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Generalized solution of boundary value problem with an inhomogeneous boundary condition

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ABSTRACT

A solution to boundary value problem is investigated for a controlled oscillation process, described by Fredholm integro-differential equation with inhomogeneous boundary conditions. An algorithm is developed for constructing a generalized solution of boundary value problem. It is proved that a weak generalized solution is an element of Hilbert space. Approximate solutions of the boundary value problem are determined and their convergence to the exact solution is proved.

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1. Introduction

Natural phenomena and technology processes are usually described by differential equations, integral equations and integro-differential equations. When investigating these processes, apart from the description by equations, additional conditions are required too [1-3]. These additional conditions could be given as an initial condition or as a boundary condition [4-7]. Therefore, equations satisfying the initial and boundary conditions, in other words, boundary value problems play an important role in practice.

Nowadays, boundary value problems with differential equations are relatively well understood [6-9], but boundary value problems with integral equations and integro-differential equations are not yet sufficiently investigated. Therefore, we study the boundary value problem for the oscillation process described by Fredholm integro-differential equations [10] with inhomogeneous boundary condition.

2. Generalized solution ofthe boundary value problem

Consider the Oscillaton process described by following integro-differential equation

$$V_{tt} = V_{xx} + \lambda \int_0^T K(t,\tau)V(\tau,x)d\tau\,, 0 < x < 1, 0 < t \le T, \tag{1}$$

subject to initial

$$V(0,x) = \psi_1(x), V_t(0,x) = \psi_2(x), 0 < x < 1, \quad (2)$$

and boundary

$$V_x(t,0) = 0, V_x(t,1) + \alpha V(t,1) = u(t), 0 < t \leq T, \quad (3)$$

conditions. Here $K(t,\tau)$ is a given function defined in domain $D = \{0 \leq t \leq T, 0 \leq \tau \leq T\}$ andsatisfies the condition

$$\int_0^T \int_0^T K^2(t,\tau)d\tau dt = K_0 < \infty; \quad (4)$$

$\psi_1(x) \in H_1(0,1)$, $\psi_2(x) \in H(0,1)$ are given functions; λ is a parameter, T is a fixed moment of time, $\alpha > 0$ is a constant. $H(Y)$ is a Hilbert space of square-integrable functions defined on the set Y.
Solution to boundary value problem [1-3] will be found in form of Fourier series:

$$V(t,x) = \sum_{n=1}^{\infty} V_n(t) z_n(x), \quad (5)$$

where

$$V_n(t) = \langle V(t,x), z_n(x) \rangle = \int_Q V(t,x) z_n(x) dx$$

is a Fouruier coefficient.

Functions $z_n(x)$are solutions to following problem

$$z_n'' + \lambda_n^2 z_n(x) = 0, \quad z_n'(0) = 0, \quad z_n'(1) + \alpha z_n(1) = 0,$$

for each fixed $n = 1,2,3, \dots$. Solving this boundary value problem, we have following functions which called *eigenfunctions*

$$z_n(x) = C_n cos\lambda_n x, \qquad n = 1,2,3, \dots \quad (6)$$

Using boundary conditions we find that

$$C_n = \sqrt{\frac{2(\lambda_n^2+\alpha^2)}{\lambda_n^2+\alpha^2+\alpha}},$$

then $z_n(x) = \sqrt{\frac{2(\lambda_n^2+\alpha^2)}{\lambda_n^2+\alpha^2+\alpha}} cos\lambda_n x$ and $\{z_n(x)\}$ is an orthonormal system.

The numbers λ_n are called an *eigenvalues* of eigenfnctions. They are positive roots of transcendental equations $tg\lambda_n = \frac{\alpha}{\lambda_n}$. It is known that we can't analytically solve these transcendental equations, so we have solved they graphically as the following Fig 1.

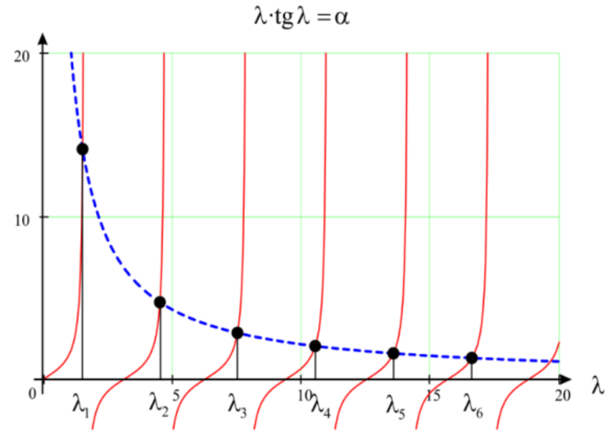

Figure 1. *Solutions of transcendental equations*

As shown in Fig. 1 values of eigenvalues $\lambda_n$are increase with growing value of *n*. To see that Eigenvalues λ_n depend on parameterα, we have calculated values of λ_n using program MATLAB for $\alpha = 0.5$; 1.5; 2.5; 3; 5and obtain following table (Table 1). Similarly, values of eigenfunctions $z_n(x) = \sqrt{\frac{2(\lambda_n^2+\alpha^2)}{(\lambda_n^2+\alpha^2+\alpha)}}\cos\lambda_n x$ corresponding to eigenvalues $\lambda_n$was calculated and given in Table 1.

Table 1. *Eigenvalues and eigenfunctions*

№	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
					α=0.5					
$\lambda_n(x)$	0.653	3.292	6.362	9.477	12.606	15.740	18.876	22.014	25.153	28.292
Z_n(x)	1.0725	1.384	1.406	1.410	1.412	1.4128	1.4132	1.4135	1.4137	1.4138
					α=1.5					
$\lambda_n(x)$	0.9882	3.5422	6.5097	9.5801	12.6841	15.8026	18.9286	22.059	25.1922	28.3272
Z_n(x)	1.1685	1.3476	1.3910	1.4031	1.4078	1.410	1.4113	1.4121	1.4125	1.4129
					α=2.5					
$\lambda_n(x)$	1.1422	3.7318	6.6431	9.6776	2.7598	15.8643	18.9805	22.1038	25.2315	28.3623
Z_n(x)	1.2558	1.3340	1.3804	1.3968	1.4039	1.4074	1.4094	1.4106	1.4115	1.4120
					α=3					
$\lambda_n(x)$	1.1925	3.8088	6.704	9.724	12.7966	15.8945	19.0061	22.1259	25.251	28.3797
Z_n(x)	1.2462	1.3318	1.3765	1.3942	1.4021	1.4062	1.4085	1.4099	1.4109	1.4116
					α=5					
$\lambda_n(x)$	1.3138	4.0336	6.9096	9.8927	12.9352	16.0107	19.1055	22.2126	25.3276	28.4483
z(x)	1.298	1.3356	1.3679	1.3863	1.3962	1.4018	1.4052	1.4074	1.4089	1.4099

From this table, we see that the more value of parameter $\alpha$the more values of eigenfunctions$z_n(x)$ and eigenvaluesλ_n.
As given boundary value problem (1) - (3) hasn't a classical solution, we use the notion of generalized solution of boundary value problem.

By multiplying both sides of tequation (4) by any function $\Phi(t,x)$ and using integral in parts we have integral identity

$$\int_0^1 [V_t\Phi(t,x) - V\Phi_t(t,x)]|_{t_1}^{t_2} dx \equiv \int_{t_1}^{t_2}\int_{t_1}^{t_2} (-V\Phi_{tt}(t,x) + V\Phi_{xx}(t,x))dxdt + \quad (7)$$

$$+\int_{t_1}^{t_2}\int_0^1(\lambda\int_0^T K(t,\tau)V(\tau,x)d\tau)\,\Phi(t,x)dxdt+$$

$$+\int_{t_1}^{t_2}(u(t)\Phi(t,1)+V(t,0)\Phi_x(t,0)-V(t,1)(\propto\Phi(t,1)+\Phi_x(t,1))dt;$$

Definition. Function $V(t,x)\in H(Q_T)$, which satisfies the integral identity (7) and initial conditions in the weak sense (i.e. for any functions $\phi_0(x)\in H(0,1),\phi_1(x)\in H(0,T)$ equalities

$$\lim_{t\to+0}\int_Q V(t,x)\phi_0(x)dx=\int_Q\psi_1(x)\phi_0(x)dx\ \lim_{t\to+0}\int_Q V_t(t,x)\phi_1(x)dx=\int_Q\psi_2(x)\phi_1(x)dx,$$

fulfile), is called a generalized solution of boundary value problem (1) - (3).

As function $\Phi(t,x)$ is arbitrary since the, we take $\Phi(t,x)\equiv z_n(x)$ in (7), and by direct calculations we have the following problem

$$V_n''(t)+\lambda_n^2V_n(t)=\lambda\int_0^T K(t,\tau)V_n(\tau)d\tau+u(t)z_n(1),$$

$$V_n(t)|_{t=t_1}=<V(t_1,x),z_n(x)>,$$

$$V_n'(t)|_{t=t_1}=<V_t(t_1,x),z_n(x)>.$$

We reduce this problem to Fredholm integral equation of the second kind

$$V_n(t)=\psi_{1n}cos\lambda_n t+\frac{1}{\lambda_n}\psi_{2n}sin\lambda_n t+$$

$$+\frac{1}{\lambda_n}\int_0^t sin\lambda_n(t-\tau)\left[\lambda\int_0^T K(t,\eta)V_n(\eta)d\eta+z_n(1)u(\tau)\right]d\tau. \tag{8}$$

Here, $\psi_{1n},\psi_{2n}$are Fourier coefficients of functions $\psi_1(x)$, $\psi_2(x)$respectively.
Using the notations

$$q_n(t)=\psi_{1n}cos\lambda_n t+\frac{1}{\lambda_n}\psi_{2n}sin\lambda_n t+\frac{1}{\lambda_n}\int_0^t sin\lambda_n(t-\tau)z_n(1)u(\tau)\,d\tau; \tag{9}$$

$$K_n(t,s)=\frac{1}{\lambda_n}\int_0^t sin\lambda_n(t-\tau)K(\tau,s)\,d\tau; \tag{10}$$

we obtain the following integral equation

$$V_n(t)=\lambda\int_0^T K_n(t,s)\,V_n(s)ds+q_n(t), \tag{11}$$

*S*olution of this equation (11) will be found by following formula [6]`

$$V_n(t) = \lambda \int_0^T B_n(t,s,\lambda) q_n(s)\, ds + q_n(t); \tag{12}$$

Here

$$B_n(t,s,\lambda) = \sum_{i=1}^{\infty} \lambda^{i-1} K_{n,i}(t,s), \tag{13}$$

is a resolvents of repeated kernels $K_{n,i}(t,s)$. Repeated kernels $K_n(t,s) \equiv K_{n,i}(t,s)$are defined by formulas

$$K_{n,i+1}(t,s) = \int_0^T K_n(t,\tau) K_{n,i}(\tau,s)\, d\tau, \quad i = 1,2,3\ldots, \qquad K_{n,1}(t,s) = K_n(t,s). \tag{14}$$

for each fixed$n = 1,2,3, \ldots$,. By (10) - (14) equations, we get the following estimates:

$$|B_n(t,s,\lambda)| \le \sqrt{T} \frac{\sqrt{\int_0^T K_n^2(\tau,s) d\tau}}{\lambda_n - |\lambda| T \sqrt{K_0}}, \tag{15}$$

which hold for values of λ satisfying the inequality

$$|\lambda| \frac{T}{\lambda_n} \sqrt{K_0} < 1. \tag{16}$$

Using inequality (15), we obtain inequalities:

$$\int_0^T |B_n(t,s,\lambda)|^2\, ds \le \frac{K_0 T}{\left(\lambda_n - |\lambda| T \sqrt{K_0}\right)^2}. \tag{17}$$

Neumann series absolutely convergence for parameters, satisfying the condition

$$|\lambda| < \frac{\lambda_n}{T\sqrt{K_0}} \tag{18}$$

for each $n = 1,2,3, \ldots$, from which we can see that the radius of convergence $|\lambda| < \frac{\lambda_n}{T\sqrt{K_0}}$ of the Neumann series increases with growth n. As the sum of an absolutely convergent series, resolvent $B_n(t,s,\lambda)$ is a continuous function. As we have seen, when the condition $|\lambda| < \frac{\lambda_1}{T\sqrt{K_0}}$is met, the Neumann series absolutely converges to the continuous function for any $n = 1,2,3$. Thus, we find the solution of the boundary value problem (1) - (3) by the formula (5), where $V_n(t)$ is defined as solution of the integral equation (12) by the formula (13).
Substituting (11) into (12) by direct calculations we obtain

$$V_n(t) = \lambda \int_0^T B_n(t,s,\lambda) q_n(s) ds + q_n(t) =$$

$$= \lambda \int_0^T B_n(t,s,\lambda)\left[\psi_{1n} cos\lambda_n s + \frac{1}{\lambda_n}\psi_{2n} sin\lambda_n s + \frac{1}{\lambda_n}\int_0^s sin\lambda_n(s-\tau) z_n(1)u(\tau)\,d\tau\right] +$$

$$+\psi_{1n} cos\lambda_n t + \frac{1}{\lambda_n}\psi_{2n} sin\lambda_n t + \frac{1}{\lambda_n}\int_0^t sin\lambda_n(t-\tau) z_n(1)u(\tau)\,d\tau =$$

$$= \psi_n(t,\lambda) + \frac{1}{\lambda_n}\left[\int_0^t\int_\tau^T B_n(t,s,\lambda) sin\lambda_n(s-\tau) z_n(1)u(\tau) ds d\tau + \right.$$

$$\left. + \int_t^T\int_\tau^T B_n(t,s,\lambda) sin\lambda_n(s-\tau) z_n(1)u(\tau) ds d\tau + \int_0^t sin\lambda_n(t-\tau) z_n(1)u(\tau) d\tau\right].$$

Denoting the expression by

$$\psi_n(t,\lambda) = \psi_n\left[cos\lambda_n s + \lambda\int_0^t B_n(t,s,\lambda) cos\lambda_n\, sds\right] + \frac{1}{\lambda_n}\psi_{2n}\left[sin\lambda_n t + \lambda\int_0^t B_n(t,s,\lambda) sin\lambda_n\, sds\right], \quad (19)$$

$$D_n(t,\tau,\lambda) = \begin{cases} sin\lambda_n(t-\tau) + \int_0^t B_n(t,s,\lambda) sin\lambda_n(s-\tau) ds, & 0 \le \tau \le t, \\ \int_0^t B_n(t,s,\lambda) sin\lambda_n(s-\tau) ds, & t \le \tau \le T, \end{cases} \quad (20)$$

we get the formula for finding the Fourier coefficient

$$V_n(t) = \psi_n(t,\lambda) + \frac{1}{\lambda_n}\int_0^T D_n(t,\tau,\lambda)\, z_n(1)u(\tau) d\tau. \quad (21)$$

Then solution $V(t,x)$to boundary value problem (1) - (3) is determined by the formula

$$V(t,x) = \sum_{n=1}^{\infty} V_n(t) z_n(x) =$$
$$= \sum_{n=1}^{\infty}\left[\psi_n(t,\lambda) + \frac{1}{\lambda_n}\int_0^T D_n(t,\tau,\lambda)\, z_n(1)u(\tau) d\tau\right] z_n(x); \quad (22)$$

Lemma 1. The solution of boundary value problem (1) - (3) defined by formula (22) is an element of the Hilbert space $H(Q)$.

Proof: To proof Lemma 1 we should show that following equality is fulfill

$$\int_0^T V^2(t,x) dx dt < \infty.$$

Taking equations (10) and (11) into account we get the following inequalities:

$$\int_{t}^{T}\int_{\tau}^{T} V^2(t,x)dxdt = \int_{t}^{T}\int_{\tau}^{T} [V_n(t)z_n(x)]^2 dxdt = \int_0^T \sum_{n=1}^{\infty} V_n^2(t)dt \le$$

$$\le 2\left[3T\lambda^2 \frac{K_0T}{\left(\lambda_1 - |\lambda| T\sqrt{K_0}\right)^2}\left(\|\psi_1(x)\|^2_{H(0,1)} + \frac{1}{\lambda_1^2}\|\psi_2(x)\|^2_{H(0,1)} + \frac{1}{\lambda^2{}_1}\|z_n(1)\|^2\|u(t)\|^2_{H(0,T)}\right) + \right.$$

$$\left. +3T\left(\|\psi_{1n}\|^2 + \frac{1}{\lambda^2{}_1}\|\psi_{2n}\|^2 + \frac{1}{\lambda^2{}_1}\|z_n(1)\|^2\|u\|^2\right)\right] \le$$

$$\le 6T\left[\|\psi_{1n}\|^2 + \frac{1}{\lambda_1^2}\|\psi_{2n}\|^2 + \frac{1}{\lambda_1^2}\|z_n(1)\|^2\|u\|^2\right]\left(\lambda^2 \frac{K_0T}{\left(\lambda_1 - |\lambda| T\sqrt{K_0}\right)^2} + 1\right) < \infty,$$

from which it follows that $V(t,x) \in H(Q)$.

3. Approximate solution of boundary value problem

When defining the functions $V_n(t), n = 1,2,3,\ldots$, by formulas (20)-(22), due to infinity of the Neumann series, it is not always possible to find the exact value of resolvent $R_n(t,s,\lambda)$. Therefore, approximations of the resolvent are usually used. Truncated series in the form

$$B_n^m(t,s,\lambda) = \sum_{i=1}^{m} \lambda^{i-1} K_{n,i}(t,s) \tag{23}$$

is called an m th *approximation of the resolvent* or a *resolvental approximation* for each fixed$n = 1,2,3,\ldots$. The function $V_n^m(t)$defined by formula

$$V_n^m(t) = \lambda \int_0^T B_n^m(t,s,\lambda) q_n(s)ds + q_n(t), \; n = 1,2,3,\ldots, \tag{24}$$

and it is called the m^{th} *approximation* of the function $V_n(t)$ for each fixed$m = 1,2,3,\ldots$.
According to formulas (5) and (23), function is defined by the formulas

$$V^m(t,x) = \sum_{n=1}^{\infty} V_n^m(t) z_n(x)$$

and it is called an m^{th} approximation of the solution to boundary value problem (1)-(3) .

To prove that mth approximate solutions $V^m(t,x)$ to boundary value problem (1) - (3) converge to their exact $V(t,x)$ solution with respect to the norm of the space, we do the following calculations :

$$\|V(t,x) - V^m(t,x)\|^2_{H(Q)} = \int_0^T\int_0^1 \left(V(t,x) - V^m(t,x)\right)^2 dxdt \le \int_0^T\int_0^1 \left(\sum_{n=1}^{\infty}[V_n(t) - V_n^m(t)]z_n(x)\right)^2 dxdt \le$$

$$\le \int_0^T \sum_{n=1}^{\infty}[V_n(t) - V_n^m(t)]^2\, dt \le \int_0^T \sum_{n=1}^{\infty}\left\{\psi_{1n} + \left(\lambda\int_0^T [B_n(t,s,\lambda) - B_n^m(t,s,\lambda)]\cos\lambda_n s ds\right)\right.$$

$$\left. + \frac{\psi_{2n}}{\lambda_n}\left(\lambda\int_0^T [B_n(t,s,\lambda) - B_n^m(t,s,\lambda)]\cos\lambda_n s ds\right) + \frac{1}{\lambda_n}\int_0^T D_n[(t,s,\lambda) - D_n^m(t,s,\lambda)]z_n(1)u(\tau)d\tau\right\}^2 dt \le$$

$$\leq 3T\sum_{n=1}^{\infty}\left(\psi_{1n}^2+\frac{\psi_{2n}^2}{\lambda_n^2}+\frac{2T}{\lambda_n^2}\|u(t)\|_{H(0,T)}^2\right)\frac{\lambda^2K_0T}{\lambda_n^2}\left(\sum_{i=m+1}^{\infty}\left[|\lambda|\frac{\sqrt{K_0T^2}}{\lambda_n}\right]^{i-1}\right)^{2m}\left(1-\frac{1}{\ln\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_n}}\right)^2\leq \quad (25)$$

$$\leq C_3(\lambda)\left(|\lambda|\frac{\sqrt{K_0T^2}}{\lambda_1}\right)^{2m}\to\infty,\quad m\to\infty,$$

$$C_3(\lambda)=3T\left(\|\psi_1(x)\|_{H(0,1)}^2+\frac{2}{\lambda_n^2}\|\psi_2(x)\|_{H(0,1)}^2+\frac{2T}{\lambda_1^2}\|u(t)\|_{H(0,T)}^2\right)\lambda^2K_0T\times\left(1-\frac{1}{\ln|\lambda|\frac{\sqrt{K_0T^2}}{\lambda_1}}\right)\left(\frac{1}{\lambda_1^2}+\frac{1}{6}\right).$$

The solution of boundary value problem, defined by truncated Fourier series,

$$V^{m,k}(t,x)=\sum_{n=1}^{k}V_n^{m,k}(t)\,z_n(x)=\sum_{n=1}^{k}(\lambda\int_0^T B_n^{m,k}(t,s,\lambda)q_n(s)ds+q_n(t)) \quad (26)$$

is called an m,k^{th} *approximate* solution of boundary value problem for each fixed $m,k=1,2,3,\ldots$. Proof of convergence these $m,k^{\,th}$ *approximate* solutions *to* $m^{\,th}$ *approximate* solutions follows from following inequality:

$$\|V^m(t,x)-V^{m,k}(t,x)\|_{H(Q)}^2=\int_0^T\int_0^1(\sum_{n=1}^{\infty}V^m(t,x)-V^{m,k}(t,x))^2\,dxdt\leq$$

$$\leq\int_0^T\int_0^1\left(\sum_{n=1}^{\infty}[V_n^m(t)-V_n^{m,k}(t)]z_n(x)\right)^2dxdt\leq\int_0^T\sum_{n=1}^{\infty}[V_n^m(t)-V_n^{m,k}(t)]^2\,dt\leq$$

$$\leq\int_0^T\sum_{n=1}^{\infty}\{\psi_{1n}+\left(\lambda\int_0^T[B_n^m(t,s,\lambda)-B_n^{m,k}(t,s,\lambda)]\cos\lambda_n sds\right)$$

$$+\frac{\psi_{2n}}{\lambda_n}\left(\lambda\int_0^T[B_n(t,s,\lambda)-B_n^m(t,s,\lambda)]\cos\lambda_n sds\right)+\frac{1}{\lambda_n}\int_0^T D_n[(t,s,\lambda)-D_n^m(t,s,\lambda)]z_n(1)u(\tau)d\tau\Big\}^2dt\leq$$

$$\leq 2\int_0^T\sum_{n=k+1}^{\infty}(\lambda^2\frac{K_0T^2}{(\lambda_n-|\lambda|T\sqrt{K_0})^2}+1)\int_0^T q_n^2(t)dt\leq$$

$$\leq 2\sum_{n=k+1}^{\infty}(\lambda^2\frac{K_0T^2}{(\lambda_n-|\lambda|T\sqrt{K_0})^2}+1)\int_0^T\left(\psi_{1n}\cos\lambda_n t+\frac{\psi_{2n}}{\lambda_n}\sin\lambda_n t++\frac{1}{\lambda_n}\int_0^t\sin\lambda_n(t-\tau)z_n(1)u(\tau)\,d\tau\right)^2dt\leq$$

$$\leq 6\left(1+\lambda^2\frac{K_0T^2}{(\lambda_n-|\lambda|T\sqrt{K_0})^2}\right)\sum_{n=k+1}^{\infty}\left(\psi_{1n}^2+\frac{\psi_{1n}^2}{\lambda}+\frac{2}{\lambda_n^2}\int_0^T\sin^2(T-\tau)d\tau\int_0^T u^2(\tau)\,d\tau\right)\underset{k\to\infty}{\to}0,\quad m=1,2,3,\ldots \quad (27)$$

Convergence of that $m,k^{\,th}$ approximate solutions $V^{m,k}(t,x)$of BVP(1)-(3) to exact solutions $V(t,x)$ is proved as follows

$$\|V(t,x)-V^{m,k}(t,x)\|_H=\|V(t,x)-V^m(t,x)\|_H+\|V^m(t,x)-V^{m,k}(t,x)\|_H\underset{k,m\to\infty}{\to}0,$$

As according to formulas (25) qnd (27), we obtain

$$||V(t,x) - V^m(t,x)||_{H(Q)} \underset{m\to\infty}{\to} 0,$$

$$||V^m(t,x) - V^{m,k}(t)||_{H(Q)} \underset{k\to\infty}{\to} 0, \qquad m = 1,2,3,\dots.$$

4. Conclusion

- An algorithm is developed for finding generalized solutions of the inhomogeneous boundary value problem with Frodholm integro-differential operator;
- Approximate solutions, which used in sientific researches and in practice, are found;
- Convergence of approximate solutions of boundary value problem to its exact solutions is proved;
- By numerical calculations using Matlab, we investigated the dependence of the solution of given boundary value problem on a parameter.

References

[1] Vladimirov V.S.," Matematicheskie zadachi odnoskorostnoi toerii perenosa chastis", Trudy MİAN, T. 61, (1961), 130-158.

[2] Volterra V., Teoriya funksionalov, integrelnyhi integro- differensiyalnyh uravneniy. Moskva, Nauka, (1982).

[3] Tyn Myint-U, Lokenath, Partial Differential Equations for Scientists and Engineers, Prentice Hall, (1987).

[4] Sharma J.N., Kehar Singh, Partial Differential Equations for Engineers and Scientists, Alpha Science İnternational Ltd. UK. (2000).

[5] Denemeyer R. Introduction to: Partial Differential Equations and Boundary Value Problems, McGraw-Hill Book Company, New York, (1968).

[6] Snedon I.N., Elements of Partial Differential Equations, dover Publications, INC.,New York , (2006).

[7] Chaglıyan M., Chelebi O., Kysmi Diferensiyel Denklemler, Uludag Üniversitesi Guchlendirme Vakfı,Yayın No:196,VİPASH A.SH.,Yayın No:72,2002.

[8] Koca K., Kısmi Diferensiyel Denklemler, Gunduz Egitim ve Yayıncılık, Ankara, (2001).

[9] Anar E., Kısmi Diferensiyel Denklemler, Palme Yayıncılık, Ankara, (2005).

[10] Kerimbekov A., Abdyldaeva E., "On the Solvability of a Nonlinear Tracking Problem Under Boundary Control for the Elastic Oscillations Described by Fredholm Integro-Differential Equations", System Modeling and Optimization. 27th IFIP TC 7 Conference, CSMO 2015. Sophia Antipolis, France, June 29–July 3, 2015. Revised Selected Papers. Sprınger. (2017). 312-322.