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On the recursive sequence $x_{n+1}=\dfrac{x_{n-20}}{1+x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-17}}$

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ABSTRACT

The behaviour of the solutions of the following system of difference equations is examined,

$$x_{n+1}=\frac{x_{n-20}}{1+x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-17}},$$

where the initial conditions are positive real numbers. The initial conditions of the equation are arbitrary positive real numbers. Also, we discuss and illustrate the stability of the solutions in the neighborhood of the critical points and the periodicity of the considered equations.

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1. Introduction

Recently there has been a lot of interest in studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations. For some results in this area, for example: [1-40].
Elabbasy et al. [8-9] investigated the global stability, periodicity character and gave the solution of some special cases of the following difference equations

$$x_{n+1}=ax_n-\frac{bx_n}{cx_n-dx_{n-1}},\quad x_{n+1}=\frac{\alpha x_{n-k}}{\beta+\gamma\prod_{i=0}^{k}x_{n-i}}.$$

In [13] Elsayed dealt with the dynamics and found the solution of the following rational recursive sequences

$$x_{n+1}=\frac{x_{n-5}}{\pm1\pm x_{n-1}x_{n-3}x_{n-5}}.$$

Simsek et. al. [28,29,30,33], studied the following problems with positive initial values

$$x_{n+1}=\frac{x_{n-3}}{1+x_{n-1}},\quad x_{n+1}=\frac{x_{n-5}}{1+x_{n-2}},\quad x_{n+1}=\frac{x_{n-5}}{1+x_{n-1}x_{n-3}},\quad x_{n+1}=\frac{x_{n-3}}{1+x_nx_{n-1}x_{n-2}}$$

respectively.

In this work the following non linear difference equation was studied

$$x_{n+1}=\frac{x_{n-20}}{1+x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-17}} \tag{1}$$

where $x_{-20},x_{-19},...,x_{-1},x_0\in(0,\infty)$ is investigated.

2. Main results

Let $\bar{x}$ be the unique positive equilibrium of the equation (1), then clearly

$$\bar{x}=\frac{\bar{x}}{1+\overline{xxxxxx}}\Rightarrow \bar{x}+\bar{x}^7=\bar{x}\Rightarrow \bar{x}^7=0\Rightarrow \bar{x}=0$$

so, $\bar{x}=0$ can be obtained.
For any $k\geq 0$ and $m>k$ notation $i=\overline{k,m}$ means $i=k,k+1,...,m$.

Theorem 1: Consider the difference equation (1). Then the following statements are true.

a) The sequences

$$(x_{21n-20}),(x_{21n-19}),\ ...,\ (x_{21n-1}),\ (x_{21n})$$

are being decreasing and

$$a_1,a_2,\ ...,\ a_{20},a_{21}\geq 0$$

are existed and such that

$$\lim_{n\to\infty}x_{21n-20+k}=a_{1+k}\ \text{ for } k=\overline{0,20}.$$

b) $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12},a_{13},a_{14},a_{15},a_{16},a_{17},a_{18},a_{19},a_{20},a_{21},...)$ is a solution of Eq. (1) having period 21.

c) $\prod_{k=0}^{6}\lim_{n\to\infty}x_{21n-20-j+3k}=0,\ j=\overline{0,2}$ or $\prod_{k=0}^{6}a_{3k+i}=0,\ i=\overline{1,3}$.

d) If there exist $n_0\in\mathbb{N}$ such that $x_{n-17}\geq x_{n+1}$ for all $n\geq n_0$, then

$$\lim_{n\to\infty}x_n=0.$$

e) The following formulas below are

$$x_{21n+1+k}=x_{-20+k}\left(1-\frac{x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}\sum_{j=0}^{n}\prod_{i=1}^{7j}\frac{1}{1+x_{3i-17+k}x_{3i-14+k}x_{3i-11+k}x_{3i-8+k}x_{3i-5+k}x_{3i-2+k}}\right),$$

$$x_{21n+4+k}=x_{-17+k}\left(1-\frac{x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-20+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}\sum_{j=0}^{n}\prod_{i=1}^{7j+1}\frac{1}{1+x_{3i-17+k}x_{3i-14+k}x_{3i-11+k}x_{3i-8+k}x_{3i-5+k}x_{3i-2+k}}\right),$$

$$x_{21n+7+k}=x_{-14+k}\left(1-\frac{x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-17+k}x_{-20+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}\sum_{j=0}^{n}\prod_{i=1}^{7j+2}\frac{1}{1+x_{3i-17+k}x_{3i-14+k}x_{3i-11+k}x_{3i-8+k}x_{3i-5+k}x_{3i-2+k}}\right),$$

$$x_{21n+10+k}=x_{-11+k}\left(1-\frac{x_{-2+k}x_{-5+k}x_{-8+k}x_{-14+k}x_{-17+k}x_{-20+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}\sum_{j=0}^{n}\prod_{i=1}^{7j+3}\frac{1}{1+x_{3i-17+k}x_{3i-14+k}x_{3i-11+k}x_{3i-8+k}x_{3i-5+k}x_{3i-2+k}}\right),$$

$$x_{21n+13+k}=x_{-8+k}\left(1-\frac{x_{-2+k}x_{-5+k}x_{-11+k}x_{-14+k}x_{-17+k}x_{-20+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}\sum_{j=0}^{n}\prod_{i=1}^{7j+4}\frac{1}{1+x_{3i-17+k}x_{3i-14+k}x_{3i-11+k}x_{3i-8+k}x_{3i-5+k}x_{3i-2+k}}\right),$$

$$x_{21n+16+k}=x_{-5+k}\left(1-\frac{x_{-2+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}x_{-20+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}\sum_{j=0}^{n}\prod_{i=1}^{7j+5}\frac{1}{1+x_{3i-17+k}x_{3i-14+k}x_{3i-11+k}x_{3i-8+k}x_{3i-5+k}x_{3i-2+k}}\right),$$

$$x_{21n+19+k}=x_{-2+k}\left(1-\frac{x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}x_{-20+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}\sum_{j=0}^{n}\prod_{i=1}^{7j+6}\frac{1}{1+x_{3i-17+k}x_{3i-14+k}x_{3i-11+k}x_{3i-8+k}x_{3i-5+k}x_{3i-2+k}}\right),$$

$k=\overline{0,2}$ holds.

f) If $x_{21n+1+k}\to a_{1+k}\neq 0$, $x_{21n+4+k}\to a_{4+k}\neq 0$, $x_{21n+7+k}\to a_{7+k}\neq 0$, $x_{21n+10+k}\to a_{10+k}\neq 0$, $x_{21n+13+k}\to a_{13+k}\neq 0$, $x_{21n+16+k}\to a_{16+k}\neq 0$ then $x_{21n+19+k}\to a_{19+k}=0$ as $n\to\infty$. $k=\overline{0,2}$.

Proof

a) Firstly, from the (1)

b)

$$x_{n+1}\left(1+x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-17}\right)=x_{n-20}$$

is obtained. If $x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-17}\in(0,\infty)$, then $1+x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-17}\in(1,\infty)$. Since

$$x_{n+1}<x_{n-20},$$

$n\in\mathbb{N}$,

$$\lim_{n\to\infty}x_{21n-20+k}=a_{1+k}, \text{ for } k=\overline{0,20}$$

existed formulas are obtained.

c) $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12},a_{13},a_{14},a_{15},a_{16},a_{17},a_{18},a_{19},a_{20},a_{21},...)$ is a solution of (1) having period 21.

d) In wiew of (1),

$$n=21n\Rightarrow x_{21n+1}=\frac{x_{21n-20}}{1+\prod_{k=0}^{5}x_{21n-17+3k}}$$

is obtained. If the limits are put on both sides of the above equality,

$$\prod_{k=0}^{6}\lim_{n\to\infty}x_{21n-20+3k}=0 \text{ or } \prod_{k=0}^{6}a_{3k+1}=0$$

is obtained. Similarly for $n=21n+1$ and $n=21n+2$, we can obtain x_{21n+2} and x_{21n+3}.

e) If there exist $n_0\in\mathbb{N}$ such that $x_{n-17}\geq x_{n+1}$ for all $n\geq n_0$, then,

$a_1\leq a_4\leq a_7\leq a_{10}\leq a_{13}\leq a_{16}\leq a_{19}\leq a_1$, $a_2\leq a_5\leq a_8\leq a_{11}\leq a_{14}\leq a_{17}\leq a_{20}\leq a_2$, $a_3\leq a_6\leq a_9\leq a_{12}\leq a_{15}\leq a_{18}\leq a_{21}\leq a_3$.

Using (c), we get

$$\prod_{k=0}^{6}a_{3k+i}=0,\ i=\overline{1,3}.$$

Then we see that

$$\lim_{n\to\infty} x_n = 0 .$$

Hence the proof of (d) completed.

f) Subracting x_{n-20} from the left and right-hand sides in (1)

$$x_{n+1} - x_{n-20} = \frac{1}{1+x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-20}}\left(x_{n-1} - x_{n-22}\right)$$

and the following formula

$$n \ge 3 \text{ for } \begin{cases} x_{3n-8} - x_{3n-29} = \left(x_1 - x_{-20}\right)\prod_{i=1}^{n-3}\frac{1}{1+x_{3i-2}x_{3i-5}x_{3i-8}x_{3i-11}x_{3i-14}x_{3i-17}} \\ x_{3n-7} - x_{3n-28} = \left(x_2 - x_{-19}\right)\prod_{i=1}^{n-3}\frac{1}{1+x_{3i-1}x_{3i-4}x_{3i-7}x_{3i-10}x_{3i-13}x_{3i-16}} \\ x_{3n-6} - x_{3n-27} = \left(x_3 - x_{-18}\right)\prod_{i=1}^{n-3}\frac{1}{1+x_{3i}x_{3i-3}x_{3i-6}x_{3i-9}x_{3i-12}x_{3i-15}} \end{cases} \quad (2)$$

hold. Replacing n by 7j in (2) and summing from $j = 0$ to $j = n$, we obtain:

$$x_{21n+1+k} - x_{-20+k} = \left(x_{1+k} - x_{-20+k}\right)\sum_{j=0}^{n}\prod_{i=1}^{7j}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}x_{3i-17+k}},\ k = \overline{0,2}$$

Also, $7j+1$ inserted in (2) by replacing n, $j = 0$ to $j = n$ is obtained by summing

$$x_{21n+4+k} - x_{-17+k} = \left(x_{4+k} - x_{-17+k}\right)\sum_{j=0}^{n}\prod_{i=1}^{7j+1}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}x_{3i-17+k}},\ k = \overline{0,2}$$

Also, $7j+2$ inserted in (2) by replacing n, $j = 0$ to $j = n$ is obtained by summing

$$x_{21n+7+k} - x_{-14+k} = \left(x_{7+k} - x_{-14+k}\right)\sum_{j=0}^{n}\prod_{i=1}^{7j+2}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}x_{3i-17+k}},\ k = \overline{0,2}.$$

Also, $7j+3$ inserted in (2) by replacing n, $j = 0$ to $j = n$ is obtained by summing

$$x_{21n+10+k} - x_{-11+k} = \left(x_{10+k} - x_{-11+k}\right)\sum_{j=0}^{n}\prod_{i=1}^{7j+3}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}x_{3i-17+k}},\ k = \overline{0,2}.$$

Also, $7j+4$ inserted in (2) by replacing n, $j = 0$ to $j = n$ is obtained by summing

$$x_{21n+13+k} - x_{-8+k} = \left(x_{13+k} - x_{-8+k}\right)\sum_{j=0}^{n}\prod_{i=1}^{7j+4}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}x_{3i-17+k}},\ k = \overline{0,2}.$$

Also, $7j+5$ inserted in (2) by replacing n, $j = 0$ to $j = n$ is obtained by summing

$$x_{21n+16+k}-x_{-5+k}=\left(x_{16+k}-x_{-5+k}\right)\sum_{j=0}^{n}\prod_{i=1}^{7j+5}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}x_{3i-17+k}}\ ,\ k=\overline{0,2}.$$

Also, $7j+6$ inserted in (2) by replacing n, $j=0$ to $j=n$ is obtained by summing

$$x_{21n+19+k}-x_{-2+k}=\left(x_{19+k}-x_{-2+k}\right)\sum_{j=0}^{n}\prod_{i=1}^{7j+6}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}x_{3i-17+k}},\ k=\overline{0,2}.$$

Now we obtained of the above formulas:

$$x_{21n+k+1}=x_{-20+k}\left(1-\frac{x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}\sum_{j=0}^{n}\prod_{i=1}^{7j}\frac{1}{1+x_{3i-17+k}x_{3i-14+k}x_{3i-11+k}x_{3i-8+k}x_{3i-5+k}x_{3i-2+k}}\right),$$

$$x_{21n+4+k}=x_{-17+k}\left(1-\frac{x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-20+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}\sum_{j=0}^{n}\prod_{i=1}^{7j+1}\frac{1}{1+x_{3i-17+k}x_{3i-14+k}x_{3i-11+k}x_{3i-8+k}x_{3i-5+k}x_{3i-2+k}}\right),$$

$$x_{21n+7+k}=x_{-14+k}\left(1-\frac{x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-17+k}x_{-20+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}\sum_{j=0}^{n}\prod_{i=1}^{7j+2}\frac{1}{1+x_{3i-17+k}x_{3i-14+k}x_{3i-11+k}x_{3i-8+k}x_{3i-5+k}x_{3i-2+k}}\right),$$

$$x_{21n+10+k}=x_{-11+k}\left(1-\frac{x_{-2+k}x_{-5+k}x_{-8+k}x_{-14+k}x_{-17+k}x_{-20+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}\sum_{j=0}^{n}\prod_{i=1}^{7j+3}\frac{1}{1+x_{3i-17+k}x_{3i-14+k}x_{3i-11+k}x_{3i-8+k}x_{3i-5+k}x_{3i-2+k}}\right),$$

$$x_{21n+13+k}=x_{-8+k}\left(1-\frac{x_{-2+k}x_{-5+k}x_{-11+k}x_{-14+k}x_{-17+k}x_{-20+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}\sum_{j=0}^{n}\prod_{i=1}^{7j+4}\frac{1}{1+x_{3i-17+k}x_{3i-14+k}x_{3i-11+k}x_{3i-8+k}x_{3i-5+k}x_{3i-2+k}}\right),$$

$$x_{21n+16+k}=x_{-5+k}\left(1-\frac{x_{-2+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}x_{-20+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}\sum_{j=0}^{n}\prod_{i=1}^{7j+5}\frac{1}{1+x_{3i-17+k}x_{3i-14+k}x_{3i-11+k}x_{3i-8+k}x_{3i-5+k}x_{3i-2+k}}\right),$$

$$x_{21n+19+k}=x_{-2+k}\left(1-\frac{x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}x_{-20+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}\sum_{j=0}^{n}\prod_{i=1}^{7j+6}\frac{1}{1+x_{3i-17+k}x_{3i-14+k}x_{3i-11+k}x_{3i-8+k}x_{3i-5+k}x_{3i-2+k}}\right),\ k=\overline{0,2}\ \text{holds.}$$

g) Suppose that $a_1=a_4=a_7=a_{10}=a_{13}=a_{16}=a_{19}=0$. By e) we have

$$\lim_{n\to\infty}x_{21n+1}=\lim_{n\to\infty}x_{-20}\left(1-\frac{x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}\sum_{j=0}^{n}\prod_{i=1}^{7j}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}}\right)$$

$$a_1=x_{-20}\left(1-\frac{x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}\sum_{j=0}^{\infty}\prod_{i=1}^{7j}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}}\right)$$

$$a_1=0\Rightarrow\frac{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}=\sum_{j=0}^{\infty}\prod_{i=1}^{7j}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}}\qquad(3)$$

Similarly;
$$a_4=0\Rightarrow\frac{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-20}}=\sum_{j=0}^{\infty}\prod_{i=1}^{7j+1}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}}\qquad(4)$$

Similarly;
$$a_7=0\Rightarrow\frac{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{x_{-2}x_{-5}x_{-8}x_{-11}x_{-17}x_{-20}}=\sum_{j=0}^{\infty}\prod_{i=1}^{7j+2}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}}\qquad(5)$$

Similarly;
$$a_{10}=0\Rightarrow\frac{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{x_{-2}x_{-5}x_{-8}x_{-14}x_{-17}x_{-20}}=\sum_{j=0}^{\infty}\prod_{i=1}^{7j+3}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}}\qquad(6)$$

Similarly; $$a_{13}=0\Rightarrow\frac{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{x_{-2}x_{-5}x_{-11}x_{-14}x_{-17}x_{-20}}=\sum_{j=0}^{\infty}\prod_{i=1}^{7j+4}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}} \tag{7}$$

Similarly; $$a_{16}=0\Rightarrow\frac{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{x_{-2}x_{-8}x_{-11}x_{-14}x_{-17}x_{-20}}=\sum_{j=0}^{\infty}\prod_{i=1}^{7j+5}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}} \tag{8}$$

Similarly; $$a_{19}=0\Rightarrow\frac{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}x_{-20}}=\sum_{j=0}^{\infty}\prod_{i=1}^{7j+6}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}} \tag{9}$$

From (3) and (4),

$$\begin{aligned}\frac{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}&=\sum_{j=0}^{\infty}\prod_{i=1}^{7j}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}}>\\ \frac{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-20}}&=\sum_{j=0}^{\infty}\prod_{i=1}^{7j+1}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}}\end{aligned} \tag{10}$$

thus, $x_{-20}>x_{-17}$. From the (4) and (5),

$$\begin{aligned}\frac{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-20}}&=\sum_{j=0}^{\infty}\prod_{i=1}^{7j+1}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}}>\\ \frac{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{x_{-2}x_{-5}x_{-8}x_{-11}x_{-17}x_{-20}}&=\sum_{j=0}^{\infty}\prod_{i=1}^{7j+2}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}}\end{aligned} \tag{11}$$

thus, $x_{-17}>x_{-14}$. From the (5) and (6),

$$\begin{aligned}\frac{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{x_{-2}x_{-5}x_{-8}x_{-11}x_{-17}x_{-20}}&=\sum_{j=0}^{\infty}\prod_{i=1}^{7j+2}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}}>\\ \frac{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{x_{-2}x_{-5}x_{-8}x_{-14}x_{-17}x_{-20}}&=\sum_{j=0}^{\infty}\prod_{i=1}^{7j+3}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}}\end{aligned} \tag{12}$$

thus, $x_{-14}>x_{-11}$ .From the (6) and (7),

$$\begin{aligned}\frac{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{x_{-2}x_{-5}x_{-8}x_{-14}x_{-17}x_{-20}}&=\sum_{j=0}^{\infty}\prod_{i=1}^{7j+3}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}}>\\ \frac{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{x_{-2}x_{-5}x_{-11}x_{-14}x_{-17}x_{-20}}&=\sum_{j=0}^{\infty}\prod_{i=1}^{7j+4}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}}\end{aligned} \tag{13}$$

thus, $x_{-11}>x_{-8}$ .From the (7) and (8),

$$\begin{aligned}\frac{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{x_{-2}x_{-5}x_{-11}x_{-14}x_{-17}x_{-20}}&=\sum_{j=0}^{\infty}\prod_{i=1}^{7j+4}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}}>\\ \frac{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{x_{-2}x_{-8}x_{-11}x_{-14}x_{-17}x_{-20}}&=\sum_{j=0}^{\infty}\prod_{i=1}^{7j+5}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}}\end{aligned} \tag{14}$$

thus, $x_{-8}>x_{-5}$ .From the (8) and (9),

$$\begin{aligned}\frac{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{x_{-2}x_{-8}x_{-11}x_{-14}x_{-17}x_{-20}}&=\sum_{j=0}^{\infty}\prod_{i=1}^{7j+5}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}}>\\ \frac{1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}}{x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}x_{-20}}&=\sum_{j=0}^{\infty}\prod_{i=1}^{7j+6}\frac{1}{1+x_{3i-17}x_{3i-14}x_{3i-11}x_{3i-8}x_{3i-5}x_{3i-2}}\end{aligned} \tag{15}$$

thus, $x_{-5}>x_{-2}$.

From here we obtain $x_{-20} > x_{-17} > x_{-14} > x_{-11} > x_{-8} > x_{-5} > x_{-2}$. Similarly, we can obtain $x_{-19} > x_{-16} > x_{-13} > x_{-10} > x_{-7} > x_{-4} > x_{-1}$ and $x_{-18} > x_{-15} > x_{-12} > x_{-9} > x_{-6} > x_{-3} > x_{0}$. We arrive at a contradiction which completes the proof of theorem.

3. Example

Example 3.1: If the initial conditions are selected in accordance with Lemma 1 and Theorem 1;

x[-20] = 0.99999999999999999999999; x[-19] = 0.99999999999999999999999; x[-18] = 0.888888888888888888888;
x[-17] = 0.99999999999999999999;
x[-16] = 0.9999999999999999999; x[-15] = 6; x[-14] = 0.99999999999999999; x[-13] = 0.9999999999999999;
x[-12] = 0.999999999999999; x[-11] = 0.77777777777777; x[-10] = 0.9999999999999; x[-9] = 0.999999999999;
x[-8] = 0.99999999999; x[-7] = 0.9999999999; x[-6] = 0.999999999; x[-5] = 0.99999999; x[-4] = 0.9999999;
x[-3] = 0.999999; x[-2] = 0.99999; x[-1] = 0.9999; x[0] = 0.999.

$$\begin{aligned} x_n = \{&0.562502,0.500025,0.127093,0.695653,0.666678,5.32403,0.766667,0.750006,0.596669,\\ &0.598291,0.800004,0.71259,0.847826,0.833336,0.776754,0.867925,0.857145,0.817496,\\ &0.883324,0.874903,0.844688,0.465877,0.400026,0.0574,0.610812,0.579722,5.25524,\\ &0.690279,0.672085,0.534988,0.530552,0.728998,0.655823,0.785099,0.767906,0.723852,\\ &0.809425,0.796355,0.76781,0.828458,0.818062,0.797762,0.416792,0.350257,0.0315814,\\ &0.565378,0.533888,5.22954,0.647651,0.629177,0.510473,0.491094,0.688476,0.632191,\\ &0.747529,0.729414,0.700968,0.773522,0.759638,0.745588,0.794046,0.782923,0.776142,\\ &0.385004,0.318321,0.0187474,0.535283,0.503755,5.21674,0.618892,0.600421,0.497984,\\ &0.463926,0.660873,0.619944,0.721314,0.702817,0.688931,0.748166,0.733941,0.733742,\\ &0.769474,0.758042,0.764474,0.361907,0.295245,0.0115556,0.513141,0.481689,...\} \end{aligned}$$

solutions are obtained and the graphs of the solutions are shown below.

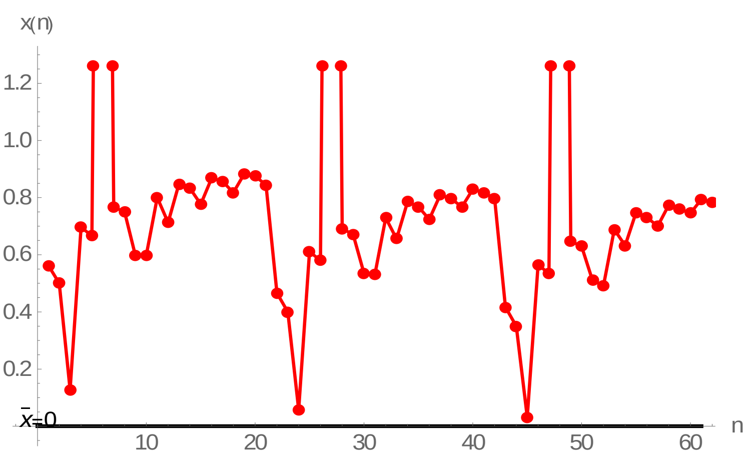

Figure 1. x_n *solutions graph*

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