doesn't depend on v^α, thus

$$\frac{1}{v^\beta}\varphi(\infty)-\frac{1}{v^\beta}\varphi(0^-)=\left(\lim_{v^\alpha\to 0}[v^\alpha\Phi(v,\alpha,\beta)]\right)-v^{-\beta}\varphi(0^-).$$

That is

$$\lim_{t\to\infty}\left[v^{-\beta}\varphi(t)\right]=\lim_{v^\alpha\to 0}[v^\alpha\Phi(v,\alpha,\beta)].$$

□

Remark 3.12. *Theorem 3.11 is satisfied for all functions except the increasing functions and oscillating functions such as sine and cosine that don't have a final value.*

Table 1: Plot of y(t) versus t (h=0.2) in Example 3.13

t	0	0.2000	0.4000	0.6000	0.8000	1.0000
y(t)	-1.1284	-0.2012	-0.0649	0.0479	0.1743	0

Example 3.13. *Let* $0 < \gamma < 1$ *and* $b \in \mathbb{R}$. *Then the problem*

$$ {}^{c}D_{0}^{\gamma}y(t) - by(t) = 0 \tag{10} $$

with the initial condition $y(0) = y_0$ *has a solution given by*

$$ y(t) = y_0 \sum_{k=0}^{\infty} \frac{(bt^{\gamma})^k}{\Gamma(\gamma k + 1)} = y_0 E_{\gamma,1}(bt^{\gamma}). $$

Applying the Sadik transform on both sides of Eq.(10), together with the Theorem 3.4, we can conclude that

$$ Y(v, \alpha, \beta) = \frac{v^{\alpha\gamma-\beta-1}}{(v^{\alpha\gamma} - b)} y_0, $$

by using the lemma 3.2 we get

$$ y(t) = y_0 \sum_{k=0}^{\infty} \frac{(bt^{\gamma})^k}{\Gamma(\gamma k + 1)} = y_0 E_{\gamma,1}(bt^{\gamma}) $$

Example 3.14. *Let* $\varphi(t) = e^t$. *Then* $\Phi(v, \alpha, \beta) = \frac{v^{-\beta}}{v^{\alpha}-1}$ *and the initial value of this function given as follows*

$$ \lim_{v^{\alpha}\to\infty} [v^{\alpha}\varphi(v, \alpha, \beta,)] = \lim_{v^{\alpha}\to\infty} \left[v^{\alpha} \frac{v^{-\beta}}{v^{\alpha} - 1} \right] = v^{-\beta} = v^{-\beta}\varphi(0^+), $$

where $\varphi(0^+) = 1$. *Therefore, Theorem (3.9) is satisfied.*

Example 3.15. *Let* $\varphi(t) = \sin(at)$. *Then* $\Phi(v, \alpha, \beta,) = \frac{av^{-\beta}}{v^{2\alpha}+a^2}$, *and the initial value of this function given as follows*

$$ \lim_{v^{\alpha}\to\infty} [v^{\alpha}\varphi(v, \alpha, \beta,)] = \lim_{v^{\alpha}\to\infty} \left[v^{\alpha} \frac{av^{-\beta}}{v^{2\alpha} + a^2} \right] = 0 = v^{-\beta}\varphi(0^+), $$

where $\varphi(0^+) = 0$. *This means that Theorem (3.9) holds.*

Example 3.16. *If we have* $\varphi(t) = 1$, *then* $\Phi(v, \alpha, \beta,) = \frac{1}{v^{\alpha+\beta}}$ *and*

$$ \lim_{v^{\alpha}\to 0} [v^{\alpha}\varphi(v, \alpha, \beta,)] = \lim_{v^{\alpha}\to 0} \left[\frac{v^{\alpha}}{v^{\alpha+\beta}} \right] = v^{-\beta} = v^{-\beta} \lim_{t\to\infty} \varphi(t), $$

where $\lim_{t\to\infty} \varphi(t) = 1$. *By using the Theorem (3.11), we get the final value of* $\varphi(t)$ *that is* $v^{-\beta}$.

Example 3.17. *Consider the Dirac delta function*

$$ \delta(t) = \begin{cases} \infty, & t = 0, \\ 0, & t \neq 0. \end{cases} $$

Then $\Phi(v, \alpha, \beta,) = \mathcal{S}[\delta(t)] =1$. *In view of Theorem (3.11), the final value of this function is*

$$ \lim_{v^{\alpha}\to 0} [v^{\alpha}\varphi(v, \alpha, \beta,)] = \lim_{v^{\alpha}\to 0} [v^{\alpha}] = 0 $$

and

$$ \lim_{t\to\infty} v^{-\beta}\delta(t) = v^{-\beta} \lim_{t\to\infty} \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} = 0. $$

So our results are satisfied.

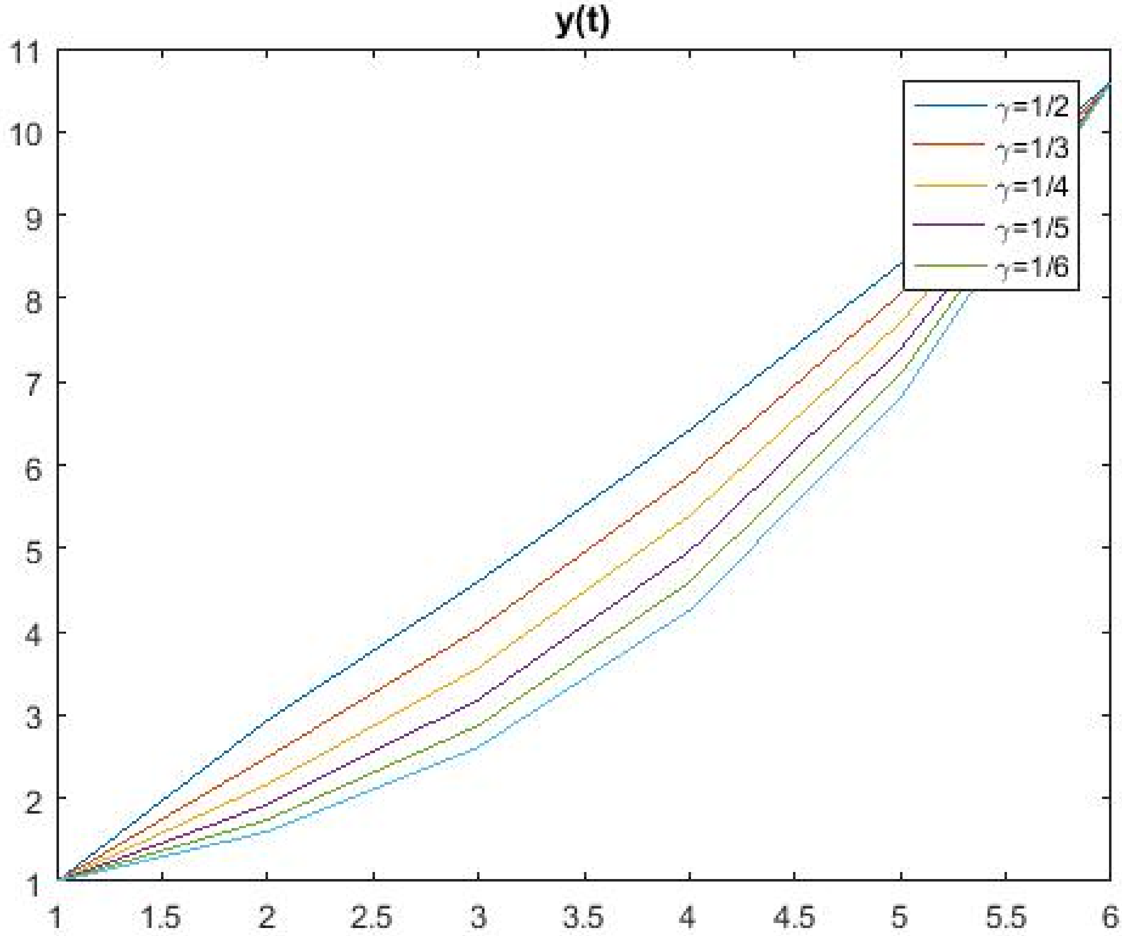

Figure 1: Plot of $y(t) = y_0 E_{\gamma,1}(bt^{\gamma})$ for the Caputo fractional problem and the initial condition $y(0) = 1$, $b = 3$ with diferent values of γ in Example 3.13.

4. Application

4.1. Fractional-order dynamical systems in control theory.

Modern and effective techniques for the time-domain analysis of dynamical systems involving fractional-order are wanted to solving systems of control theory. As a modern generalization of the ordinary PID-controller, the idea of $PI^{\lambda}D^{\mu}$-controller, including fractional-order integrator and fractional-order differentiator, has been lead to be a more efficient control dynamical systems of fractional-order. In his series of papers and books (see references of Podlubny's book [12]), successfully applied the fractional-order controller to improve the so-called CRONE-controller (Commande Robuste d'Ordre Non-En-trier controller) which is an enjoyable example of the application of fractional derivatives in control theory. He proves the advantage of the CRONE-controller compared to the classical PID-controller and also showed that the $PI^{\lambda}D^{\mu}$-controller has a better rendering record when applied for the control of fractional-order systems than the classical PID-controller. In the time domain, he described a dynamical system by the fractional-order differential equation (FDE)

$$\left[\sum_{k=0}^{n} r_{n-k}\ {}^{c}D_{0^+}^{\gamma_{n-k}}\right]\varphi(t)=f(t), \tag{11}$$

where $\gamma_{n-k}>\gamma_{n-k-1}$, $(k=0,1,2,...,n)$ are arbitrary real numbers, r_{n-k} are arbitrary constants, and ${}^{c}D_{0^+}^{\gamma_{n-k}}$ is the standard Caputo fractional derivative of order γ_{n-k}. Now, by the Sadik transform, we get

$$\left[\sum_{k=0}^{n} r_{n-k}\ v^{(\gamma_{n-k})\alpha}\right]\phi(v,\alpha,\beta)=F(v,\alpha,\beta)$$

The transfer function of fractional differential equation Eq.(11) is given by

$$K_n(v,\alpha,\beta)=\frac{F(v,\alpha,\beta)}{\phi(v,\alpha,\beta)}=\left[\sum_{k=0}^{n} r_{n-k}\ v^{(\gamma_{n-k})\alpha}\right]^{-1}.$$

The unit-impulse response $\varphi_i(t)$ of the system is defined by the inverse Sadik transform of $K_n(v,\alpha,\beta)$ so that

$$\varphi_i(t)=\mathcal{S}^{-1}\left[K_n(v,\alpha,\beta)\right]=k_n(t),$$

and the unit-step response function is given by the integral of $k_n(t)$ so that

$$\varphi_s(t)=I_{0^+}^{1}k_n(t).$$

We give a simple example to illustrate the above system

Example 4.1. *We consider a simple fractional-order transfer function*

$$K_2(v,\alpha,\beta)=\left(r\left(v^{\alpha}\right)^{\gamma}+d\right)^{-1},\quad \gamma>0, \tag{12}$$

where r and d are arbitrary constants. The fractional differential equation in the time domain identical to the transfer function (12) is

$$r\ {}^{c}D_{0^+}^{\gamma}\varphi(t)+d\varphi(t)=f(t),$$

with the initial conditions

$$\varphi(0)=0.$$

The unit-impulse response $\varphi_i(t)$ to the system is given by

$$\varphi_i(t)=\mathcal{S}^{-1}\left[K_2(v,\alpha,\beta)\right]=k_2(t),$$

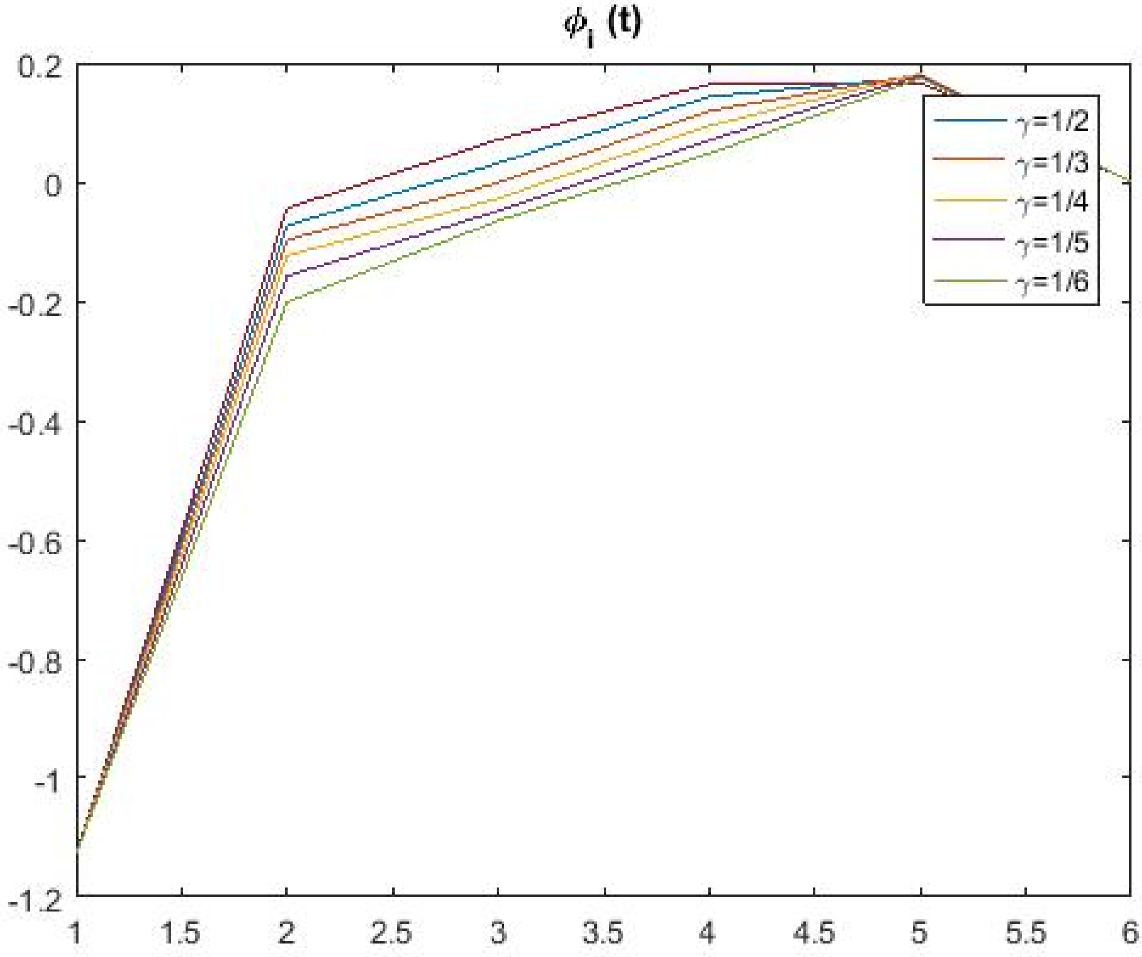

Figure 2: Plot of unit-impulse response $\varphi_i(t)$ in Example 4.1

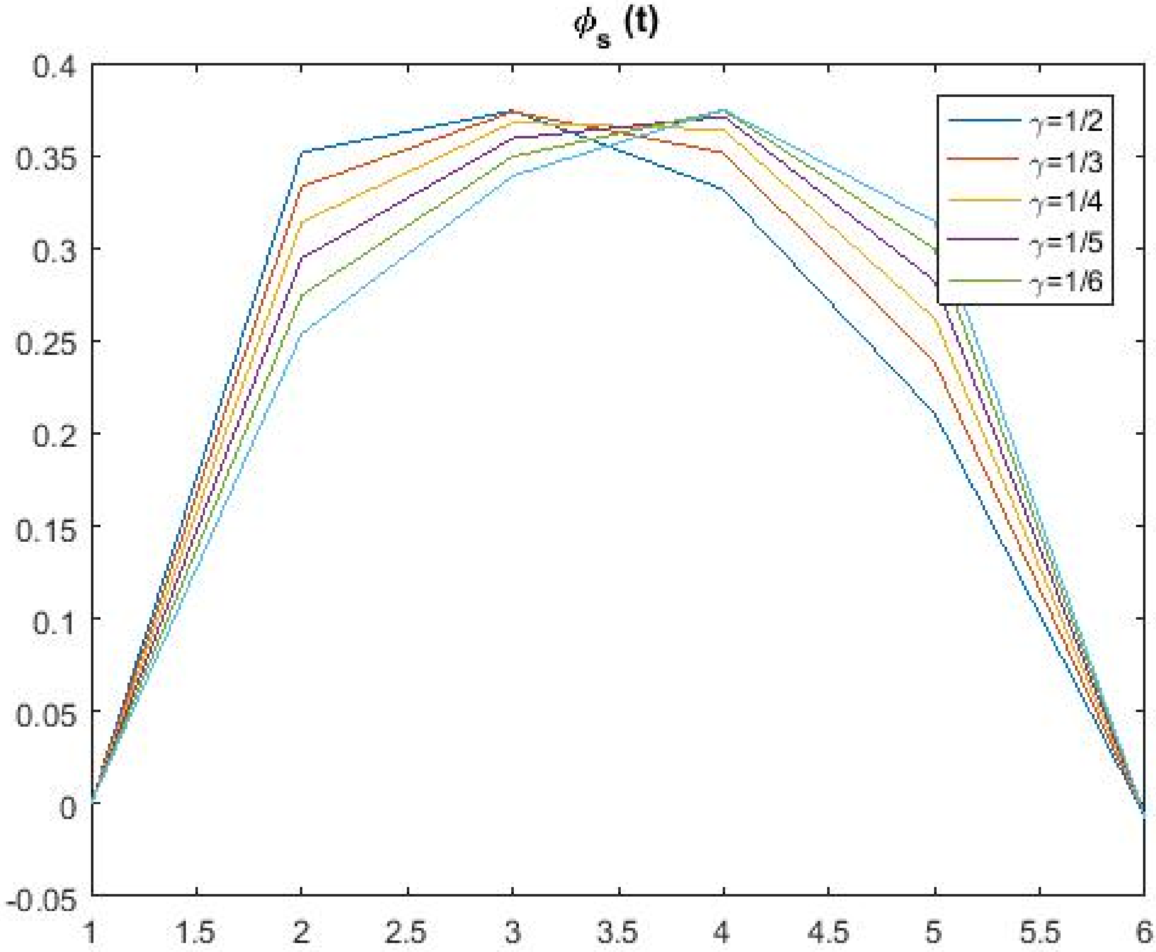

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Figure 3: Plot of unit-step response $\varphi_s(t)$ in Example 4.1

where

$$k_2(t) = \mathcal{S}^{-1}\left[\frac{1}{r\,(v^{\alpha})^{\gamma}+d}\right] = \frac{1}{r}\left[t^{\gamma-1}E_{\gamma,\gamma}(-\frac{d}{r}t^{\gamma})\right],$$

and the unit-step response to the system is

$$\begin{aligned}\varphi_s(t) &= {}^{c}D_{0+}^{-1}k_2(t) = I_{0+}^{1}\left(\frac{1}{r}\left[t^{\gamma-1}E_{\gamma,\gamma}(-\frac{d}{r}t^{\gamma})\right]\right)\\ &= \frac{1}{r}t^{\gamma}E_{\gamma,\gamma+1}(-\frac{d}{r}t^{\gamma}).\end{aligned}$$

conclusion

There are a lot of the integral transforms of exponential type kernels, the Sadik transform is new and very powerful among them and there are many problems in engineering and applied sciences can be solved by Sadik transform. So we have provided Sadik transform of Caputo fractional derivative, and we also gave a sufficient condition to guarantee the rationality of solving Caputo fractional differential equations by the Sadik transform method. Moreover, The Sadik transform of time delay, initial value theorem, and final value theorem are obtained. Some numerical examples to justify our results are presented. An application of fractional-order dynamical systems in control theory by Sadik transform is used. Finally, we obtained some illustrative figures with the help of the Matlab software.

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