Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (4) (2020), 1334 – 1345
DOI : 10.15672/hujms.541773

Research Article

Some new oscillation criteria for second-order hybrid differential equations

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Abstract

In this paper, we consider the second order hybrid differential equations. For this class of equations, we establish a new criterion to check whether all solutions of an equation, in this class, oscillate. We prove this criterion, using a generalized Riccati technique and an averaging method. The established oscillatory criteria have a distinct form, from all other relevant criteria, in the literature. We illustrate the validity of our results by means of various examples.

Mathematics Subject Classification (2010). 34A38, 34C10, 34K11

Keywords. Oscillation, hybrid differential equation, Riccati technique.

1. Introduction

The problem of oscillation or non-oscillation of the solutions of differential equations has been discussed by numerous authors and several techniques have been developed to deal with this problem. For the fundamental theory and preliminary results, we refer the reader to the books and articles in [2, 3, 8, 11, 13, 17, 18, 22, 24]. In the recent years, there has been much attention on various aspects of quadratic perturbations of nonlinear differential equations. The hybrid differential equation is an especially interesting type of nonlinear differential equations that is open to research. The reason is that hybrid differential equations include several dynamic systems, as special cases. There has been considerable work on the theory of hybrid differential equations. We refer the readers to the articles in [7, 9, 20, 23, 27]. Applications with numerical solutions have been studied by several authors, see for example, [10, 14, 16, 25].

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Received: 19.03.2019; Accepted: 12.09.2019

In a review of the literature on hybrid differential equations, Dhage and Lakshmikantham [4] discussed the existence of extremal solutions and comparison result for first order hybrid differential equation with linear perturbations of the following type:

$$\frac{d}{dt}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)),\ a.e.\ t \in J,\ x(t_0) = x_0 \in \mathbb{R},$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} - \{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$.

On the other hand, there has been no work on the qualitative theory of hybrid differential equations. This has motivated us to extend the oscillation theory to hybrid differential equations of second order. In the present paper, we initiate the oscillation theory for hybrid differential equations of the form

$$\left(\frac{x(t)}{f(t,x(t))}\right)^{''} + q(t)x(t) = g(t,x(t)),\ t \geq t_0. \tag{1.1}$$

Throughout this paper, we assume the following conditions hold:
(A_1) $q(t) \in C([t_0,\infty), \mathbb{R}_+)$;
(A_2) $f(t,x(t)) \in C([t_0,\infty) \times \mathbb{R}, \mathbb{R}_+)$, there exists a function $r(t) \in C^{'}([t_0,\infty), \mathbb{R}_+)$ such that $f(t,x(t)) \geq h(x)r(t) \geq Mr(t)$, where $h(x)$ is not identically zero on $[t_0,\infty)$ and moreover, $|h(x)| \geq M > 0$ and $\frac{d}{dt}f(t,x(t)) > 0$;
(A_3) $g(t,x(t)) \in C([t_0,\infty) \times \mathbb{R}, \mathbb{R})$, there exists a function $p(t) \in C([t_0,\infty), \mathbb{R}_+)$ such that $\frac{g(t,x(t))}{x(t)} \leq p(t)$ for $x \neq 0$, $t \geq t_0$ and $q(t) \geq p(t)$.

Note that if $f(t,x(t)) = 1$, $g(t,x(t)) = 0$, then equation (1.1) is reduced to the linear differential equations of second order

$$x^{''}(t) + q(t)x(t) = 0,\ t \geq t_0, \tag{1.2}$$

which include several equations, namely, the famous Euler equation that has been studied by many authors [1, 5, 6, 12, 15, 19, 21, 26].

By a solution of (1.1), we mean a nontrivial function $x(t) \in C^2([T_0,\infty)), T_0 \geq t_0$ which satisfies (1.1) on $[T_0,\infty)$. We only consider those solutions $x(t)$ of (1.1) satisfying $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_0$, and we assume that (1.1) possesses such solutions. A solution of (1.1) is called oscillatory if it has arbitrarily many zeros on $[t_0,\infty)$, and is called nonoscillatory otherwise. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

The aim in this paper is to present some new oscillation criteria for (1.1) by using generalized Riccati technique and an integral averaging method. The contribution is original, as no results on the oscillation of nonlinear hybrid differential equations having been reported in the literature.

The paper is divided in three sections. In Section 2, we establish some new oscillation criteria for (1.1) while in the final section, we present some examples to illustrate the effectiveness of our main results.

2. Main results

In this section, we present sufficient conditions, which guarantee the oscillatory behavior of the solutions of equation (1.1). We begin with the following theorem.

Theorem 2.1. *Suppose that the assumptions* $(A_1) - (A_3)$ *hold. Moreover, assume that there exists a positive nondecreasing function* $\delta \in C^1([t_0,\infty);(0,\infty))$ *such that for all sufficiently large* $t_1 \geq t_0$*, we have*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(\delta(s)(q(s) - p(s)) - \frac{1}{4M}\frac{(\delta^{'}(s))^2}{r(s)\delta(s)}\right) ds = \infty. \tag{2.1}$$

Then every solution $x(t)$ *of* (1.1) *is oscillatory.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality we may assume that $x(t) > 0$ for $t \geq t_1 \geq t_0$, since similar arguments can be made, for the case $x(t) < 0$, eventually. Now, from (1.1), we have

$$\left(\frac{x(t)}{f(t,x(t))}\right)'' \leq 0 \; for \; t \geq t_1. \tag{2.2}$$

Therefore $\left(\frac{x(t)}{f(t,x(t))}\right)'$ is a decreasing function. We now claim that $\left(\frac{x(t)}{f(t,x(t))}\right)' > 0$ for $t \geq t_1$. If not, then there exists $t_2 \geq t_1$ such that

$$\left(\frac{x(t)}{f(t,x(t))}\right)' \leq \left(\frac{x(t)}{f(t,x(t))}\right)'|_{t=t_2} := c < 0, \; t \geq t_2.$$

Integrating from t_2 to t, we get

$$x(t) \leq (c(t-t_2)+d)f(t,x(t)) \to -\infty \; as \; t \to \infty,$$

where $d = \frac{x(t_2)}{f(t_2,x(t_2))}$, which contradicts the fact that $x(t) > 0$ for $t \geq t_1$.
Define the function $w(t)$ by the generalized Riccati substitution

$$w(t) = \delta(t)\left(\frac{x(t)}{f(t,x(t))}\right)' \frac{1}{x(t)}, t \geq t_1. \tag{2.3}$$

Then $w(t) > 0$ for $t \geq t_1$. Differentiating (2.3) and using (1.1) and (A_3), we have

$$\begin{aligned} w'(t) &= \frac{\delta'(t)}{\delta(t)}w(t) + \frac{\delta(t)}{x(t)}(g(t,x(t)) - q(t)x(t)) - w(t)\frac{x'(t)}{x(t)} \\ &\leq \frac{\delta'(t)}{\delta(t)}w(t) + \delta(t)p(t) - q(t)\delta(t) - w^2(t)\frac{x'(t)}{\delta(t)\left(\frac{x(t)}{f(t,x(t))}\right)'}. \end{aligned}$$

By (A_2), the last inequality becomes

$$\begin{aligned} w'(t) &\leq \frac{\delta'(t)}{\delta(t)}w(t) + \delta(t)(p(t)-q(t)) - \frac{w^2(t)}{\delta(t)}f(t,x(t)) \\ &\leq \frac{\delta'(t)}{\delta(t)}w(t) - \delta(t)(q(t)-p(t)) - M\frac{r(t)}{\delta(t)}w^2(t). \end{aligned} \tag{2.4}$$

Using the inequality, $Bu - Au^2 \leq \frac{B^2}{4A}$, we have

$$w'(t) \leq -\delta(t)(q(t)-p(t)) + \frac{1}{4M}\frac{(\delta'(t))^2}{r(t)\delta(t)}.$$

Integrating the last inequality from t_1 to t and taking the limit supremum on both sides, yields

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(\delta(s)(q(s)-p(s)) - \frac{1}{4M}\frac{(\delta'(s))^2}{r(s)\delta(s)}\right) ds \leq w(t_1), \; t \geq t_1,$$

which contradicts hypothesis (2.1). The proof of the theorem is complete. □

Next we present some new oscillation results for (1.1).
We introduce the class of functions Ω. Let $D = \{(t,s) : t_0 \leq s \leq t\}$. The function $H \in C(D,\mathbb{R})$ is said to belong to the class Ω, if
(T_1) $H(t,t) = 0$ for $t \geq t_0$ and $H(t,s) > 0$ for $t > s \geq t_0$.
(T_2) H has continuous and nonpositive partial derivatives on D with respect to s and there exists a function $h_1(t,s) \in C(D,\mathbb{R})$ such that

(i) $h_1(t,s)\sqrt{H(t,s)} = -\frac{\partial H}{\partial s}(t,s)$,
(ii) $h_2(t,s) = h_1(t,s) - \sqrt{H(t,s)}\frac{\delta'(s)}{\delta(s)}$.

Theorem 2.2. *Assume that* $(A_1) - (A_3)$ *hold. In addition, assume that there exists a positive function* $\delta \in C^1([t_0,\infty);(0,\infty))$ *such that for all sufficiently large* $t_1 \geq t_0$*, we have*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s)\delta(s)(q(s)-p(s)) - \frac{1}{4M}\frac{h_2^2(t,s)\delta(s)}{r(s)} \right) ds = \infty. \tag{2.5}$$

Then every solution $x(t)$ *of* (1.1) *is oscillatory.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality we may assume that $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Multiplying both sides of (2.4) by $H(t,s)$, integrating it with respect to s from t_1 to t and using the properties of the function $H(t,s)$ for all $t \geq t_1 \geq t_0$, we get

$$\begin{aligned}
&\int_{t_1}^{t} H(t,s)\delta(s)(q(s)-p(s))ds \\
&\leq -\int_{t_1}^{t} H(t,s)w'(s)ds + \int_{t_1}^{t} H(t,s)\frac{\delta'(s)}{\delta(s)}w(s)ds - \int_{t_1}^{t} MH(t,s)\frac{r(s)}{\delta(s)}w^2(s)ds \\
&\leq H(t,t_1)w(t_1) - \int_{t_1}^{t} h_2(t,s)\sqrt{H(t,s)}w(s)ds - \int_{t_1}^{t} MH(t,s)\frac{r(s)}{\delta(s)}w^2(s)ds \\
&\leq H(t,t_1)w(t_1) - \int_{t_1}^{t} \left(\sqrt{MH(t,s)\frac{r(s)}{\delta(s)}}w(s) + \frac{1}{2}\frac{h_2(t,s)\sqrt{\delta(s)}}{\sqrt{Mr(s)}} \right)^2 ds \\
&\quad + \int_{t_1}^{t} \frac{1}{4M}\frac{\delta(s)}{r(s)}h_2^2(t,s)ds.
\end{aligned}$$

Thus, we conclude that for every $t \geq t_0$,

$$\begin{aligned}
&\int_{t_1}^{t} \left(H(t,s)\delta(s)(q(s)-p(s)) - \frac{1}{4M}\frac{\delta(s)}{r(s)}h_2^2(t,s) \right) ds \\
&\qquad \leq H(t,t_1)w(t_1) - \int_{t_1}^{t} \left(\sqrt{MH(t,s)\frac{r(s)}{\delta(s)}}w(s) + \frac{1}{2}\frac{h_2(t,s)\sqrt{\delta(s)}}{\sqrt{Mr(s)}} \right)^2 ds \\
&\qquad \leq H(t,t_1)w(t_1) \leq H(t,t_0)|w(t_0)|,
\end{aligned} \tag{2.6}$$

which implies that

$$\begin{aligned}
&\int_{t_0}^{t} \left(H(t,s)\delta(s)(q(s)-p(s)) - \frac{1}{4M}\frac{\delta(s)}{r(s)}h_2^2(t,s) \right) ds \\
&\qquad \leq H(t,t_0)\left(\int_{t_0}^{t_1} (\delta(s)(q(s)-p(s)))ds + |w(t_0)| \right).
\end{aligned} \tag{2.7}$$

Inequality (2.7) yields

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Big(H(t,s)\delta(s)(q(s)-p(s))-\frac{1}{4M}\frac{\delta(s)}{r(s)}h_2^2(t,s)\Big)ds$$
$$\leq\int_{t_0}^{t_1}(\delta(s)(q(s)-p(s)))ds+|w(t_0)|<\infty.$$

which contradicts (2.5). The proof of the theorem is complete. □

The following corollaries can easily be derived, from Theorem 2.2.

Corollary 2.3. *Assume that the conditions of Theorem 2.2 hold with (2.5) replaced by*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left(H(t,s)\delta(s)(q(s)-p(s))\right)ds=\infty \tag{2.8}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{1}{4M}\frac{\delta(s)}{r(s)}h_2^2(t,s)ds<\infty. \tag{2.9}$$

Then every solution $x(t)$ of (1.1) *is oscillatory.*

Theorem 2.2 enables us to derive many sufficient conditions for (1.1) with different choices of the function H.
Consider $H(t,s)=(t-s)^{n-1},\ (t,s)\in D$ for some integer $n>2$. Then, Theorem 2.2 leads to the following result.

Corollary 2.4. *Assume that the conditions of Theorem 2.2 hold, equation (2.5) can be written as*

$$\limsup_{t\to\infty}\frac{1}{(t-t_0)^{n-1}}\int_{t_0}^{t}\Bigg((t-s)^{n-1}\delta(s)\Bigg((q(s)-p(s))$$
$$-\frac{1}{4Mr(s)}\left(\frac{\delta'(s)}{\delta(s)}-\frac{n-1}{t-s}\right)^2\Bigg)\Bigg)ds=\infty, \tag{2.10}$$

for some integer $n>2$. Then every solution $x(t)$ of (1.1) *is oscillatory.*

Let $H(t,s)=(R(t)-R(s))^{\lambda}$, where λ is a constant, $R(t)=\int_{t_1}^{t}\frac{1}{r(s)}ds$ and $lim_{t\to\infty}R(t)=\infty$. Then, Theorem 2.2 implies the following result.

Corollary 2.5. *Assume that the conditions of Theorem 2.2 hold, equation (2.5) can be written as*

$$\limsup_{t\to\infty}\frac{1}{(R(t)-R(t_0))^{\lambda}}\int_{t_0}^{t}\Bigg((R(t)-R(s))^{\lambda}\delta(s)\Bigg((q(s)-p(s))$$
$$-\frac{1}{4Mr(s)}\left(\frac{\delta'(s)}{\delta(s)}-\frac{\lambda}{r(s)(R(t)-R(s))}\right)^2\Bigg)\Bigg)ds=\infty. \tag{2.11}$$

Then every solution $x(t)$ of (1.1) *is oscillatory.*

Let $H(t,s)=(log(\frac{t}{s}))^n$, $t>s>t_1,\ n>1$ is an integer. Then, from Theorem 2.2, we get the following result.

Corollary 2.6. *Assume that the conditions of Theorem 2.2 hold, equation (2.5) can be written as*

$$\limsup_{t\to\infty} \frac{1}{(log(\frac{t}{t_0}))^n} \int_{t_0}^{t} \Big((log(\frac{t}{s}))^n \delta(s) \Big((q(s)-p(s)) - \frac{1}{4Mr(s)} \Big(\frac{\delta'(s)}{\delta(s)} - \frac{n}{s(log(\frac{t}{s}))} \Big)^2 \Big) \Big) ds = \infty. \tag{2.12}$$

Then every solution $x(t)$ of (1.1) *is oscillatory.*

Let $H(t,s) = (\int_s^t \frac{du}{\theta(u)})^n$, $t > s > t_0$, where $n > 1$ is an integer and $\theta : [t_0, \infty) \to \mathbb{R}_+$ is a continuous function such that $lim_{t\to\infty}(\int_{t_0}^t \frac{du}{\theta(u)}) = \infty$. Then Theorem 2.2 yields the following result.

Corollary 2.7. *Assume that the conditions of Theorem 2.2 hold, equation (2.5) can be written as*

$$\limsup_{t\to\infty} \Big(\int_{t_0}^{t} \frac{du}{\theta(u)} \Big)^{-n} \int_{t_0}^{t} \Big(\Big(\Big(\int_s^t \frac{du}{\theta(u)} \Big)^n \delta(s) \Big((q(s)-p(s)) - \frac{1}{4Mr(s)} \Big(\frac{\delta'(s)}{\delta(s)} - \frac{n}{\theta(s)(\int_s^t \frac{du}{\theta(u)})} \Big)^2 \Big) \Big) ds = \infty. \tag{2.13}$$

Then every solution $x(t)$ of (1.1) *is oscillatory.*

Theorem 2.8. *Assume that*

$$0 < \inf_{s\ge t_0} \Big(\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \Big) \le \infty \tag{2.14}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{\delta(s)}{r(s)} h_2^2(t,s) ds < \infty. \tag{2.15}$$

Then (1.1) is oscillatory if there exists a continuous function ψ on $[t_0, \infty)$ with

$$\int_{t_0}^{\infty} \frac{r(s)}{\delta(s)} \psi_+^2(s) ds = \infty, \tag{2.16}$$

where $\psi_+(t) = \max\{\psi(t), 0\}$ and such that

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \Big(H(t,s)\delta(s)(q(s)-p(s)) - \frac{1}{4M} \frac{\delta(s)}{r(s)} h_2^2(t,s) \Big) ds \ge \psi(T), \tag{2.17}$$

for every $T \ge t_0$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Then there exists a $T_0 \ge t_0$ such that $x(t) \ne 0$ for all $t \ge T_0$. We define $w(t)$ as in (2.3) on $[T_0, \infty)$. As in the proof of Theorem 2.2, inequality (2.6) is satisfied for all t, t_1 with $t \ge t_1 = T \ge T_0$. So, for $t \ge T \ge T_0$, we obtain the inequality

$$\frac{1}{H(t,T)} \int_{T}^{t} \Big(H(t,s)\delta(s)(q(s)-p(s)) - \frac{1}{4M} \frac{\delta(s)}{r(s)} h_2^2(t,s) \Big) ds$$
$$\le w(T) - \frac{1}{H(t,T)} \int_{T}^{t} \Big(\sqrt{MH(t,s)\frac{r(s)}{\delta(s)}} w(s) + \frac{1}{2} \frac{h_2(t,s)\sqrt{\delta(s)}}{\sqrt{Mr(s)}} \Big)^2 ds$$

and therefore

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left(H(t,s)\delta(s)(q(s)-p(s))-\frac{1}{4M}\frac{\delta(s)}{r(s)}h_2^2(t,s)\right)ds$$
$$\le w(T)-\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left(\sqrt{MH(t,s)\frac{r(s)}{\delta(s)}}w(s)+\frac{1}{2}\frac{h_2(t,s)\sqrt{\delta(s)}}{\sqrt{Mr(s)}}\right)^2ds. \tag{2.18}$$

Thus, by (2.17), we get

$$w(T)\ge\psi(T)+\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left(\sqrt{MH(t,s)\frac{r(s)}{\delta(s)}}w(s)+\frac{1}{2}\frac{h_2(t,s)\sqrt{\delta(s)}}{\sqrt{Mr(s)}}\right)^2ds,$$

for all $t>T\ge T_0$. This implies that

$$w(T)\ge\psi(T)\ \textit{for all}\ T\ge T_0, \tag{2.19}$$

and

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left(\sqrt{MH(t,s)\frac{r(s)}{\delta(s)}}w(s)+\frac{1}{2}\frac{h_2(t,s)\sqrt{\delta(s)}}{\sqrt{Mr(s)}}\right)^2ds$$
$$\le w(T_0)-\psi(T_0)<\infty,$$

that is,

$$\liminf_{t\to\infty}(\theta(t)+\eta(t))<\infty, \tag{2.20}$$

where

$$\theta(t)=\frac{1}{H(t,T_0)}\int_{T_0}^t M\frac{r(s)}{\delta(s)}H(t,s)w^2(s)ds,\ t>T_0, \tag{2.21}$$

$$\eta(t)=\frac{1}{H(t,T_0)}\int_{T_0}^t \sqrt{H(t,s)}h_2(t,s)w(s)ds,\ t>T_0. \tag{2.22}$$

In order to show that

$$\int_{T_0}^t\frac{r(s)}{\delta(s)}w^2(s)ds<\infty, \tag{2.23}$$

suppose that

$$\int_{T_0}^t\frac{r(s)}{\delta(s)}w^2(s)ds=\infty. \tag{2.24}$$

Indeed, let α be a positive constant. Then, by condition (2.14), we get

$$\inf_{s\ge t_0}\left(\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right)>\alpha>0. \tag{2.25}$$

On the other hand, for any positive constant β, due to (2.24), there exists a $T_1>T_0$ such that

$$\int_{T_0}^t\frac{r(s)}{\delta(s)}w^2(s)ds\ge\frac{\beta}{\alpha}\ \textit{for all}\ t\ge T_1.$$

Therefore

$$\begin{aligned}\theta(t) &= \frac{M}{H(t,T_0)}\int_{T_0}^{t} H(t,s)d\Big(\int_{T_0}^{s}\frac{r(u)}{\delta(u)}w^2(u)du\Big)\\ &= \frac{M}{H(t,T_0)}\int_{T_0}^{t}\Big(\int_{T_0}^{s}\frac{r(u)}{\delta(u)}w^2(u)du\Big)\Big(-\frac{\partial H(t,s)}{\partial s}\Big)ds\\ &\geq \frac{M\beta}{\alpha H(t,T_0)}\int_{T_1}^{t}\Big(-\frac{\partial H(t,s)}{\partial s}\Big)ds = \frac{M\beta}{\alpha}\frac{H(t,T_1)}{H(t,T_0)}.\end{aligned}$$

But

$$\frac{H(t,T_1)}{H(t,t_0)} > \alpha,$$

then there exists $T_2 \geq T_1$ such that $\liminf_{t\to\infty}\frac{H(t,T_1)}{H(t,t_0)} \geq \alpha$, for all $t \geq T_2$. So, we have $\theta(t) \geq M\beta$ for all $t \geq T_2$. Then,

$$\lim_{t\to\infty}\theta(t) = \infty, \tag{2.26}$$

since β is an arbitrary positive constant.
Next, let us consider a sequence $\{\tau_l\}_{l=1,2,...}$ in (T_0,∞) with $\tau_l \to \infty$ as $l \to \infty$ and such that

$$\lim_{l\to\infty}(\theta(\tau_l)-\eta(\tau_l)) = \liminf_{t\to\infty}(\theta(t)-\eta(t)).$$

Now, by (2.20), there exists a constant μ such that

$$\theta(\tau_l)-\eta(\tau_l) \leq \mu \ (l = 1,2,...). \tag{2.27}$$

Furthermore, (2.26) guarantees that

$$\lim_{l\to\infty}\theta(\tau_l) = \infty, \tag{2.28}$$

and hence (2.27) gives

$$\lim_{l\to\infty}\eta(\tau_l) = -\infty. \tag{2.29}$$

Taking into account (2.28) and using (2.27), we derive

$$1+\frac{\eta(\tau_l)}{\theta(\tau_l)} \leq \frac{\mu}{\theta(\tau_l)} < k,\ 0<k<1 \text{ for large } l. \tag{2.30}$$

From (2.29), this implies that

$$\frac{\eta^2(\tau_l)}{\theta(\tau_l)} > (k-1)\eta(\tau_l),$$

$$\lim_{l\to\infty}\frac{\eta^2(\tau_l)}{\theta(\tau_l)} = \infty. \tag{2.31}$$

By Schwarz's inequality, we have

$$\eta^2(\tau_l) = \frac{1}{H^2(\tau_l, T_0)}\Big(\int_{T_0}^{\tau_l} \sqrt{H(\tau_l, s)} h_2^2(\tau_l, s) w(s) ds\Big)^2$$
$$\leq \Big(\frac{1}{H(\tau_l, T_0)}\int_{T_0}^{\tau_l} \frac{\delta(s)}{r(s)} h_2^2(\tau_l, s) ds\Big)\Big(\frac{1}{H(\tau_l, T_0)}\int_{T_0}^{\tau_l} H(\tau_l, s)\frac{r(s)}{\delta(s)} w^2(s) ds\Big)$$
$$\leq \frac{\theta(\tau_l)}{M}\Big(\frac{1}{H(\tau_l, T_0)}\int_{T_0}^{\tau_l} \frac{\delta(s)}{r(s)} h_2^2(\tau_l, s) ds\Big),$$

and therefore

$$\frac{\eta^2(\tau_l)}{\theta(\tau_l)} \leq \frac{1}{M\alpha}\Big(\frac{1}{H(\tau_l, t_0)}\int_{t_0}^{\tau_l} \frac{\delta(s)}{r(s)} h_2^2(\tau_l, s) ds\Big).$$

Using (2.31), we have

$$\lim_{l\to\infty} \frac{1}{H(\tau_l, t_0)}\int_{t_0}^{\tau_l} \frac{\delta(s)}{r(s)} h_2^2(\tau_l, s) ds = \infty,$$

which gives

$$\limsup_{l\to\infty} \frac{1}{H(t, t_0)}\int_{t_0}^{t} \frac{\delta(s)}{r(s)} h_2^2(t, s) ds = \infty,$$

which contradicts (2.15). Hence (2.24) fails to hold. Finally,

$$\int_{t_0}^{\infty} \frac{r(s)}{\delta(s)} \psi_+^2(s) ds \leq \int_{t_0}^{\infty} \frac{r(s)}{\delta(s)} w^2(s) ds < \infty,$$

which contradicts (2.16). Therefore (1.1) is oscillatory. □

Theorem 2.9. *Let all the assumptions of Theorem 2.8 hold, except for (2.15) which is replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t, t_0)}\int_{t_0}^{t} H(t, s)\delta(s)(q(s) - p(s)) ds < \infty. \tag{2.32}$$

Then every solution of (1.1) is oscillatory.

The proof is similar to that of Theorem 2.8, and hence is omitted.
We conclude with the following remarks that suggest some open problems, for future research.

Remark 2.10. The results obtained in this article can be extended for the more general differential equation having a damped term

$$\Big(\frac{x(t)}{f(t, x(t))}\Big)'' + p(t)\Big(\frac{x(t)}{f(t, x(t))}\Big)' + q(t)h(x(t)) = g(t, x(t)), \; t \geq t_0,$$

where $p(t)$ is a continuous function on $[t_0, \infty)$.

Remark 2.11. All above results can be extended to the following form

$$\Big(b(t)\Big(a(t)\frac{x(t)}{f(t, x(t))}\Big)'\Big)' + q(t)h(x(t)) = g(t, x(t)), \; t \geq t_0.$$

Remark 2.12. The results obtained in this paper merely initiate the study of the oscillations of hybrid differential equations. Therefore, this problem remains largely open, for future research.

3. Examples

In this section, we provide some examples to illustrate our main results.

Example 3.1. Consider the hybrid differential equation

$$\left(\frac{x(t)}{e^t}\right)'' + x(t) = (e^t - 1)\sin t, \quad t \geq t_0. \tag{3.1}$$

Here $q(t) = 1, M = 1, f(t, x(t)) = e^t, r(t) = 1$, $g(t, x(t)) = (e^t - 1)\sin t$ and $\frac{g(t,x(t))}{x(t)} = 1 - \frac{1}{e^t}$. Now, choose $\epsilon > 0$ such that $1 - \frac{1}{e^t} < \epsilon < 1$ and $p(t) = \epsilon$ with $\delta(t) = 1$. Consider

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(\delta(s)(q(s) - p(s)) - \frac{1}{4M}\frac{(\delta'(s))^2}{r(s)\delta(s)}\right) ds$$
$$= \limsup_{t\to\infty} \int_{t_1}^{t} (1 - \epsilon) ds \to \infty \ as \ t \to \infty.$$

Hence, all the conditions of Theorem 2.1 are satisfied. Therefore, every solution of (3.1) is oscillatory. In fact, $x(t) = e^t \sin t$ is one such solution of (3.1).

Example 3.2. Consider the second-order hybrid differential equation

$$\left(\frac{x(t)}{t}\right)'' + \frac{1}{t^2}x(t) = (t + 3)e^t, \quad t \geq 1. \tag{3.2}$$

Here $q(t) = \frac{1}{t^2}, g(t, x(t)) = (t + 3)e^t, M = 1, f(t, x(t)) = t, r(t) = 1$ and $p(t) = \frac{t+4}{t^2}$ with $\delta(t) = 1$. Consider

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(\delta(s)(q(s) - p(s)) - \frac{1}{4M}\frac{(\delta'(s))^2}{r(s)\delta(s)}\right) ds$$
$$= \limsup_{t\to\infty} \int_{t_1}^{t} \left(\frac{1}{s^2} - \frac{s+4}{s^3}\right) ds \leq \limsup_{t\to\infty} \int_{t_1}^{t} \frac{1}{s^2} ds = \infty.$$

Thus, the conditions of Theorem 2.1 are not satisfied. In fact, $x(t) = t^2 e^t$ is a nonoscillatory solution of (3.2).

Example 3.3. Consider the hybrid differential equation of second order

$$\left(\frac{x(t)}{e^t}\right)'' + x(t) = (e^t - 1)\sin t, \quad t \geq t_0. \tag{3.3}$$

Here $q(t) = 1, M = 1, f(t, x(t)) = e^t, r(t) = 1$, $g(t, x(t)) = (e^t - 1)\sin t$ and $\frac{g(t,x(t))}{x(t)} = 1 - \frac{1}{e^t}$. Now, choose $\epsilon > 0$ such that $1 - \frac{1}{e^t} < \epsilon < 1$, $p(t) = \epsilon$ and $\delta(t) = 1$ with $H(t, s) = t - s$ and

$h_1(t,s) = h_2(t,s) = \frac{1}{\sqrt{t-s}}$. Consider

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s)\delta(s)(q(s)-p(s)) - \frac{1}{4M} \frac{h_2^2(t,s)\delta(s)}{r(s)} \right) ds$$
$$= \limsup_{t\to\infty} \frac{1}{(t-t_0)} \int_{t_0}^{t} \left((t-s)(1-\epsilon) - \frac{1}{4}\frac{1}{t-s} \right) ds \to \infty \ as \ t \to \infty.$$

Thus, all the conditions of Theorem 2.2 are satisfied. Hence every solution of (3.3) is oscillatory. For example, $x(t) = e^t \cos t$ is one such solution.

Acknowledgment. The authors would like to thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of the manuscript.

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